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Continuum Mechanics for Engineers

Theory and Problems

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Translation by Ester Comellas

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Foreword

This book was born with the vocation of being a tool for the training of engineers in continuum mechanics. In fact, it is the fruit of the experience in teaching this discipline during many years at the Civil Engineering School of the Technical University of Catalonia (UPC/BarcelonaTech), both in undergraduate degrees (Civil Engineering and Geological Engineering) and postgraduate degrees (Master and PhD courses). Unlike other introductory texts to the mechanics of continuous media, the work presented here is specifically aimed at engineering students. We try to maintain a proper balance between the rigor of the mathematical formulation used and the clarity of the physical principles addressed, although always putting the former at the service of the latter. In this sense, the essential vector and tensor operations use simultaneously the indicial notation (more useful for rigorous mathematical proof) and the compact notation (which allows for a better understanding of the physics of the problem). However, as the text progresses, there is a clear trend towards compact notation in an attempt to focus the reader's attention on the physical component of continuum mechanics.

The text content is intentionally divided into two specific parts, which are presented sequentially. The first part (Chapters 1-5) introduces fundamental and descriptive aspects common to all continuous media (motion, deformation, stress and conservation-balance equations). In the second (Chapters 6 to 11), specific families of the continuous medium are studied, such as solids and fluids, in an approach that starts with the corresponding constitutive equation and ends with the classical formulations of solid mechanics (elastic-linear and elasto-plastic) and fluid mechanics (laminar regime). Finally, a brief incursion into the variational principles (principle of virtual work and minimization of potential energy) is attempted, to provide the initial ingredients needed to solve continuum mechanics problems using numerical methods. This structure allows the use of this text for teaching purposes both in a single course of about 100 teaching hours or as two different courses: the first based on the first five chapters dedicated to the introduction of the fundamentals of continuum mechanics and, the second, specifically dedicated to solid and fluid mechanics. The theoretical part in every chapter is followed by a number of solved problems and proposed exercises so as to help the reader in the understanding and consolidation of those theoretical aspects.

Finally, the authors wish to thank Dr. Ester Comellas for her translation work, from previous versions of the theoretical part of the book in Spanish and Catalan languages, as well as for her compilation of the book's problems and exercises from the authors' collection.

Barcelona, September 2016

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Chapter 1
Description of Motion

1.1 Definition of the Continuous Medium

A *continuous medium* is understood as an infinite set of particles (which form part of, for example, solids or fluids) that will be studied macroscopically, that is, without considering the possible discontinuities existing at microscopic level (atomic or molecular level). Accordingly, one admits that there are no discontinuities between the particles and that the mathematical description of this medium and its properties can be described by continuous functions.

1.2 Equations of Motion

The most basic description of the motion of a continuous medium can be achieved by means of mathematical functions that describe the position of each particle along time. In general, these functions and their derivatives are required to be continuous.

Definition 1.1. Consider the following definitions:

- *Spatial point*: Fixed point in space.
- *Material point*: A particle. It may occupy different spatial points during its motion along time.
- *Configuration*: Locus of the positions occupied in space by the particles of the continuous medium at a given time t.

The continuous medium is assumed to be composed of an infinite number of particles (*material points*) that occupy different positions in the physical space during its motion along time (see Figure 1.1). The *configuration* of the contin-

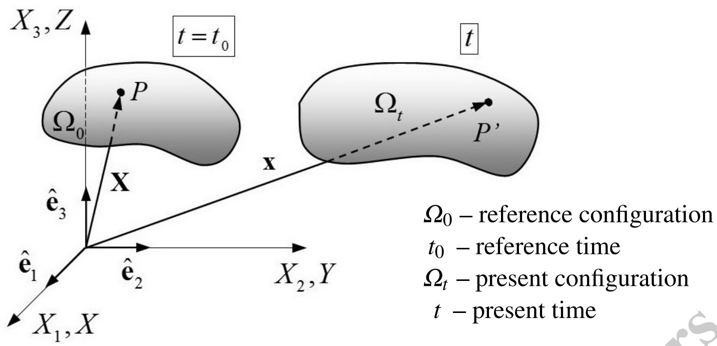

Figure 1.1: Configurations of the continuous medium.

uous medium at time t, denoted by Ω_t, is defined as the locus of the positions occupied in space by the material points (particles) of the continuous medium at the given time.

A certain time $t = t_0$ of the time interval of interest is referred to as the *reference time* and the configuration at this time, denoted by Ω_0, is referred to as *initial*, *material* or *reference configuration*[1].

Consider now the Cartesian coordinate system (X,Y,Z) in Figure 1.1 and the corresponding orthonormal basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$. In the reference configuration Ω_0, the position vector $\mathbf{X}$ of a particle occupying a point P in space (at the reference time) is given by[2,3]

$$\mathbf{X} = X_1 \hat{\mathbf{e}}_1 + X_2 \hat{\mathbf{e}}_2 + X_3 \hat{\mathbf{e}}_3 = X_i \hat{\mathbf{e}}_i, \tag{1.1}$$

where the components (X_1, X_2, X_3) are referred to as *material coordinates* (of the particle) and can be collected in a vector of components denoted as[4]

$$\mathbf{X} \overset{not}{\equiv} [\mathbf{X}] = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \overset{def}{=} \text{material coordinates.} \tag{1.2}$$

[1] In general, the time $t_0 = 0$ will be taken as the reference time.

[2] Notations (X,Y,Z) and (X_1, X_2, X_3) will be used indistinctly to designate the Cartesian coordinate system.

[3] *Einstein* or *repeated index notation* will be used in the remainder of this text. Every repetition of an index in the *same* monomial of an algebraic expression represents the sum over that index. For example,

$$\sum_{i=1}^{i=3} X_i \hat{\mathbf{e}}_i \overset{not}{\equiv} X_i \hat{\mathbf{e}}_i \quad , \qquad \sum_{k=1}^{k=3} a_{ik} b_{kj} \overset{not}{\equiv} a_{ik} b_{kj} \qquad \text{and} \qquad \sum_{i=1}^{i=3} \sum_{j=1}^{j=3} a_{ij} b_{ij} \overset{not}{\equiv} a_{ij} b_{ij}.$$

[4] Here, the vector (physical entity) $\mathbf{X}$ is distinguished from its vector of components $[\mathbf{X}]$. Henceforth, the symbol $\overset{not}{\equiv}$ (equivalent notation) will be used to indicate that the tensor and component notations at either side of the symbol are equivalent when the system of coordinates used remains unchanged.

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In the present configuration Ω_t [5], a particle originally located at a material point P (see Figure 1.1) occupies a spatial point P' and its position vector $\mathbf{x}$ is given by

$$\mathbf{x} = x_1 \hat{\mathbf{e}}_1 + x_2 \hat{\mathbf{e}}_2 + x_3 \hat{\mathbf{e}}_3 = x_i \hat{\mathbf{e}}_i , \tag{1.3}$$

where (x_1, x_2, x_3) are referred to as *spatial coordinates* of the particle at time t,

$$\mathbf{x} \stackrel{not}{\equiv} [\mathbf{x}] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \stackrel{def}{=} \text{spatial coordinates.} \tag{1.4}$$

The motion of the particles of the continuous medium can now be described by the evolution of their spatial coordinates (or their position vector) along time. Mathematically, this requires the definition of a function that provides for each particle (identified by its *label*) its spatial coordinates x_i (or its spatial position vector $\mathbf{x}$) at successive instants of time. The material coordinates X_i of the particle can be chosen as the label that univocally characterizes it and, thus, the *equation of motion*

$$\begin{cases} \mathbf{x} = \varphi\left(\text{particle}, t\right) = \varphi\left(\mathbf{X}, t\right) \stackrel{not}{=} \mathbf{x}\left(\mathbf{X}, t\right) \\ x_i = \varphi_i\left(X_1, X_2, X_3, t\right) \quad i \in \{1,2,3\} \end{cases} \tag{1.5}$$

is obtained, which provides the spatial coordinates in terms of the material ones. The spatial coordinates x_i of the particle can also be chosen as label, defining the *inverse equation of motion*[6] as

$$\begin{cases} \mathbf{X} = \varphi^{-1}\left(\mathbf{x}, t\right) \stackrel{not}{=} \mathbf{X}\left(\mathbf{x}, t\right) , \\ X_i = \varphi_i^{-1}\left(x_1, x_2, x_3, t\right) \quad i \in \{1,2,3\} , \end{cases} \tag{1.6}$$

which provides the material coordinates in terms of the spatial ones.

Remark 1.1. There are different alternatives when choosing the label that characterizes a particle, even though the option of using its material coordinates is the most common one. When the equation of motion is written in terms of the material coordinates as label (as in (1.5)), one refers to it as the *equation of motion in canonical form*.

---

[5] Whenever possible, uppercase letters will be used to denote variables relating to the reference configuration Ω_0 and lowercase letters to denote the variables referring to the current configuration Ω_t.

[6] With certain abuse of notation, the function will be frequently confused with its image. Hence, the equation of motion will be often written as $\mathbf{x} = \mathbf{x}\left(\mathbf{X}, t\right)$ and its inverse equation as $\mathbf{X} = \mathbf{X}\left(\mathbf{x}, t\right)$.

There exist certain mathematical restrictions to guarantee the existence of φ and φ^{-1}, as well as their correct physical meaning. These restrictions are:

- $\varphi(\mathbf{X},0) = \mathbf{X}$ since, by definition, $\mathbf{X}$ is the position vector at the reference time $t = 0$ (consistency condition).
- $\varphi \in C^1$ (function φ is continuous with continuous derivatives at each point and at each instant of time).
- φ is biunivocal (to guarantee that two particles do not occupy simultaneously the same point in space and that a particle does not occupy simultaneously more than one point in space).
- The Jacobian of the transformation $J = \det\left[\dfrac{\partial\varphi(\mathbf{X},t)}{\partial\mathbf{X}}\right] \overset{not}{=} \left|\dfrac{\partial\varphi(\mathbf{X},t)}{\partial\mathbf{X}}\right| > 0.$

The physical interpretation of this condition (which will be studied later) is that every differential volume must always be positive or, using the *principle of mass conservation* (which will be seen later), the density of the particles must always be positive.

> *Remark 1.2.* The equation of motion at the reference time $t = 0$ results in $\mathbf{x}(\mathbf{X},t)|_{t=0} = \mathbf{X}$. Accordingly, $x = X$, $y = Y$, $z = Z$ is the equation of motion at the reference time and the Jacobian at this instant of time is[7]
>
> $$J(\mathbf{X},0) = \left|\frac{\partial(xyz)}{\partial(XYZ)}\right| = \det\left[\frac{\partial x_i}{\partial X_j}\right] = \det[\delta_{ij}] = \det\mathbf{1} = 1.$$

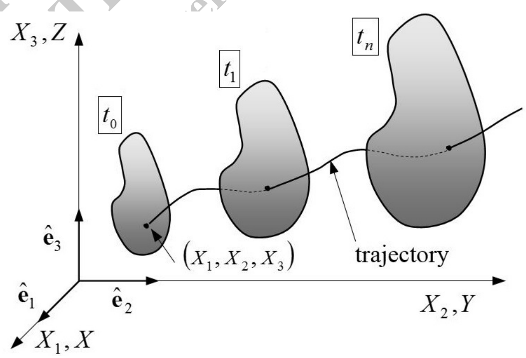

Figure 1.2: Trajectory or pathline of a particle.

[7] The two-index operator Delta Kronecker $\overset{not}{=} \delta_{ij}$ is defined as $\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ij} = 1$ when $i = j$. Then, the unit tensor $\mathbf{1}$ is defined as $[\mathbf{1}]_{ij} = \delta_{ij}$.

Remark 1.3. The expression $\mathbf{x} = \varphi(\mathbf{X},t)$, particularized for a fixed value of the material coordinates $\mathbf{X}$, provides the equation of the *trajectory* or *pathline* of a particle (see Figure 1.2).

Example 1.1 – *The spatial description of the motion of a continuous medium is given by*

$$\mathbf{x}(\mathbf{X},t) \stackrel{not}{\equiv} \begin{bmatrix} x_1 = X_1 e^{2t} \\ x_2 = X_2 e^{-2t} \\ x_3 = 5X_1 t + X_3 e^{2t} \end{bmatrix} = \begin{bmatrix} x = X e^{2t} \\ y = Y e^{-2t} \\ z = 5Xt + Z e^{2t} \end{bmatrix}$$

Obtain the inverse equation of motion.

Solution

The determinant of the Jacobian is computed as

$$J = \left| \frac{\partial x_i}{\partial X_j} \right| = \begin{vmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{vmatrix} = \begin{vmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 5t & 0 & e^{2t} \end{vmatrix} = e^{2t} \neq 0.$$

The sufficient (but not necessary) condition for the function $\mathbf{x} = \varphi(\mathbf{X},t)$ to be biunivocal (that is, for its inverse to exist) is that the determinant of the Jacobian of the function is not null. In addition, since the Jacobian is positive, the motion has physical sense. Therefore, the inverse of the given spatial description exists and is determined by

$$\mathbf{X} = \varphi^{-1}(\mathbf{x},t) \stackrel{not}{\equiv} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} x_1 e^{-2t} \\ x_2 e^{2t} \\ x_3 e^{-2t} - 5t x_1 e^{-4t} \end{bmatrix}.$$

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1.3 Descriptions of Motion

The mathematical description of the properties of the particles of the continuous medium can be addressed in two alternative ways: the *material* description (typically used in solid mechanics) and the *spatial* description (typically used in fluid mechanics). Both descriptions essentially differ in the type of argument (material coordinates or spatial coordinates) that appears in the mathematical functions that describe the properties of the continuous medium.

1.3.1 Material Description

In the material description[8], a given property (for example, the density ρ) is described by a certain function $\overline{\rho}\,(\bullet,t):\ R^3 \times R^+ \to R^+$, where the argument $(\bullet)$ in $\overline{\rho}\,(\bullet,t)$ represents the material coordinates,

$$\rho = \overline{\rho}\,(\mathbf{X},t) = \overline{\rho}\,(X_1,X_2,X_3,t)\,. \tag{1.7}$$

Here, if the three arguments $\mathbf{X} \equiv (X_1,X_2,X_3)$ are fixed, a specific particle is being followed (see Figure 1.3) and, hence, the name of material description.

1.3.2 Spatial Description

In the spatial description[9], the focus is on a point in space. The property is described as a function $\rho\,(\bullet,t):\ R^3 \times R^+ \to R^+$ of the point in space and of time,

$$\rho = \rho\,(\mathbf{x},t) = \rho\,(x_1,x_2,x_3,t)\,. \tag{1.8}$$

Then, when the argument $\mathbf{x}$ in $\rho = \rho\,(\mathbf{x},t)$ is assigned a certain value, the evolution of the density for the *different* particles that occupy the point in space along time is obtained (see Figure 1.3). Conversely, fixing the time argument in (1.8) results in an instantaneous distribution (like a *snapshot*) of the property in space. Obviously, the direct and inverse equations of motion allow shifting from one

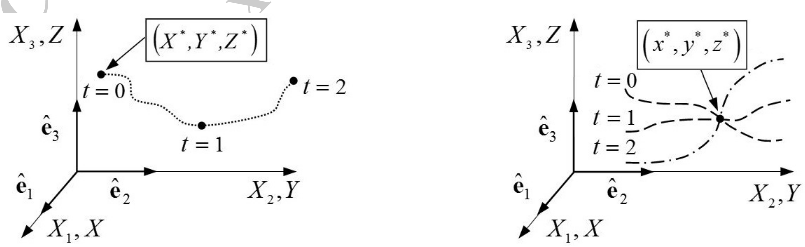

Figure 1.3: Material description (left) and spatial description (right) of a property.

[8] Literature on this topic also refers to the material description as *Lagrangian description*.
[9] The spatial description is also referred to as *Eulerian description*.

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description to the other as follows.

$$\begin{cases} \rho\left(\mathbf{x},t\right) = \rho\left(\mathbf{x}\left(\mathbf{X},t\right),t\right) = \overline{\rho}\left(\mathbf{X},t\right) \\ \overline{\rho}\left(\mathbf{X},t\right) = \overline{\rho}\left(\mathbf{X}\left(\mathbf{x},t\right),t\right) = \rho\left(\mathbf{x},t\right) \end{cases} \tag{1.9}$$

Example 1.2 – *The equation of motion of a continuous medium is*

$$\mathbf{x} = \mathbf{x}\left(\mathbf{X},t\right) \stackrel{not}{\equiv} \begin{bmatrix} x = X - Yt \\ y = Xt + Y \\ z = -Xt + Z \end{bmatrix} .$$

Obtain the spatial description of the property whose material description is

$$\overline{\rho}\left(X,Y,Z,t\right) = \frac{X+Y+Z}{1+t^2} .$$

Solution

The equation of motion is given in the canonical form since in the reference configuration Ω_0 its expression results in

$$\mathbf{x} = \mathbf{X}\left(\mathbf{X},0\right) \stackrel{not}{\equiv} \begin{bmatrix} x = X \\ y = Y \\ z = Z \end{bmatrix} .$$

The determinant of the Jacobian is

$$J = \left| \frac{\partial x_i}{\partial X_j} \right| = \begin{vmatrix} \dfrac{\partial x}{\partial X} & \dfrac{\partial x}{\partial Y} & \dfrac{\partial x}{\partial Z} \\ \dfrac{\partial y}{\partial X} & \dfrac{\partial y}{\partial Y} & \dfrac{\partial y}{\partial Z} \\ \dfrac{\partial z}{\partial X} & \dfrac{\partial z}{\partial Y} & \dfrac{\partial z}{\partial Z} \end{vmatrix} = \begin{vmatrix} 1 & -t & 0 \\ t & 1 & 0 \\ -t & 0 & 1 \end{vmatrix} = 1 + t^2 \neq 0$$

and the inverse equation of motion is given by

$$\mathbf{X}\left(\mathbf{x},t\right) \stackrel{not}{\equiv} \begin{bmatrix} X = \dfrac{x+yt}{1+t^2} \\ Y = \dfrac{y-xt}{1+t^2} \\ Z = \dfrac{z+zt^2+xt+yt^2}{1+t^2} \end{bmatrix} .$$

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Consider now the material description of the property,

$$\overline{\rho}\,(X,Y,Z,t) = \frac{X+Y+Z}{1+t^2},$$

its spatial description is obtained by introducing the inverse equation of motion into the expression above,

$$\overline{\rho}\,(X,Y,Z,t) \equiv \frac{x+yt+y+z+zt^2+yt^2}{\left(1+t^2\right)^2} = \rho\,(x,y,z,t)\,.$$

1.4 Time Derivatives: Local, Material and Convective

The consideration of different descriptions (material and spatial) of the properties of the continuous medium leads to diverse definitions of the time derivatives of these properties. Consider a certain property and its material and spatial descriptions,

$$\Gamma\,(\mathbf{X},t) = \gamma(\mathbf{x},t)\,, \tag{1.10}$$

in which the change from the spatial to the material description and vice versa is performed by means of the equation of motion (1.5) and its inverse equation (1.6).

Definition 1.2. The *local derivative* of a property is its variation along time at a fixed point in space. If the *spatial description* $\gamma(\mathbf{x},t)$ of the property is available, the local derivative is mathematically written as[10]

$$\text{local derivative} \overset{not}{=} \frac{\partial \gamma(\mathbf{x},t)}{\partial t}\,.$$

The *material derivative* of a property is its variation along time following a specific particle (material point) of the continuous medium. If the *material description* $\Gamma\,(\mathbf{X},t)$ of the property is available, the material derivative is mathematically written as

$$\text{material derivative} \overset{not}{=} \frac{d}{dt}\Gamma = \frac{\partial \Gamma\,(\mathbf{X},t)}{\partial t}\,.$$

[10] The expression $\partial\,(\bullet,t)/\partial t$ is understood in the classical sense of partial derivative with respect to the variable t.

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However, taking the spatial description of the property $\gamma(\mathbf{x},t)$ and considering the equation of motion is implicit in this expression yields

$$\gamma(\mathbf{x},t) = \gamma(\mathbf{x}(\mathbf{X},t),t) = \Gamma(\mathbf{X},t) \ . \tag{1.11}$$

Then, the material derivative (following a particle) is obtained from the *spatial description* of the property as

$$\text{material derivative} \overset{not}{=} \frac{d}{dt}\gamma(\mathbf{x}(\mathbf{X},t),t) = \frac{\partial \Gamma(\mathbf{X},t)}{\partial t} \ . \tag{1.12}$$

Expanding (1.12) results in[11]

$$\begin{aligned} \frac{d\gamma(\mathbf{x}(\mathbf{X},t),t)}{dt} &= \frac{\partial \gamma(\mathbf{x},t)}{\partial t} + \frac{\partial \gamma}{\partial x_i}\frac{\partial x_i}{\partial t} = \frac{\partial \gamma(\mathbf{x},t)}{\partial t} + \frac{\partial \gamma}{\partial \mathbf{x}} \cdot \underbrace{\frac{\partial \mathbf{x}}{\partial t}}_{\mathbf{v}(\mathbf{x},t)} = \\ &= \frac{\partial \gamma(\mathbf{x},t)}{\partial t} + \frac{\partial \gamma}{\partial \mathbf{x}} \cdot \mathbf{v}(\mathbf{x},t) \ , \end{aligned} \tag{1.13}$$

where the definition of velocity as the derivative of the equation of motion (1.5) with respect to time has been taken into account,

$$\frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} = \mathbf{V}(\mathbf{X}(\mathbf{x},t),t) = \mathbf{v}(\mathbf{x},t) \ . \tag{1.14}$$

The deduction of the material derivative from the spatial description can be generalized for any property $\chi(\mathbf{x},t)$ (of scalar, vectorial or tensorial character) as[12]

$$\underbrace{\frac{d\chi(\mathbf{x},t)}{dt}}_{\substack{\text{material}\\ \text{derivative}}} = \underbrace{\frac{\partial \chi(\mathbf{x},t)}{\partial t}}_{\substack{\text{local}\\ \text{derivative}}} + \underbrace{\mathbf{v}(\mathbf{x},t) \cdot \nabla \chi(\mathbf{x},t)}_{\substack{\text{convective}\\ \text{derivative}}} \ . \tag{1.15}$$

Remark 1.4. The expression in (1.15) implicitly defines the *convective derivative* $\mathbf{v} \cdot \nabla(\bullet)$ as the difference between the material and spatial derivatives of the property. In continuum mechanics, the term *convection* is applied to phenomena that are related to mass (or particle) transport. Note that, if there is no convection ($\mathbf{v} = \mathbf{0}$), the convective derivative disappears and the local and material derivatives coincide.

[11] In literature, the notation $D(\bullet)/Dt$ is often used as an alternative to $d(\bullet)/dt$.
[12] The symbolic form of the *spatial Nabla* operator, $\nabla \equiv \partial \hat{\mathbf{e}}_i / \partial x_i$, is considered here.

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Example 1.3 – *Given the equation of motion*

$$\mathbf{x}(\mathbf{X},t) \overset{not}{\equiv} \begin{bmatrix} x = X + Yt + Zt \\ y = Y + 2Zt \\ z = Z + 3Xt \end{bmatrix} ,$$

and the spatial description of a property, $\rho(\mathbf{x},t) = 3x + 2y + 3t$, *obtain the material derivative of this property.*

Solution

The material description of the property is obtained introducing the equation of motion into its spatial description,

$$\overline{\rho}(X,Y,Z,t) = 3(X + Yt + Zt) + 2(Y + 2Zt) + 3t = 3X + 3Yt + 7Zt + 2Y + 3t \, .$$

The material derivative is then calculated as the derivative of the material description with respect to time,

$$\frac{\partial \overline{\rho}}{\partial t} = 3Y + 7Z + 3 \, .$$

An alternative way of deducing the material derivative is by using the concept of material derivative of the spatial description of the property,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho \quad \text{with}$$

$$\frac{\partial \rho}{\partial t} = 3 \, , \quad \mathbf{v} = \frac{\partial \mathbf{x}}{\partial t} = [Y + Z, \; 2Z, \; 3X]^T \quad \text{and} \quad \nabla \rho = [3, \; 2, \; 0]^T \, .$$

Replacing in the expression of the material derivative operator,

$$\frac{d\rho}{dt} = 3 + 3Y + 7Z$$

is obtained. Note that the expressions for the material derivative obtained from the material description, $\partial \overline{\rho} / \partial t$, and the spatial description, $d\rho / dt$, coincide.

1.5 Velocity and Acceleration

Definition 1.3. The *velocity* is the time derivative of the equation of motion.

The material description of velocity is, consequently, given by

$$\begin{cases} \mathbf{V}(\mathbf{X},t) = \dfrac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} \\ V_i(\mathbf{X},t) = \dfrac{\partial x_i(\mathbf{X},t)}{\partial t} \quad i \in \{1,\ 2,\ 3\} \end{cases} \tag{1.16}$$

and, if the inverse equation of motion $\mathbf{X} = \varphi^{-1}(\mathbf{x},t)$ is known, the spatial description of the velocity can be obtained as

$$\mathbf{v}(\mathbf{x},t) = \mathbf{V}(\mathbf{X}(\mathbf{x},t),t) \ . \tag{1.17}$$

Definition 1.4. The *acceleration* is the time derivative of the velocity field.

If the velocity is described in material form, the material description of the acceleration is given by

$$\begin{cases} \mathbf{A}(\mathbf{X},t) = \dfrac{\partial \mathbf{V}(\mathbf{X},t)}{\partial t} \\ A_i(\mathbf{X},t) = \dfrac{\partial V_i(\mathbf{X},t)}{\partial t} \quad i \in \{1,\ 2,\ 3\} \end{cases} \tag{1.18}$$

and, through the inverse equation of motion $\mathbf{X} = \varphi^{-1}(\mathbf{x},t)$, the spatial description is obtained, $\mathbf{a}(\mathbf{x},t) = \mathbf{A}(\mathbf{X}(\mathbf{x},t),t)$. Alternatively, if the spatial description of the velocity is available, applying (1.15) to obtain the material derivative of $\mathbf{v}(\mathbf{x},t)$,

$$\mathbf{a}(\mathbf{x},t) = \frac{d\mathbf{v}(\mathbf{x},t)}{dt} = \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t) \cdot \nabla \mathbf{v}(\mathbf{x},t) \ , \tag{1.19}$$

directly yields the spatial description of the acceleration.

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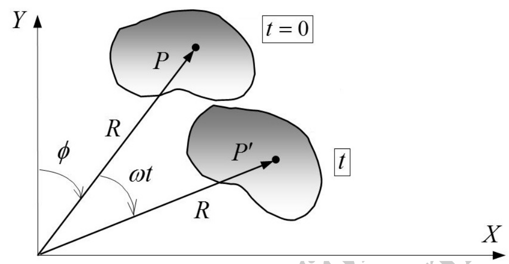
Example 1.4 – *Consider the solid in the figure below, which rotates at a constant angular velocity* ω *and has the expression*

$$\begin{cases} x = R\sin(\omega t + \phi) \\ y = R\cos(\omega t + \phi) \end{cases}$$

as its equation of motion. Find the velocity and acceleration of the motion described both in material and spatial forms.



Solution

The equation of motion can be rewritten as

$$\begin{cases} x = R\sin(\omega t + \phi) = R\sin(\omega t)\cos\phi + R\cos(\omega t)\sin\phi \\ y = R\cos(\omega t + \phi) = R\cos(\omega t)\cos\phi - R\sin(\omega t)\sin\phi \end{cases}$$

and, since for $t = 0$, $X = R\sin\phi$ and $Y = R\cos\phi$, the canonical form of the equation of motion and its inverse equation result in

$$\begin{cases} x = X\cos(\omega t) + Y\sin(\omega t) \\ y = -X\sin(\omega t) + Y\cos(\omega t) \end{cases} \quad \text{and} \quad \begin{cases} X = x\cos(\omega t) - y\sin(\omega t) \\ Y = x\sin(\omega t) + y\cos(\omega t) \end{cases} .$$

Velocity in material description:

$$\mathbf{V}(\mathbf{X},t) = \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t}) \stackrel{not}{\equiv} \begin{bmatrix} \dfrac{\partial x}{\partial t} = -X\omega\sin(\omega t) + Y\omega\cos(\omega t) \\ \dfrac{\partial y}{\partial t} = -X\omega\cos(\omega t) - Y\omega\sin(\omega t) \end{bmatrix}$$

Velocity in spatial description:
Replacing the canonical form of the equation of motion into the material description of the velocity results in

$$\mathbf{v}(\mathbf{x},t) = \mathbf{V}(\mathbf{X}(\mathbf{x},t),t) \stackrel{not}{\equiv} \begin{bmatrix} \omega y \\ -\omega x \end{bmatrix} .$$

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Acceleration in material description:

$$\mathbf{A}(\mathbf{X},t) = \frac{\partial \mathbf{V}(\mathbf{X},t)}{\partial t}$$

$$\mathbf{A}(\mathbf{X},t) \overset{not}{\equiv} \begin{bmatrix} \dfrac{\partial \mathrm{v}_x}{\partial t} = -X\omega^2 \cos(\omega t) - Y\omega^2 \sin(\omega t) \\ \dfrac{\partial \mathrm{v}_y}{\partial t} = X\omega^2 \sin(\omega t) - Y\omega^2 \cos(\omega t) \end{bmatrix} =$$

$$= -\omega^2 \begin{bmatrix} X\cos(\omega t) + Y\sin(\omega t) \\ -X\sin(\omega t) + Y\cos(\omega t) \end{bmatrix} .$$

Acceleration in spatial description:
Replacing the canonical form of the equation of motion into the material description of the acceleration results in

$$\mathbf{a}(\mathbf{x},t) = \mathbf{A}(\mathbf{X}(\mathbf{x},t),t) \overset{not}{\equiv} \begin{bmatrix} -\omega^2 x \\ -\omega^2 y \end{bmatrix} .$$

This same expression can be obtained if the expression for the velocity $\mathbf{v}(\mathbf{x},t)$ and the definition of material derivative in (1.15) are taken into account,

$$\mathbf{a}(\mathbf{x},t) = \frac{d\mathbf{v}(\mathbf{x},t)}{dt} = \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t)\cdot\nabla\mathbf{v}(\mathbf{x},t) =$$

$$\overset{not}{\equiv} \frac{\partial}{\partial t}\begin{bmatrix} \omega y \\ -\omega x \end{bmatrix} + \begin{bmatrix} \omega y \,, & -\omega x \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} \omega y \,, & -\omega x \end{bmatrix} ,$$

$$\mathbf{a}(\mathbf{x},t) \overset{not}{\equiv} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \omega y \,, & -\omega x \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x}(\omega y) & \dfrac{\partial}{\partial x}(-\omega x) \\ \dfrac{\partial}{\partial y}(\omega y) & \dfrac{\partial}{\partial y}(-\omega x) \end{bmatrix} = \begin{bmatrix} -\omega^2 x \\ -\omega^2 y \end{bmatrix} .$$

Note that the result obtained using both procedures is identical.

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1.6 Stationarity

Definition 1.5. A property is *stationary* when its spatial description does not depend on time.

According to the above definition, and considering the concept of local derivative, any stationary property has a null local derivative. For example, if the velocity for a certain motion is stationary, it can be described in spatial form as

$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}(\mathbf{x}) \iff \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} = 0 \ . \tag{1.20}$$

Remark 1.5. The non-dependence on time of the spatial description (stationarity) assumes that, *for a same point in space*, the property being considered does not vary along time. This does not imply that, *for a same particle*, such property does not vary along time (the material description may depend on time). For example, if the velocity $\mathbf{v}(\mathbf{x},t)$ is stationary,

$$\mathbf{v}(\mathbf{x},t) \equiv \mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{x}(\mathbf{X},t)) = \mathbf{V}(\mathbf{X},t) \ ,$$

and, thus, the material description of the velocity depends on time. In the case of stationary density (see Figure 1.4), for two particles labeled $\mathbf{X}_1$ and $\mathbf{X}_2$ that have varying densities along time, when occupying a same spatial point $\mathbf{x}$ (at two different times t_1 and t_2) their density value will coincide,

$$\overline{\rho}(\mathbf{X}_1,t_1) = \overline{\rho}(\mathbf{X}_2,t_2) = \rho(\mathbf{x}) \ .$$

That is, for an observer placed outside the medium, the density of the fixed point in space $\mathbf{x}$ will always be the same.

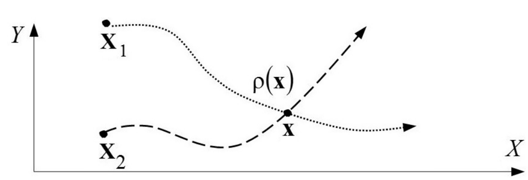

Figure 1.4: Motion of two particles with stationary density.

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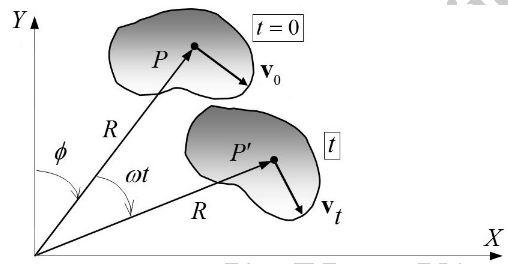
Example 1.5 – *Justify if the motion described in Example 1.4 is stationary or not.*

Solution

The velocity field in Example 1.4 is $\mathbf{v}(\mathbf{x}) \overset{not}{\equiv} [\omega y\, ,\; -\omega x]^T$. Therefore, it is a case in which the spatial description of the velocity is not dependent on time and, thus, the velocity is stationary. Obviously, this implies that the velocity of the particles (whose motion is a *uniform rotation* with respect to the origin, with angular velocity ω) does not depend on time (see figure below). The direction of the velocity vector for a same particle is tangent to its circular trajectory and changes along time.



The acceleration (material derivative of the velocity),

$$\mathbf{a}(\mathbf{x}) = \frac{d\mathbf{v}(\mathbf{x})}{dt} = \frac{\partial \mathbf{v}(\mathbf{x})}{\partial t} + \mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{x}) \ ,$$

appears due to the change in direction of the velocity vector of the particles and is known as the *centripetal acceleration*.

1.7 Trajectory

Definition 1.6. A *trajectory* (or *pathline*) is the locus of the positions occupied in space by a given particle along time.

The parametric equation of a trajectory as a function of time is obtained by particularizing the equation of motion for a given particle (identified by its material coordinates $\mathbf{X}^*$, see Figure 1.5),

$$\mathbf{x}(t) = \varphi(\mathbf{X}, t)\Big|_{\mathbf{X}=\mathbf{X}^*} \ . \tag{1.21}$$

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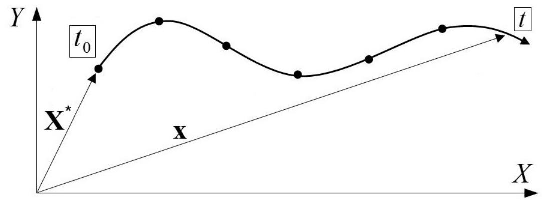

Figure 1.5: Trajectory or pathline of a particle.

Given the equation of motion $\mathbf{x} = \varphi(\mathbf{X},t)$, each point in space is occupied by a trajectory characterized by the value of the label (material coordinates) $\mathbf{X}$. Then, the equation of motion defines a family of curves whose elements are the trajectories of the various particles.

1.7.1 Differential Equation of the Trajectories

Given the velocity field in spatial description $\mathbf{v}(\mathbf{x},t)$, the family of trajectories can be obtained by formulating the system of differential equations that imposes that, for each point in space $\mathbf{x}$, the velocity vector is the time derivative of the parametric equation of the trajectory defined in (1.21), i.e.,

$$\text{Find } \mathbf{x}(t) := \begin{cases} \dfrac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x}(t),t) \ , \\ \dfrac{dx_i(t)}{dt} = \mathrm{v}_i(x(t),t) \quad i \in \{1,\, 2,\, 3\} \ . \end{cases} \tag{1.22}$$

The solution to this first-order system of differential equations depends on three integration constants (C_1,C_2,C_3),

$$\begin{cases} \mathbf{x} = \phi(C_1,C_2,C_3,t) \ , \\ x = \phi_i(C_1,C_2,C_3,t) \quad i \in \{1,\, 2,\, 3\} \ . \end{cases} \tag{1.23}$$

These expressions constitute a family of curves in space parametrized by the constants (C_1,C_2,C_3). Assigning a particular value to these constants yields a member of the family, which is the trajectory of a particle characterized by the label (C_1,C_2,C_3).

To obtain the equation in canonical form, the consistency condition is imposed in the reference configuration,

$$\mathbf{x}(t)\Big|_{t=0} = \mathbf{X} \implies \mathbf{X} = \phi(C_1,C_2,C_3,0) \implies C_i = \chi_i(\mathbf{X}) \ \ i \in \{1,\, 2,\, 3\}, \tag{1.24}$$

and, replacing into (1.23), the canonical form of the equation of the trajectory,

$$\mathbf{X} = \phi(C_1(\mathbf{X}),C_2(\mathbf{X}),C_3(\mathbf{X}),t) = \varphi(\mathbf{X},t) \ , \tag{1.25}$$

is obtained.

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Example 1.6 – *Given the velocity field in Example 1.5,* $\mathbf{v}(\mathbf{x}) \overset{not}{\equiv} [\omega y\ ,\ -\omega x]^T$, *obtain the equation of the trajectory.*

Solution

Using expression (1.22), one can write

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{v}(\mathbf{x},t) \quad \Longrightarrow \quad \begin{cases} \dfrac{dx(t)}{dt} = \mathrm{v}_x(\mathbf{x},t) = \omega y\ , \\ \dfrac{dy(t)}{dt} = \mathrm{v}_y(\mathbf{x},t) = -\omega x\ . \end{cases}$$

This system of equations is a system with crossed variables. Differentiating the second equation and replacing the result obtained into the first equation yields

$$\frac{d^2y(t)}{dt^2} = -\omega \frac{dx(t)}{dt} = -\omega^2 y(t) \quad \Longrightarrow \quad y'' + \omega^2 y = 0\ .$$

The *characteristic equation* of this second-order differential equation is $r^2 + \omega^2 = 0$ and its *characteristic solutions* are $r_j = \pm i\omega \quad j \in \{1,\ 2\}$. Therefore, the y component of the equation of the trajectory is

$$y(t) = \text{Real Part}\left\{C_1 e^{iwt} + C_2 e^{-iwt}\right\} = C_1 \cos(\omega t) + C_2 \sin(\omega t)\ .$$

The solution for $x(t)$ is obtained from $dy/dt = -\omega x$, which results in $x = -dy/(\omega\, dt)$ and, therefore,

$$\begin{cases} x(C_1,C_2,t) = C_1 \sin(\omega t) - C_2 \cos(\omega t)\ , \\ y(C_1,C_2,t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)\ . \end{cases}$$

This equation provides the expressions of the trajectories in a non-canonical form. The canonical form is obtained considering the initial condition,

$$\mathbf{x}(C_1,C_2,0) = \mathbf{X}\ ,$$

that is,

$$\begin{cases} x(C_1,C_2,0) = -C_2 = X\ , \\ y(C_1,C_2,0) = C_1 = Y\ . \end{cases}$$

Finally, the equation of motion, or the equation of the trajectory, in canonical form

$$\begin{cases} x = Y \sin(\omega t) + X \cos(\omega t) \\ y = Y \cos(\omega t) - X \sin(\omega t) \end{cases}$$

is obtained.

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1.8 Streamline

Definition 1.7. The *streamlines* are a family of curves that, for every instant of time, are the velocity field envelopes[13].

According to its definition, the tangent at each point of a streamline has the same direction (though not necessarily the same magnitude) as the velocity vector at that same point in space.

Remark 1.6. In general, the velocity field (in spatial description) will be different for each instant of time ($\mathbf{v} \equiv \mathbf{v}(\mathbf{x},t)$). Therefore, one must speak of a different family of streamlines for each instant of time (see Figure 1.6).

1.8.1 *Differential Equation of the Streamlines*

Consider a given time t^* and the spatial description of the velocity field at this time $\mathbf{v}(\mathbf{x},t^*)$. Let $\mathbf{x}(\lambda)$ be the equation of a streamline parametrized in terms of a certain parameter λ. Then, the vector tangent to the streamline is defined, for

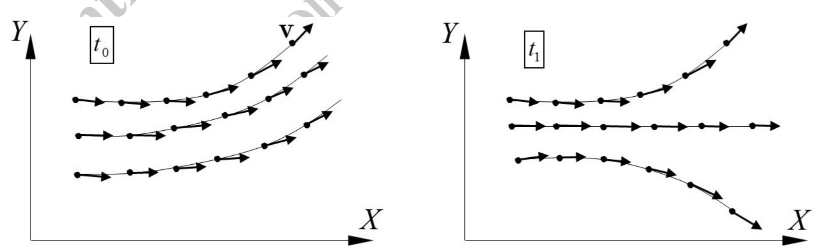

Figure 1.6: Streamlines at two different instants of time.

[13] The envelopes of a vector field are the family of curves whose tangent vector has, at each point, the same direction as the corresponding vector of the vector field.

each value of λ [14], by $d\mathbf{x}(\lambda)/d\lambda$ and the vector field tangency condition can be written as follows.

$$\text{Find} \quad \mathbf{x}(\lambda) := \begin{cases} \dfrac{d\mathbf{x}(\lambda)}{d\lambda} = \mathbf{v}(\mathbf{x}(\lambda), t^*) \,, \\ \dfrac{dx_i(\lambda)}{d\lambda} = \mathrm{v}_i(\mathbf{x}(\lambda), t^*) \quad i \in \{1, 2, 3\} \,. \end{cases} \tag{1.26}$$

The expressions in (1.26) constitute a system of first-order differential equations whose solution for each time t^*, which will depend on three integration constants (C_1', C_2', C_3'), provides the parametric expression of the streamlines,

$$\begin{cases} \mathbf{x} = \phi(C_1', C_2', C_3', \lambda, t^*) \,, \\ x_i = \phi_i(C_1', C_2', C_3', \lambda, t^*) \quad i \in \{1, 2, 3\} \,. \end{cases} \tag{1.27}$$

Each triplet of integration constants (C_1', C_2', C_3') identifies a streamline whose points, in turn, are obtained by assigning values to the parameter λ. For each time t^* a new family of streamlines is obtained.

Remark 1.7. In a stationary velocity field $(\mathbf{v}(\mathbf{x},t) \equiv \mathbf{v}(\mathbf{x}))$ the *trajectories* and *streamlines coincide*. This can be proven from two different viewpoints:

- The fact that the time variable does not appear in (1.22) or (1.26) means that the differential equations defining the trajectories and those defining the streamlines only differ in the *denomination* of the integration parameter (t or λ, respectively). The solution to both systems must be, therefore, the same, except for the *name* of the parameter used in each type of curves.
- From a more physical point of view: a) If the velocity field is stationary, its envelopes (the streamlines) do not change along time; b) a given particle moves in space keeping the trajectory in the direction tangent to the velocity field it encounters along time; c) consequently, if a trajectory starts at a certain point in a streamline, it will stay on this streamline throughout time.

[14] It is assumed that the value of the parameter λ is chosen such that, at each point in space $\mathbf{x}$, not only does $d\mathbf{x}(\lambda)/d\lambda$ have the same direction as the vector $\mathbf{v}(\mathbf{x},t)$, but it coincides therewith.

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1.9 Streamtubes

Definition 1.8. A *streamtube* is a surface formed by a bundle of streamlines that occupy the points of a closed line, fixed in space, and that *does not* constitute a streamline.

In non-stationary cases, even though the closed line does not vary in space, the streamtube and streamlines do change. On the contrary, in a stationary case, the streamtube remains fixed in space along time.

1.9.1 Equation of the Streamtube

Streamlines constitute a family of curves of the type

$$\mathbf{x} = \mathbf{f}(C_1, C_2, C_3, \lambda, t) \ . \tag{1.28}$$

The problem consists in determining, for each instant of time, which curves of the family of curves of the streamlines cross a closed line, which is fixed in the space Γ, whose mathematical expression parametrized in terms of a parameter s is

$$\Gamma := \mathbf{x} = \mathbf{g}(s) \ . \tag{1.29}$$

To this aim, one imposes, in terms of the parameters λ^* and s^*, that a same point belong to both curves,

$$\begin{cases} \mathbf{g}(s^*) = \mathbf{f}(C_1, C_2, C_3, \lambda^*, t) \ , \\ g_i(s^*) = f_i(C_1, C_2, C_3, \lambda^*, t) \quad i \in \{1,\ 2,\ 3\} \ . \end{cases} \tag{1.30}$$

A system of three equations is obtained from which, for example, s^*, λ^* and C_3 can be isolated,

$$\begin{aligned} s^* &= s^*(C_1, C_2, t) \ , \\ \lambda^* &= \lambda^*(C_1, C_2, t) \ , \\ C_3 &= C_3(C_1, C_2, t) \ . \end{aligned} \tag{1.31}$$

Introducing (1.31) into (1.30) yields

$$\mathbf{x} = \mathbf{f}(C_1,\ C_2,\ C_3(C_1, C_2, t),\ \lambda^*(C_1, C_2, t),\ t) = \mathbf{h}(C_1, C_2, t) \ , \tag{1.32}$$

which constitutes the parametrized expression (in terms of the parameters C_1 and C_2) of the streamtube for each time t (see Figure 1.7).

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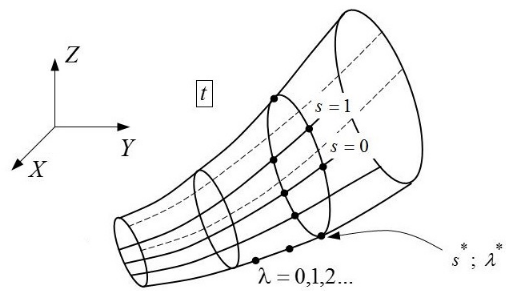

Figure 1.7: Streamtube at a given time t.

1.10 Streaklines

> **Definition 1.9.** A *streakline*, relative to a fixed point in space $\mathbf{x}^*$ named *spill point* and at a time interval $\left[t_i, t_f\right]$ named *spill period*, is the locus of the positions occupied at time t by all the particles that have occupied $\mathbf{x}^*$ over the time $\tau \in [t_i, t] \bigcap \left[t_i, t_f\right]$.

The above definition corresponds to the physical concept of the color line (streak) that would be observed in the medium at time t if a tracer fluid were injected at spill point $\mathbf{x}^*$ throughout the time interval $\left[t_i, t_f\right]$ (see Figure 1.8).

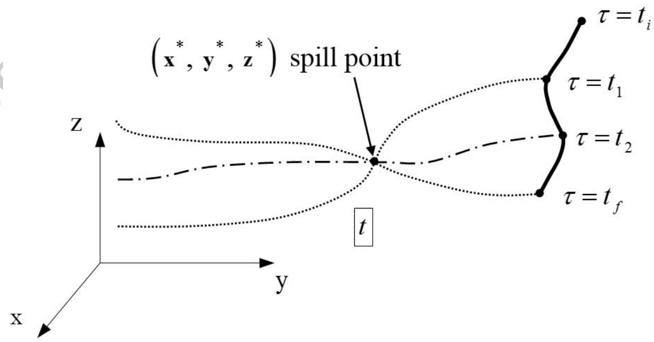

Figure 1.8: Streakline corresponding to the spill period $\tau \in \left[t_i, t_f\right]$.

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1.10.1 Equation of the Streakline

To determine the equation of a streakline one must identify all the particles that occupy point $\mathbf{x}^*$ in the corresponding times τ. Given the equation of motion (1.5) and its inverse equation (1.6), the label of the particle which at time τ occupies the spill point must be identified. Then,

$$\left.\begin{array}{l}\mathbf{x}^* = \mathbf{x}(\mathbf{X},\tau) \\ x_i^* = x_i(\mathbf{X},\tau) \quad i\in\{1,\,2,\,3\}\end{array}\right\} \implies \mathbf{X} = \mathbf{f}(\tau) \tag{1.33}$$

and replacing (1.33) into the equation of motion (1.5) results in

$$\mathbf{x} = \varphi(\mathbf{f}(\tau),t) = \mathbf{g}(\tau,t) \quad \tau \in [t_i,t]\bigcap\left[t_i,t_f\right] . \tag{1.34}$$

Expression (1.34) is, for each time t, the parametric expression (in terms of parameter τ) of a curvilinear segment in space which is the streakline at that time.

Example 1.7 – *Given the equation of motion*

$$\begin{cases} x = (X+Y)\,t^2 + X\cos t \,, \\ y = (X+Y)\cos t - X \,, \end{cases}$$

obtain the equation of the streakline associated with the spill point $\mathbf{x}^* = (0,1)$ *for the spill period* $[t_0,+\infty)$.

Solution

The material coordinates of a particle that has occupied the spill point at time τ are given by

$$\left.\begin{array}{l} 0 = (X+Y)\,\tau^2 + X\cos\tau \\ 1 = (X+Y)\cos\tau - X \end{array}\right\} \implies \begin{cases} X = \dfrac{-\tau^2}{\tau^2+\cos^2\tau} \,, \\[2ex] Y = \dfrac{\tau^2+\cos\tau}{\tau^2+\cos^2\tau} \,. \end{cases}$$

Therefore, the label of the particles that have occupied the spill point from the initial spill time t_0 until the present time t is defined by

$$\left.\begin{array}{l} X = \dfrac{-\tau^2}{\tau^2+\cos^2\tau} \\[2ex] Y = \dfrac{\tau^2+\cos\tau}{\tau^2+\cos^2\tau} \end{array}\right\} \quad \tau \in [t_0,t]\bigcap[t_0,\infty) = [t_0,t] \ .$$

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Then, replacing these into the equation of motion, the equation of the streakline is obtained,

$$\mathbf{x} = \mathbf{g}(\tau, t) \stackrel{not}{\equiv} \begin{bmatrix} x = \dfrac{\cos\tau}{\tau^2 + \cos^2\tau} t^2 + \dfrac{-\tau^2}{\tau^2 + \cos^2\tau} \cos t \\ y = \dfrac{\cos\tau}{\tau^2 + \cos^2\tau} \cos t - \dfrac{-\tau^2}{\tau^2 + \cos^2\tau} \end{bmatrix} \qquad \tau \in [t_0, t] \ .$$

Remark 1.8. In a *stationary* problem, the streaklines are segments of the trajectories (or of the streamlines). The rationale is based on the fact that, in the stationary case, the trajectory follows the envelope of the velocity field, which remains constant along time. If one considers a spill point $\mathbf{x}^*$, all the particles that occupy this point will follow portions (segments) of the same trajectory.

1.11 Material Surface

Definition 1.10. A *material surface* is a mobile surface in space always constituted by the same particles (material points).

In the reference configuration Ω_0, surface Σ_0 can be defined in terms of a function of the material coordinates $F(X,Y,Z)$ as

$$\Sigma_0 := \{X,Y,Z \mid F(X,Y,Z) = 0\} \ . \tag{1.35}$$

Remark 1.9. The function $F(X,Y,Z)$ does not depend on time, which guarantees that the particles, identified by their label, that satisfy equation $F(X,Y,Z) = 0$ are always the same in accordance with the definition of material surface.

The spatial description of the surface is obtained from the spatial description of $F(\mathbf{X}(\mathbf{x},t)) = f(x,y,z,t)$ as

$$\Sigma_t := \{x,y,z \mid f(x,y,z,t) = 0\} \ . \tag{1.36}$$

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Remark 1.10. The function $f(x,y,z,t)$ depends explicitly on time, which indicates that the *points in space* that are on the surface will vary along time. This time dependence of the spatial description of the surface confers the character of mobile surface in space to the surface (see Figure 1.9).

Remark 1.11. The necessary and sufficient condition for a mobile surface in space, defined implicitly by a function $f(x,y,z,t)=0$, to be material (to be always constituted by the same particles) is that *the material derivative of $f(x,y,z,t)$ is null*,

$$\frac{df(\mathbf{x},t)}{dt}=\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla f=0 \qquad \forall\mathbf{x}\in\Sigma_t \quad \forall t\ .$$

The condition is *necessary* because, if the surface is a material surface, its material description will not depend on time $(F\equiv F(\mathbf{X}))$ and, therefore, its spatial description will have a null material derivative. The condition of *sufficiency* is based on the fact that, if the material derivative of $f(\mathbf{x},t)$ is zero, the corresponding material description will not depend on time $(F\equiv F(\mathbf{X}))$ and, therefore, the set of particles (identified by their material coordinates) that satisfy the condition $F(\mathbf{X})=0$ is always the same.

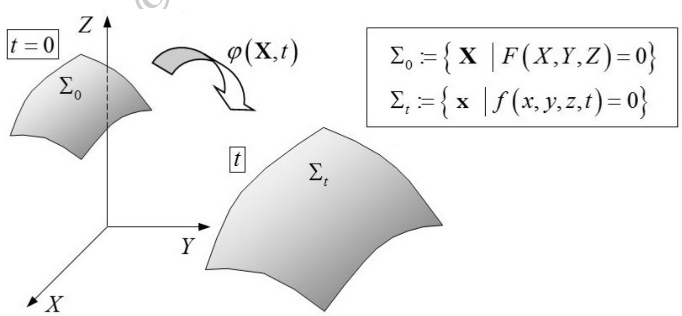

Figure 1.9: A material surface at two different instants of time.

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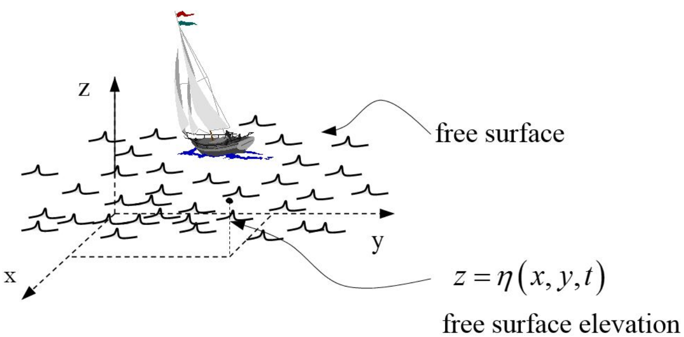
Example 1.8 – *In ocean waves theory, the condition that the free surface of the fluid in contact with the atmosphere is a material surface is imposed. This restriction implies that the free surface is always composed of the same particles, which is a reasonable hypothesis (especially in deep waters). Determine how this condition is stated in terms of the velocity field of the fluid.*

Solution

Assuming that $z = \eta(x,y,t)$ defines the elevation of the sea surface with respect to a reference level, the free surface of the water will be given by

$$f(x,y,z,t) \equiv z - \eta(x,y,t) = 0 \ .$$



The condition $df/dt = 0$ can be written as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f \quad \text{where} \quad \frac{\partial f}{\partial t} = -\frac{\partial \eta}{\partial t} \quad \text{and}$$

$$\mathbf{v} \cdot \nabla f \overset{not}{\equiv} [\mathrm{v}_x,\ \mathrm{v}_y,\ \mathrm{v}_z] \left[\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z} \right]^T = \mathrm{v}_x \frac{\partial f}{\partial x} + \mathrm{v}_y \frac{\partial f}{\partial y} + \mathrm{v}_z \frac{\partial f}{\partial z} \ .$$

Then,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = -\frac{\partial \eta}{\partial t} - \mathrm{v}_x \frac{\partial \eta}{\partial x} - \mathrm{v}_y \frac{\partial \eta}{\partial y} + \mathrm{v}_z = 0$$

and, isolating v_z leads to

$$\mathrm{v}_z = \frac{\partial \eta}{\partial t} + \mathrm{v}_x \frac{\partial \eta}{\partial x} + \mathrm{v}_y \frac{\partial \eta}{\partial y} \ .$$

Therefore, the material surface condition results in a condition on the vertical component of the velocity field.

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1.12 Control Surface

Definition 1.11. A *control surface* is a fixed surface in space.

The mathematical description of a control surface is given by

$$\Sigma := \{\mathbf{x} \mid f(x,y,z) = 0\} \ . \tag{1.37}$$

Obviously, a control surface is occupied by the different particles of the continuous medium along time (see Figure 1.10).

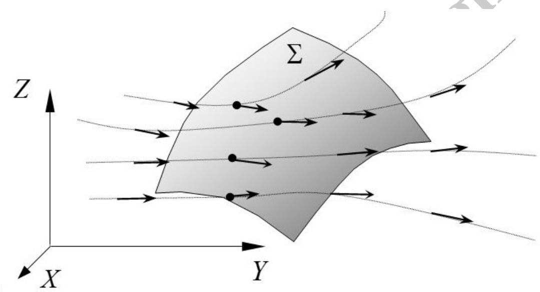

Figure 1.10: Movement of particles through a control surface along time.

1.13 Material Volume

Definition 1.12. A *material volume* is a volume enclosed by a closed material surface.

The mathematical description of a material volume (see Figure 1.11) is given, in the material description, by[15]

$$V_0 := \{\mathbf{X} \mid F(\mathbf{X}) \leq 0\} \tag{1.38}$$

and, in the spatial description, by

$$V_t := \{\mathbf{x} \mid f(\mathbf{x},t) \leq 0\} , \tag{1.39}$$

[15] It is assumed that function $F(\mathbf{X})$ is defined such that $F(\mathbf{X}) < 0$ corresponds to points in the interior of V_0.

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where $F(\mathbf{X}) = f(\mathbf{x}(\mathbf{X},t),t)$ is the function that describes the material surface that encloses the volume.

> *Remark 1.12.* A material volume is always constituted by the same particles. This is proven by *reductio ad absurdum* as follows. If a certain particle could enter or exit the material volume, it would be incorporated into the material surface during its motion (at least, for an instant of time). This would be contrary to the fact that the surface, being a material surface, is always constituted by the same particles.

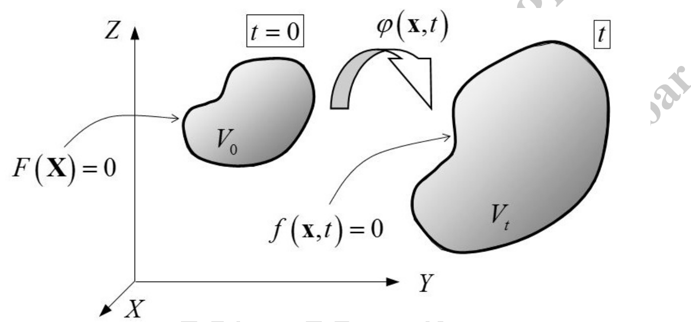

Figure 1.11: A material volume at two different instants of time.

1.14 Control Volume

> **Definition 1.13.** A *control volume* is a group of points in space situated in the interior of a closed control surface.

It is a volume fixed in space that is occupied by the particles of the medium during its motion. The mathematical description of the control volume (see Figure 1.12) is[16]

$$V := \{\mathbf{x} \mid f(\mathbf{x}) \leq 0\} \ . \tag{1.40}$$

[16] It is assumed that function $f(\mathbf{x})$ is defined such that $f(\mathbf{x}) < 0$ corresponds to points in the interior of V.

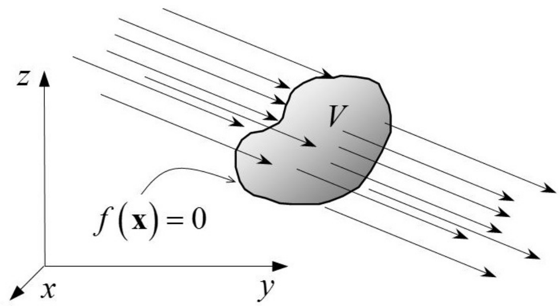

Figure 1.12: A control volume is occupied by different particles along time.

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PROBLEMS

Problem 1.1 – *Justify whether the following statements are true or false.*

a) If the velocity field is stationary, the acceleration field is also stationary.

b) If the velocity field is uniform, the acceleration field is always null.

Solution

a) A stationary velocity field implies that the spatial description of velocity does not depend on time,

$$\frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} = 0 \quad \Longrightarrow \quad \mathbf{v}(\mathbf{x}) \; .$$

The acceleration is the material derivative of the velocity, therefore

$$\mathbf{a}(\mathbf{x},t) = \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t) \cdot \nabla \mathbf{v}(\mathbf{x},t) = \mathbf{v}(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{x}) \; .$$

The resulting expression does not depend on time. Thus, *the statement is true*.

b) A uniform velocity field implies that the spatial description of velocity does not depend on the spatial coordinates,

$$\mathbf{v}(\mathbf{x},t) \quad \Longrightarrow \quad \mathbf{v}(t) \; .$$

The material derivative of the velocity results in

$$\mathbf{a}(\mathbf{x},t) = \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t) \cdot \nabla \mathbf{v}(\mathbf{x},t) = \frac{\partial \mathbf{v}(t)}{\partial t} \; ,$$

where the expression used for the gradient of the velocity field is

$$[\nabla \mathbf{v}(t)]_{ij} = \frac{\partial \mathrm{v}_i(t)}{\partial \mathrm{x}_j} = 0 \; .$$

Therefore, *the statement is false* because $\partial \mathbf{v}(t)/\partial t$ is not necessarily zero.

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Problem 1.2 – *Calculate the acceleration at time $t = 2$ in point $(1,1,1)$ of the velocity field*

$$\mathbf{v} \stackrel{not}{\equiv} \left[x - z \,, \quad z \left(e^{t} + e^{-t} \right) \,, \quad 0 \right]^T \,.$$

Solution

Since the velocity field is given in its spatial expression and the acceleration is requested for a point $\mathbf{x}^* = (1,1,1)^T$, the equation of motion is not needed. One can simply apply

$$\mathbf{a}(\mathbf{x},t) = \frac{d\mathbf{v}(\mathbf{x},t)}{dt} = \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t) \cdot \nabla \mathbf{v}(\mathbf{x},t) \,,$$

where

$$\frac{\partial \mathbf{v}}{\partial t} \stackrel{not}{\equiv} \left[0 \,, \quad z \left(e^{t} - e^{-t} \right) \,, \quad 0 \right]^T \quad \text{and}$$

$$\nabla \mathbf{v} \stackrel{not}{\equiv} \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{bmatrix} \left[x - z \,, \quad z \left(e^{t} + e^{-t} \right) \,, \quad 0 \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & \left(e^{t} + e^{-t} \right) & 0 \end{bmatrix} \,, \text{ such that}$$

$$\mathbf{v} \cdot \nabla \mathbf{v} \stackrel{not}{\equiv} \left[x - z \,, \quad 0 \,, \quad 0 \right]^T \,.$$

Therefore, the spatial expression for the acceleration field is

$$\mathbf{a} \stackrel{not}{\equiv} \left[x - z \,, \quad z \left(e^{t} - e^{-t} \right) \,, \quad 0 \right]^T$$

and, for the given point at the given instant of time, the acceleration is

$$\boxed{\mathbf{a}(\mathbf{x} = \mathbf{x}^*,\, t = 2) \stackrel{not}{\equiv} \left[0 \,, \quad e^{2} - e^{-2} \,, \quad 0 \right]^T} \,.$$

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Problem 1.3 – *The equation of a certain motion is*

$$x = X \ , \quad y = \frac{1}{2}\left((Y+Z)\,\mathrm{e}^{\,t} + (Y-Z)\,\mathrm{e}^{-t} \right) \ , \quad z = \frac{1}{2}\left((Y+Z)\,\mathrm{e}^{\,t} - (Y-Z)\,\mathrm{e}^{-t} \right) \ .$$

Calculate the accelerations that would be observed along time by:

a) An observer located in the fixed point $(1,1,1)$.

b) An observer traveling with the particle that at time $t=0$ *occupied position* $(1,1,1)$.

c) An observer located in point $(1,1,1)$ *that measures the accelerations as the difference between velocities at this point per unit of time.*

Solution

a) The spatial description of the acceleration in point $\mathbf{x}^* = (1,1,1)$ must be obtained,

$$\mathbf{a}\left(\mathbf{x} = \mathbf{x}^*,\, t\right) = \mathbf{A}\left(\mathbf{X}\left(\mathbf{x}^*, t\right),\, t\right) = \frac{\partial \mathbf{V}\left(\mathbf{X}\left(\mathbf{x}^*, t\right), t\right)}{\partial t} \ .$$

The material expression of the velocity field is

$$\mathbf{V}(\mathbf{X},t) = \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} \quad \Longrightarrow \quad \mathbf{V}(\mathbf{X},t) \stackrel{not}{\equiv} \begin{bmatrix} 0 \\ \dfrac{1}{2}\left((Y+Z)\,\mathrm{e}^{\,t} - (Y-Z)\,\mathrm{e}^{-t}\right) \\ \dfrac{1}{2}\left((Y+Z)\,\mathrm{e}^{\,t} + (Y-Z)\,\mathrm{e}^{-t}\right) \end{bmatrix} \ .$$

Then, the material description of the acceleration is

$$\mathbf{A}(\mathbf{X},t) = \frac{\partial \mathbf{V}(\mathbf{X},t)}{\partial t} \stackrel{not}{\equiv} \begin{bmatrix} 0 \\ \dfrac{1}{2}\left((Y+Z)\,\mathrm{e}^{\,t} + (Y-Z)\,\mathrm{e}^{-t}\right) \\ \dfrac{1}{2}\left((Y+Z)\,\mathrm{e}^{\,t} - (Y-Z)\,\mathrm{e}^{-t}\right) \end{bmatrix} \ .$$

Careful observation of the expression obtained reveals that

$$A_y = \frac{1}{2}\left((Y+Z)\,\mathrm{e}^{\,t} + (Y-Z)\,\mathrm{e}^{-t} \right) = y \quad \text{and}$$

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$$A_z = \frac{1}{2}\left((Y+Z)\,\mathrm{e}^{\,t} - (Y-Z)\,\mathrm{e}^{-t}\right) = z\,.$$

Therefore, the spatial description of the acceleration field is

$$\mathbf{a}(\mathbf{x},t) \overset{not}{\equiv} [0\,, \quad y\,, \quad z]^T$$

and, for $\mathbf{x} = \mathbf{x}^*$,

$$\boxed{\mathbf{a}(\mathbf{x}^*,t) \overset{not}{\equiv} [0\,,\ 1\,,\ 1]^T}\,.$$

NOTE: In case one does not realize that $A_y = y$ and $A_z = z$, this same result can be obtained by replacing into the material expression of the acceleration field the inverse equation of motion as follows.

$$\left.\begin{aligned} y+z &= (Y+Z)\,\mathrm{e}^{\,t} \\ y-z &= (Y-Z)\,\mathrm{e}^{-t} \end{aligned}\right\} \quad \Longrightarrow \quad \left\{\begin{aligned} Y+Z &= (y+z)\,\mathrm{e}^{-t} \\ Y-Z &= (y-z)\,\mathrm{e}^{\,t} \end{aligned}\right.$$

$$\begin{cases} X = x \\[2mm] Y = \dfrac{1}{2}\left((y+z)\,\mathrm{e}^{-t} + (y-z)\,\mathrm{e}^{\,t}\right) \\[2mm] Z = \dfrac{1}{2}\left((y+z)\,\mathrm{e}^{-t} - (y-z)\,\mathrm{e}^{\,t}\right) \end{cases}$$

b) The material description of the acceleration in point $\mathbf{X}^* = (1,1,1)$ must be obtained. Replacing point $\mathbf{X}^*$ into the expression obtained in *a*) yields

$$\boxed{\mathbf{A}(\mathbf{X}^*,t) \overset{not}{\equiv} \left[0\,,\ \mathrm{e}^{\,t}\,,\ \mathrm{e}^{\,t}\right]^T}\,.$$

c) The difference between the spatial velocities per unit of time must be obtained, for point $\mathbf{x}^* = (1,1,1)$,

$$\frac{\Delta\mathbf{v}(\mathbf{x}^*,t)}{\Delta t} \quad \longrightarrow \quad \frac{\partial\mathbf{v}(\mathbf{x}^*,t)}{\partial t}\,.$$

The spatial description of the velocity field is

$$\mathbf{v}(\mathbf{x}=\mathbf{x}^*,\ t) = \mathbf{V}(\mathbf{X}(\mathbf{x}^*,t),t)\ .$$

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Careful observation of the material expression of the velocity field obtained in *a*) reveals that $V_y = z$ and $V_z = y$, therefore

$$\mathbf{v}(\mathbf{x},t) \overset{not}{\equiv} [0\ ,\ \ z\ ,\ \ y]^T \quad \Longrightarrow \quad \boxed{\frac{\partial \mathbf{v}(\mathbf{x}^*,t)}{\partial t} \overset{not}{\equiv} [0\ ,\ \ 0\ ,\ \ 0]^T}\ .$$

Problem 1.4 – *Given the spatial description of the velocity field in Cartesian coordinates,*

$$\mathbf{v} \overset{not}{\equiv} [x\ ,\ \ y\ ,\ \ z\varphi(t)]^T$$

and the surface

$$\Sigma_t := \left\{ \mathbf{x} \mid F(x,y,z,t) = \mathrm{e}^{-2t}\left(x^2+y^2\right) + z^2\mathrm{e}^{-t^2} - C = 0 \right\}\ ,$$

where $C \neq 0$ is a constant, determine $\varphi(t)$ considering that the particles on this surface are always the same.

Solution

The function F defines the material surface $\Sigma_t := \{\mathbf{x} \mid F(x,y,z,t) = 0\}$. The necessary and sufficient condition for this surface to be a material surface is

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \mathbf{v}\cdot\nabla F = 0 \qquad \forall \mathbf{x} \in \Sigma_t \quad \forall t\ ,$$

where

$$\begin{aligned}
&\frac{\partial F}{\partial t} = -2\mathrm{e}^{-2t}\left(x^2+y^2\right) - 2tz^2\mathrm{e}^{-t^2}\ , \\
&\nabla F \overset{not}{\equiv} \left[2x\mathrm{e}^{-2t}\ ,\ \ 2y\mathrm{e}^{-2t}\ ,\ \ 2z\mathrm{e}^{-t^2}\right]^T\ ,\ \text{ and} \\
&\mathbf{v}\cdot\nabla F = 2x^2\mathrm{e}^{-2t} + 2y^2\mathrm{e}^{-2t} + 2z^2\mathrm{e}^{-t^2}\varphi(t)\ .
\end{aligned}$$

Then, the necessary and sufficient condition above is reduced to

$$2z^2(\varphi(t)-t)\mathrm{e}^{-t^2} = 0 \qquad \forall \mathbf{x} \in \Sigma_t \quad \forall t\ .$$

Moreover, for $\mathbf{x} \in \Sigma_t$, the term z^2 can be isolated from the expression of the function defining the material surface $F(x,y,z,t)$ given in the statement, $z^2 = \left(C - \mathrm{e}^{-2t}\left(x^2+y^2\right)\right)\mathrm{e}^{t^2}$. Replacing this expression into the previous equation yields

$$2\left(C - \mathrm{e}^{-2t}\left(x^2+y^2\right)\right)(\varphi(t)-t) = 0 \qquad \forall \mathbf{x}\ \forall t\ .$$

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Since $\left(C - \mathrm{e}^{-2t}\left(x^2+y^2\right)\right) = 0$ cannot be satisfied for $\forall \mathbf{x}$ and $\forall t$ because C is a constant, the only possibility left is

$$\boxed{\varphi(t) = t} \ .$$

Problem 1.5 – *Given the velocity field of a perfect fluid*

$$\mathbf{v}(\mathbf{x},t) \stackrel{not}{\equiv} \left[z\mathrm{e}^{\,t} \ , \quad \frac{y}{1+t} \ , \quad \mathrm{v}_z \right]^T$$

and the surface $\varphi(\mathbf{x},t) = x - z(1+t)\,\mathrm{e}^{\,t} + k = 0$ *(where k is a constant), which is known to be a material surface, determine:*

a) The equation of the trajectory in canonical form and the equation of the streamlines.

b) The equation of the streakline and the position of its initial and final points if the spill point is $\mathbf{x}^*$ *and the spill period is* $t \in [t_1, t_2]$.

Solution

a) To be able to calculate the trajectories and streamlines, the expression for the velocity field must be completed. To find v_z, the information given about surface φ is used. The necessary and sufficient condition for this surface to be a material surface is

$$\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial t} + \mathbf{v} \cdot \nabla \varphi = 0 \quad \forall \mathbf{x} \in \Sigma_t \qquad \forall t \ ,$$

$$\text{where} \quad \frac{\partial \varphi}{\partial t} = -z\left(\mathrm{e}^{\,t} + \mathrm{e}^{\,t}(1+t)\right) \ , \quad \nabla\varphi \stackrel{not}{\equiv} \left[1 \ , \ 0 \ , \ -\mathrm{e}^{\,t}(1+t)\right]^T$$

$$\text{and} \quad \mathbf{v}\cdot\nabla\varphi = z\mathrm{e}^{\,t} - \mathrm{v}_z\mathrm{e}^{\,t}(1+t) \ .$$

Then, the material derivative of φ is

$$\frac{d\varphi}{dt} = -z\mathrm{e}^{\,t} - z\mathrm{e}^{\,t}(1+t) + z\mathrm{e}^{\,t} - \mathrm{v}_z\mathrm{e}^{\,t}(1+t) = 0$$

which results in $\mathrm{v}_z = -z$. Therefore, the spatial description of velocity field is

$$\mathbf{v}(\mathbf{x},t) \stackrel{not}{\equiv} \left[z\mathrm{e}^{\,t} \ , \quad \frac{y}{1+t} \ , \quad -z \right]^T \ .$$

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Now, this field must be integrated to obtain the equation of the trajectory since $d\mathbf{x}/dt = \mathbf{v}(\mathbf{x},t)$. Applying the equality for each component and particularizing for the velocity field determined yields

$$\frac{dx}{dt} = z\mathrm{e}^{\,t} \,, \quad \frac{dy}{dt} = \frac{y}{1+t} \quad \text{and} \quad \frac{dz}{dt} = -z \,.$$

Note that the x-component depends on the z-coordinate. Then, the z-coordinate must be determined first,

$$\frac{dz}{dt} = -z \quad \Longrightarrow \quad z = C_1 \mathrm{e}^{-t} \,.$$

Replacing the expression found for z into the x-component and integrating the expression results in

$$\frac{dx}{dt} = C_1 \mathrm{e}^{-t} \mathrm{e}^{\,t} = C_1 \quad \Longrightarrow \quad x = C_1 t + C_2 \,.$$

Finally, the y-component is

$$\frac{dy}{dt} = \frac{y}{1+t} \quad \Longrightarrow \quad y = C_3 (1+t) \,.$$

To obtain the canonical form of the expression, $\mathbf{x} = \mathbf{X}$ for $t = 0$ is imposed,

$$\begin{cases} x(0) = C_2 = X \\ y(0) = C_3 = Y \\ z(0) = C_1 = Z \end{cases}$$

and, finally, the equation of the trajectory in canonical form is

$$\boxed{\begin{array}{l} x = X + Zt \\ y = Y(1+t) \\ z = Z\mathrm{e}^{-t} \end{array}} \,.$$

The equation of the streamlines is found by integrating the velocity field with respect to λ, that is, $d\mathbf{x}(\lambda)/d\lambda = \mathbf{v}(\mathbf{x}(\lambda),t)$. As in the case of the equation of the trajectory, the z-component must be determined before the x-component,

$$\frac{dz}{d\lambda} = -z \quad \Longrightarrow \quad z = C_1 \mathrm{e}^{-\lambda} \,.$$

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Replacing into the x-component yields

$$\frac{dx}{d\lambda} = C_1 \mathrm{e}^{(t-\lambda)} \quad \Longrightarrow \quad x = -C_1 \mathrm{e}^{(t-\lambda)} + C_2$$

and the remaining component results in

$$\frac{dy}{d\lambda} = \frac{y}{1+t} \quad \Longrightarrow \quad y = C_3 \mathrm{e}^{\frac{\lambda}{1+t}} \ .$$

Then, the equation of the streamlines is

$$\boxed{\begin{array}{l} x = -C_1 \mathrm{e}^{(t-\lambda)} + C_2 \\ y = C_3 \mathrm{e}^{\frac{\lambda}{1+t}} \\ z = C_1 \mathrm{e}^{-\lambda} \end{array}} \ .$$

b) To obtain the equation of the streakline it is enough to take the equation of motion and impose $\mathbf{x}^* = \mathbf{x}(\mathbf{X}, \tau)$, where τ is a time belonging to the spill period.

$$\begin{cases} x^* = X + Z\tau \\ y^* = Y(1+\tau) \\ z^* = Z\mathrm{e}^{-\tau} \end{cases}$$

And the inverse of this equation is

$$\begin{cases} X = x^* - Z\tau = x^* - z^* \tau \mathrm{e}^{\tau} \\ Y = \dfrac{y^*}{1+\tau} \\ Z = z^* \mathrm{e}^{\tau} \end{cases}$$

Replacing these into the equation of motion results in the equation of the streakline,

$$\boxed{\begin{array}{l} x = x^* - z^* (\tau - t) \mathrm{e}^{\tau} \\ y = y^* \dfrac{1+t}{1+\tau} \\ z = z^* \mathrm{e}^{(\tau - t)} \end{array}} \ .$$

Consider the physical concept of the streakline as the color line that would be observed in the medium if a tracer fluid were injected at the spill point through-

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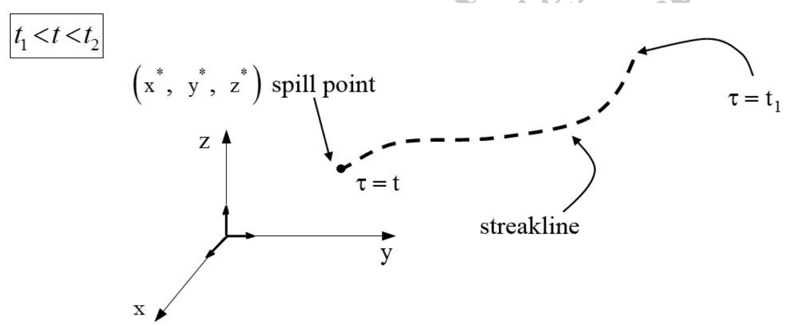
out the spill period. Then, for each time t, the streakline can be visualized in terms of the parameter τ, which gives the position in space of the colored particles. It is verified that, as expected, $x = x^*$ for $t = \tau$, since it corresponds to the time in which the streakline is crossing the spill point. Now, the streakline must be delimited for each time t.

There are two distinct cases:

i) $\boxed{t_1 < t < t_2}$

The first colored point in the streakline is the one crossing the spill point at $\tau = t_1$ while the last one is the one crossing the spill point at $\tau = t$.

$$\text{Initial point:} \begin{cases} x = x^* - z^* (t_1 - t) \, \mathrm{e}^{\,t_1} \\ y = y^* \dfrac{1+t}{1+t_1} \\ z = z^* \mathrm{e}^{\,(t_1 - t)} \end{cases} \qquad \text{Final point:} \begin{cases} x = x^* \\ y = y^* \\ z = z^* \end{cases}$$



ii) $\boxed{t \geq t_2}$

The first colored point in the streakline is the same as in the previous case, $\tau = t_1$, but the last point is now $\tau = t_2$. The streakline has now "moved away" from the spill point.

$$\text{Initial point:} \begin{cases} x = x^* - z^* (t_1 - t) \, \mathrm{e}^{\,t_1} \\ y = y^* \dfrac{1+t}{1+t_1} \\ z = z^* \mathrm{e}^{\,(t_1 - t)} \end{cases} \qquad \text{Final point:} \begin{cases} x = x^* - z^* (t_2 - t) \, \mathrm{e}^{\,t_2} \\ y = y^* \dfrac{1+t}{1+t_2} \\ z = z^* \mathrm{e}^{\,(t_2 - t)} \end{cases}$$

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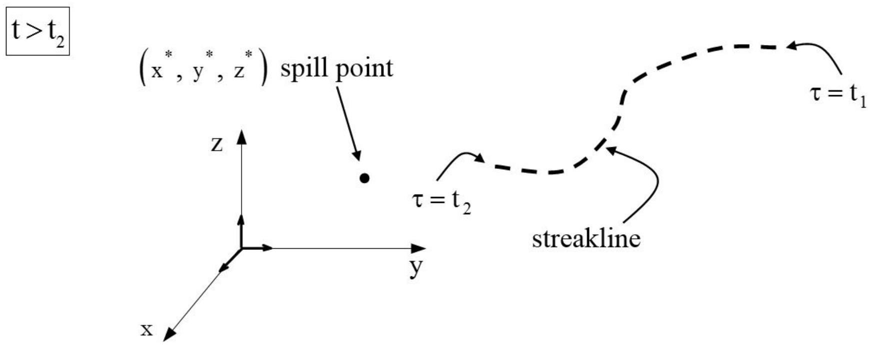

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EXERCISES

1.1 – *Justify if the following statements are true or false.*

a) *Two streamlines corresponding to a same instant of time can never cross each other unless the velocity field at the cross point is zero.*

b) *Two different trajectories can never cross each other.*

c) *Two streaklines corresponding to two spill points with the same spill period can cross each other at one or more points.*

1.2 – *Given the following velocity field in material description*

$$\mathbf{v} \stackrel{not}{\equiv} \left[A\mathrm{e}^{At}X_1, \quad BtX_1, \quad CX_3 \right]^T ,$$

with A, B and C constants, obtain its spatial description and the conditions A, B and C must fulfill for the motion to be feasible for $0 < t < \infty$.

1.3 – *Tracer fluid is injected at point* $(1,1,1)$ *of the interior of a fluid from time* $t = 1$ *to time* $t = 2$. *If the equation of the streamlines is*

$$x = C_1\mathrm{e}^{\lambda t} , \quad y = C_2\mathrm{e}^{\lambda t} , \quad z = C_3\mathrm{e}^{2\lambda t}$$

determine the equation of the streakline, indicating its initial and final points for $t = 5$.

1.4 – *The spatial description of the velocity field of a fluid is*

$$\mathbf{v} \stackrel{not}{\equiv} \left[y\mathrm{e}^{-t}, \quad z\mathrm{e}^{\,t}, \quad 0 \right]^T .$$

Tracer fluid is injected on plane $y = 0$ *at time* $t = 1$. *Obtain the spatial equation of the stain along time.*

1.5 – *A certain motion is defined by the velocity field*

$$\mathrm{v}_x = 2ax \quad ; \qquad \mathrm{v}_y = -by \quad ; \qquad \mathrm{v}_z = -\frac{z}{t+c} .$$

Determine:

a) *The equation of the trajectory in canonical form and the equation of the streamlines.*

b) *The possible values of a, b and c such that the motion has physical sense for* $t \in [0,\infty)$.

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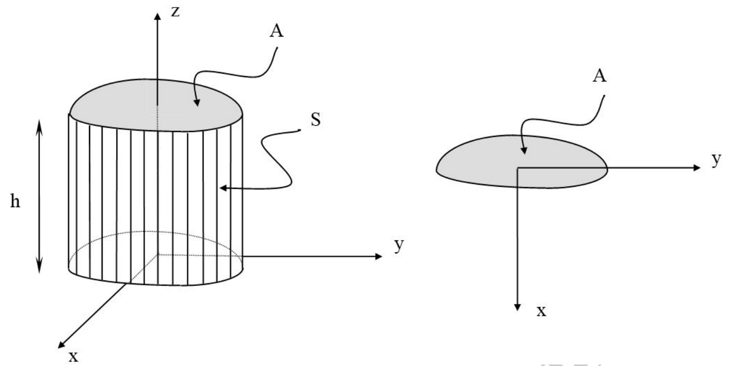
c) *The spatial description of the material surface that, at time $t = 1$, was a sphere with center at $(0,0,0)$ and radius R (consider $a = b = c = 1$).*



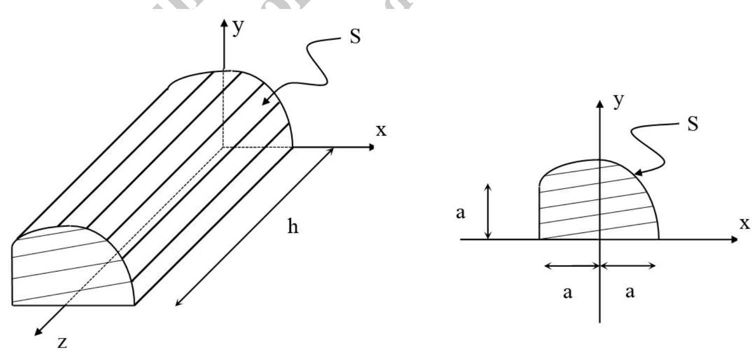
1.6 – *A certain motion is defined by the velocity field*

$$\mathrm{v}_x = y\mathrm{e}^{-t} \quad ; \qquad \mathrm{v}_y = y \quad ; \qquad \mathrm{v}_z = 0 \ .$$

Determine:

a) *The equation of the trajectory in canonical form and the equation of the streamlines.*

b) *The spatial description of the material surface that, at time $t = 1$, was a sphere with center at $(0,0,0)$ and radius R.*



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Chapter 2
Strain

2.1 Introduction

> **Definition 2.1.** In the broader context, the concept of *deformation* no longer refers to the study of the absolute motion of the particles as seen in Chapter 1, but to the study of the *relative motion*, with respect to a given particle, *of the particles in its differential neighborhood*.

2.2 Deformation Gradient Tensor

Consider the continuous medium in motion of Figure 2.1. A particle P in the reference configuration Ω_0 occupies the point in space P' in the present configuration Ω_t, and a particle Q situated in the differential neighborhood of P has relative positions with respect to this particle in the reference and present times given by $d\mathbf{X}$ and $d\mathbf{x}$, respectively. The equation of motion is given by

$$\begin{cases} \mathbf{x} = \varphi(\mathbf{X},t) \stackrel{not}{=} \mathbf{x}(\mathbf{X},t) \\ x_i = \varphi_i(X_1,X_2,X_3,t) \stackrel{not}{=} x_i(X_1,X_2,X_3,t) \qquad i \in \{1,2,3\} \end{cases} . \tag{2.1}$$

Differentiating (2.1) with respect to the material coordinates $\mathbf{X}$ results in the

$$\boxed{\text{Fundamental equation of deformation} \quad \begin{cases} d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} \\ dx_i = \dfrac{\partial x_i}{\partial X_j}\, dX_j = F_{ij}\, dX_j \qquad i,j \in \{1,2,3\} \end{cases}} \tag{2.2}$$

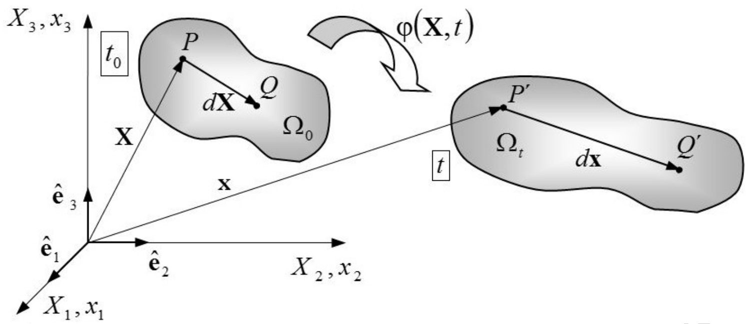

Figure 2.1: Continuous medium in motion.

Equation (2.2) defines the *material deformation gradient tensor* $\mathbf{F}(\mathbf{X},t)$ [1].

Material deformation gradient tensor	$\begin{cases} \mathbf{F} \overset{not}{=} \mathbf{x} \otimes \overline{\nabla} \\ F_{ij} = \dfrac{\partial x_i}{\partial X_j} \qquad i,j \in \{1,2,3\} \end{cases}$ (2.3)

The explicit components of tensor $\mathbf{F}$ are given by

$$[\mathbf{F}] = \left[\mathbf{x}\otimes\overline{\nabla}\right] = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{[\mathbf{x}]} \underbrace{\left[\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3}\right]}_{\left[\overline{\nabla}\right]^T} = \begin{bmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_1}{\partial X_3} \\ \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_2}{\partial X_3} \\ \dfrac{\partial x_3}{\partial X_1} & \dfrac{\partial x_3}{\partial X_2} & \dfrac{\partial x_3}{\partial X_3} \end{bmatrix} . \quad (2.4)$$

Remark 2.1. The *deformation gradient tensor* $\mathbf{F}(\mathbf{X},t)$ contains the information of the relative motion, along time t, of all the material particles in the differential neighborhood of a given particle, identified by its material coordinates $\mathbf{X}$. In effect, equation (2.2) provides the evolution of the relative position vector $d\mathbf{x}$ in terms of the corresponding relative position in the reference time, $d\mathbf{X}$. Thus, if the value of $\mathbf{F}(\mathbf{X},t)$ is known, the information associated with the general concept of deformation defined in Section 2.1 is also known.

[1] Here, the symbolic form of the *material Nabla operator*, $\overline{\nabla} \equiv \partial \hat{\mathrm{e}}_i / \partial X_i$, applied to the expression of the open or tensor product, $[\mathbf{a}\otimes\mathbf{b}]_{ij} \overset{not}{=} [\mathbf{a}\,\mathbf{b}]_{ij} = a_i b_j$, is considered.

2.2.1 Inverse Deformation Gradient Tensor

Consider now the inverse equation of motion

$$\begin{cases} \mathbf{X} = \varphi^{-1}(\mathbf{x},t) \overset{not}{=} \mathbf{X}(\mathbf{x},t) \ , \\ X_i = \varphi_i^{-1}(x_1,x_2,x_3,t) \overset{not}{=} X_i(x_1,x_2,x_3,t) \qquad i \in \{1,2,3\} \ . \end{cases} \tag{2.5}$$

Differentiating (2.5) with respect to the spatial coordinates x_i results in

$$\begin{cases} d\mathbf{X} = \mathbf{F}^{-1} \cdot d\mathbf{x} \ , \\ dX_i = \dfrac{\partial X_i}{\partial x_j}\, dx_j = F_{ij}^{-1}\, dx_j \qquad i,j \in \{1,2,3\} \ . \end{cases} \tag{2.6}$$

The tensor defined in (2.6) is named *spatial deformation gradient tensor* or *inverse (material) deformation gradient tensor* and is characterized by[2]

$$\boxed{\text{Spatial deformation gradient tensor} \quad \begin{cases} \mathbf{F}^{-1} \overset{not}{=} \mathbf{X} \otimes \nabla \\ F_{ij}^{-1} = \dfrac{\partial X_i}{\partial x_j} \qquad i,j \in \{1,2,3\} \end{cases}} \tag{2.7}$$

Remark 2.2. The spatial deformation gradient tensor, denoted in (2.6) and (2.7) as $\mathbf{F}^{-1}$, is in effect the inverse of the (material) deformation gradient tensor $\mathbf{F}$. The verification is immediate since[3]

$$\underbrace{\frac{\partial x_i}{\partial X_k}}_{F_{ik}} \underbrace{\frac{\partial X_k}{\partial x_j}}_{F_{kj}^{-1}} = \frac{\partial x_i}{\partial x_j} \overset{not}{=} \delta_{ij} \quad \Longrightarrow \quad \mathbf{F} \cdot \mathbf{F}^{-1} = \mathbf{1} \ ,$$

$$\underbrace{\frac{\partial X_i}{\partial x_k}}_{F_{ik}^{-1}} \underbrace{\frac{\partial x_k}{\partial X_j}}_{F_{kj}} = \frac{\partial X_i}{\partial X_j} \overset{not}{=} \delta_{ij} \quad \Longrightarrow \quad \mathbf{F}^{-1} \cdot \mathbf{F} = \mathbf{1} \ .$$

[2] Here, the symbolic form of the *spatial Nabla operator*, $\nabla \equiv \partial \hat{\mathbf{e}}_i / \partial x_i$, is considered. Note the difference in notation between this spatial operator ∇ and the *material Nabla* $\overline{\nabla}$.

[3] The two-index operator *Delta Kronecker* δ_{ij} is defined as $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The second-order unit tensor $\mathbf{1}$ is given by $[\mathbf{1}]_{ij} = \delta_{ij}$.

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The explicit components of tensor $\mathbf{F}^{-1}$ are given by

$$\left[\mathbf{F}^{-1}\right]=[\mathbf{X}\otimes\nabla]=\underbrace{\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}}_{[\mathbf{X}]}\underbrace{\left[\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \frac{\partial}{\partial x_3}\right]}_{[\nabla]^T}=\begin{bmatrix}\dfrac{\partial X_1}{\partial x_1} & \dfrac{\partial X_1}{\partial x_2} & \dfrac{\partial X_1}{\partial x_3}\\ \dfrac{\partial X_2}{\partial x_1} & \dfrac{\partial X_2}{\partial x_2} & \dfrac{\partial X_2}{\partial x_3}\\ \dfrac{\partial X_3}{\partial x_1} & \dfrac{\partial X_3}{\partial x_2} & \dfrac{\partial X_3}{\partial x_3}\end{bmatrix}. \quad (2.8)$$

Example 2.1 – *At a given time, the motion of a continuous medium is defined by*

$$\begin{cases}x_1=X_1-AX_3\\x_2=X_2-AX_3\\x_3=-AX_1+AX_2+X_3\end{cases}.$$

Obtain the material deformation gradient tensor $\mathbf{F}(\mathbf{X},t)$ at this time. By means of the inverse equation of motion, obtain the spatial deformation gradient tensor $\mathbf{F}^{-1}(\mathbf{x})$. Using the results obtained, verify that $\mathbf{F}\cdot\mathbf{F}^{-1}=\mathbf{1}$.

Solution

The material deformation gradient tensor is

$$\mathbf{F}=\mathbf{x}\otimes\overline{\nabla}\stackrel{not}{\equiv}[\mathbf{x}]\left[\overline{\nabla}\right]^T=\begin{bmatrix}X_1-AX_3\\X_2-AX_3\\-AX_1+AX_2+X_3\end{bmatrix}\left[\frac{\partial}{\partial X_1},\ \frac{\partial}{\partial X_2},\ \frac{\partial}{\partial X_3}\right]$$

$$\mathbf{F}\stackrel{not}{\equiv}\begin{bmatrix}1 & 0 & -A\\0 & 1 & -A\\-A & A & 1\end{bmatrix}.$$

The inverse equation of motion is obtained directly from the algebraic inversion of the equation of motion,

$$\mathbf{X}(\mathbf{x},t)\stackrel{not}{\equiv}\begin{bmatrix}X_1=\left(1+A^2\right)x_1-A^2x_2+Ax_3\\X_2=A^2x_1+\left(1-A^2\right)x_2+Ax_3\\X_3=Ax_1-Ax_2+x_3\end{bmatrix}.$$

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Then, the spatial deformation gradient tensor is

$$\mathbf{F}^{-1} = \mathbf{X} \otimes \nabla \overset{not}{\equiv} [\mathbf{X}]\,[\nabla]^T = \begin{bmatrix} \left(1+A^2\right)x_1 - A^2 x_2 + A x_3 \\ A^2 x_1 + \left(1-A^2\right)x_2 + A x_3 \\ A x_1 - A x_2 + x_3 \end{bmatrix} \left[\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \frac{\partial}{\partial x_3}\right]$$

$$\mathbf{F}^{-1} \overset{not}{\equiv} \begin{bmatrix} 1+A^2 & -A^2 & A \\ A^2 & 1-A^2 & A \\ A & -A & 1 \end{bmatrix} .$$

Finally, it is verified that

$$\mathbf{F} \cdot \mathbf{F}^{-1} \overset{not}{\equiv} \begin{bmatrix} 1 & 0 & -A \\ 0 & 1 & -A \\ -A & A & 1 \end{bmatrix} \begin{bmatrix} 1+A^2 & -A^2 & A \\ A^2 & 1-A^2 & A \\ A & -A & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{not}{\equiv} \mathbf{1} .$$

2.3 Displacements

Definition 2.2. A *displacement* is the difference between the position vectors in the present and reference configurations of a same particle.

The displacement of a particle P at a given time is defined by vector $\mathbf{u}$, which joins the points in space P (initial position) and P' (position at the present time t) of the particle (see Figure 2.2). The displacement of all the particles in the continuous medium defines a *displacement vector field* which, as all properties of the continuous medium, can be described in material form $\mathbf{U}(\mathbf{X},t)$ or in spatial form $\mathbf{u}(\mathbf{x},t)$ as follows.

$$\begin{cases} \mathbf{U}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X} \\ U_i(\mathbf{X},t) = x_i(\mathbf{X},t) - X_i \qquad i \in \{1,2,3\} \end{cases} \tag{2.9}$$

$$\begin{cases} \mathbf{u}(\mathbf{x},t) = \mathbf{x} - \mathbf{X}(\mathbf{x},t) \\ u_i(\mathbf{x},t) = x_i - X_i(\mathbf{x},t) \qquad i \in \{1,2,3\} \end{cases} \tag{2.10}$$

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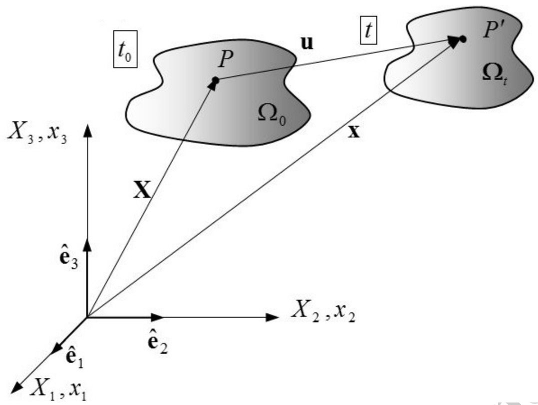

Figure 2.2: Displacement of a particle.

2.3.1 Material and Spatial Displacement Gradient Tensors

Differentiation with respect to the material coordinates of the displacement vector U_i defined in (2.9) results in

$$\frac{\partial U_i}{\partial X_j} = \underbrace{\frac{\partial x_i}{\partial X_j}}_{F_{ij}} - \underbrace{\frac{\partial X_i}{\partial X_j}}_{\delta_{ij}} = F_{ij} - \delta_{ij} \stackrel{def}{=} J_{ij} \ , \tag{2.11}$$

which defines the *material displacement gradient tensor* as follows.

$$\text{Material displacement gradient tensor} \quad \begin{cases} \mathbf{J}(\mathbf{X},t) \stackrel{def}{=} \mathbf{U}(\mathbf{X},t) \otimes \overline{\nabla} = \mathbf{F} - \mathbf{1} \\ J_{ij} = \dfrac{\partial U_i}{\partial X_j} = F_{ij} - \delta_{ij} \qquad i,j \in \{1,2,3\} \end{cases} \tag{2.12}$$

$$\begin{cases} \mathbf{U} = \mathbf{J} \cdot d\mathbf{X} \\ dU_i = \dfrac{\partial U_i}{\partial X_j} dX_j = J_{ij}\, dX_j \qquad i,j \in \{1,2,3\} \end{cases} \tag{2.13}$$

Similarly, differentiation with respect to the spatial coordinates of the expression of u_i given in (2.10) yields

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{\partial x_i}{\partial x_j}}_{\delta_{ij}} - \underbrace{\frac{\partial X_i}{\partial x_j}}_{F_{ij}^{-1}} = \delta_{ij} - F_{ij}^{-1} \stackrel{def}{=} j_{ij} \ , \tag{2.14}$$

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which defines the *spatial displacement gradient tensor* as follows.

$$\boxed{\text{Spatial displacement gradient tensor} \quad \begin{cases} \mathbf{j}(\mathbf{x},t) \stackrel{def}{=} \mathbf{u}(\mathbf{x},t) \otimes \nabla = \mathbf{1} - \mathbf{F}^{-1} \\ j_{ij} = \dfrac{\partial u_i}{\partial x_j} = \delta_{ij} - F_{ij}^{-1} \qquad i,j \in \{1,2,3\} \end{cases}} \tag{2.15}$$

$$\begin{cases} \mathbf{u} = \mathbf{j} \cdot d\mathbf{x} \\ du_i = \dfrac{\partial u_i}{\partial x_j} dx_j = j_{ij}\, dx_j \qquad i,j \in \{1,2,3\} \end{cases} \tag{2.16}$$

2.4 Strain Tensors

Consider now a particle of the continuous medium that occupies the point in space P in the material configuration, and another particle Q in its differential neighborhood separated a segment $d\mathbf{X}$ (with length $dS = \sqrt{d\mathbf{X} \cdot d\mathbf{X}}$) from the previous paticle, being $d\mathbf{x}$ (with length $ds = \sqrt{d\mathbf{x} \cdot d\mathbf{x}}$) its counterpart in the present configuration (see Figure 2.3). Both differential vectors are related through the deformation gradient tensor $\mathbf{F}(\mathbf{X},t)$ by means of equations (2.2) and (2.6),

$$\begin{cases} d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} \quad \text{and} \quad d\mathbf{X} = \mathbf{F}^{-1} \cdot d\mathbf{x}\,, \\ dx_i = F_{ij}\, dX_j \quad \text{and} \quad dX_i = F_{ij}^{-1} dx_j \qquad i,j \in \{1,2,3\}\ . \end{cases} \tag{2.17}$$

Then,

$$\begin{cases} (ds)^2 = d\mathbf{x} \cdot d\mathbf{x} \stackrel{not}{\equiv} [d\mathbf{x}]^T [d\mathbf{x}] = [\mathbf{F} \cdot d\mathbf{X}]^T [\mathbf{F} \cdot d\mathbf{X}] \stackrel{not}{\equiv} d\mathbf{X} \cdot \mathbf{F}^T \cdot \mathbf{F} \cdot d\mathbf{X} \\ (ds)^2 = dx_k\, dx_k = F_{ki}\, dX_i\, F_{kj}\, dX_j = dX_i\, F_{ki}\, F_{kj}\, dX_j = dX_i\, F_{ik}^T F_{kj}\, dX_j \end{cases} \tag{2.18}$$

or, alternatively[4],

$$\begin{cases} (dS)^2 = d\mathbf{X} \cdot d\mathbf{X} \stackrel{not}{\equiv} [d\mathbf{X}]^T [d\mathbf{X}] = \left[\mathbf{F}^{-1} \cdot d\mathbf{x}\right]^T \left[\mathbf{F}^{-1} \cdot d\mathbf{x}\right] = \\ \qquad \stackrel{not}{\equiv} d\mathbf{x} \cdot \mathbf{F}^{-T} \cdot \mathbf{F}^{-1} \cdot d\mathbf{x}\,, \\ (dS)^2 = dX_k\, dX_k = F_{ki}^{-1} dx_i\, F_{kj}^{-1} dx_j = dx_i\, F_{ki}^{-1} F_{kj}^{-1} dx_j = \\ \qquad = dx_i\, F_{ik}^{-T} F_{kj}^{-1} dx_j\ . \end{cases} \tag{2.19}$$

[4] The convention $\left[(\bullet)^{-1}\right]^T \stackrel{not}{\equiv} (\bullet)^{-T}$ is used.

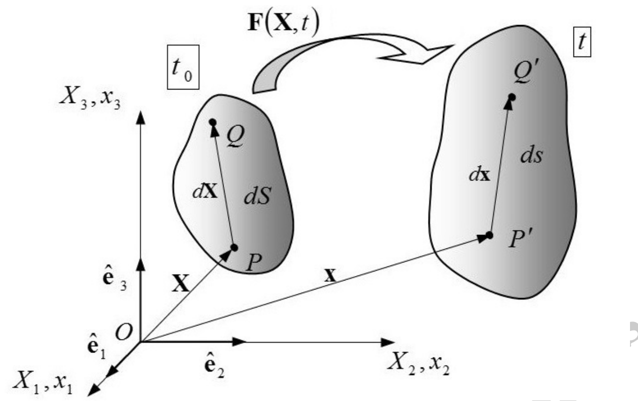

Figure 2.3: Differential segments in a continuous medium.

2.4.1 Material Strain Tensor (Green-Lagrange Strain Tensor)

Subtracting expressions (2.18) and (2.19) results in

$$\begin{aligned}(ds)^2-(dS)^2 &= d\mathbf{X}\cdot\mathbf{F}^T\cdot\mathbf{F}\cdot d\mathbf{X}-d\mathbf{X}\cdot d\mathbf{X}= \\ &= d\mathbf{X}\cdot\mathbf{F}^T\cdot\mathbf{F}\cdot d\mathbf{X}-d\mathbf{X}\cdot\mathbf{1}\cdot d\mathbf{X}= \\ &= d\mathbf{X}\cdot\underbrace{\left(\mathbf{F}^T\cdot\mathbf{F}-\mathbf{1}\right)}_{\overset{def}{=}\,2\mathbf{E}}\cdot d\mathbf{X}=2\,d\mathbf{X}\cdot\mathbf{E}\cdot d\mathbf{X}\,,\end{aligned} \tag{2.20}$$

which implicitly defines the *material strain tensor* or *Green-Lagrange strain tensor* as follows.

$$\begin{array}{c}\text{Material}\\ \text{(Green-Lagrange)}\\ \text{strain tensor}\end{array}\quad \begin{cases}\mathbf{E}(\mathbf{X},t)=\dfrac{1}{2}\left(\mathbf{F}^T\cdot\mathbf{F}-\mathbf{1}\right)\\ E_{ij}(\mathbf{X},t)=\dfrac{1}{2}\left(F_{ki}\,F_{kj}-\delta_{ij}\right)\quad i,j\in\{1,2,3\}\end{cases} \tag{2.21}$$

Remark 2.3. The material strain tensor $\mathbf{E}$ is symmetric. Proof is obtained directly from (2.21), observing that

$$\begin{cases}\mathbf{E}^T=\dfrac{1}{2}\left(\mathbf{F}^T\cdot\mathbf{F}-\mathbf{1}\right)^T=\dfrac{1}{2}\left(\mathbf{F}^T\cdot\left(\mathbf{F}^T\right)^T-\mathbf{1}^T\right)=\dfrac{1}{2}\left(\mathbf{F}^T\cdot\mathbf{F}-\mathbf{1}\right)=\mathbf{E}\,,\\ E_{ij}=E_{ji}\quad i,j\in\{1,2,3\}\,.\end{cases}$$

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2.4.2 *Spatial Strain Tensor (Almansi Strain Tensor)*

Subtracting expressions (2.18) and (2.19) in an alternative form yields

$$\begin{aligned}(ds)^2-(dS)^2 &= d\mathbf{x}\cdot d\mathbf{x}-d\mathbf{x}\cdot\mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\cdot d\mathbf{x}= \\ &= d\mathbf{x}\cdot\mathbf{1}\cdot d\mathbf{x}-d\mathbf{x}\cdot\mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\cdot d\mathbf{x}= \\ &= d\mathbf{x}\cdot\underbrace{\left(\mathbf{1}-\mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\right)}_{\overset{def}{=}\,2\mathbf{e}}\cdot d\mathbf{x}=2\,d\mathbf{x}\cdot\mathbf{e}\cdot d\mathbf{x}\,,\end{aligned} \tag{2.22}$$

which implicitly defines the *spatial strain tensor* or *Almansi strain tensor* as follows.

$$\text{Spatial (Almansi) strain tensor}\quad\begin{cases}\mathbf{e}(\mathbf{x},t)=\dfrac{1}{2}\left(\mathbf{1}-\mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\right)\\ e_{ij}(\mathbf{x},t)=\dfrac{1}{2}\left(\delta_{ij}-F^{-1}_{ki}F^{-1}_{kj}\right)\quad i,j\in\{1,2,3\}\end{cases} \tag{2.23}$$

Remark 2.4. The spatial strain tensor $\mathbf{e}$ is symmetric. Proof is obtained directly from (2.23), observing that

$$\begin{cases}\mathbf{e}^T=\dfrac{1}{2}\left(\mathbf{1}-\mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\right)^T=\dfrac{1}{2}\left(\mathbf{1}^T-\left(\mathbf{F}^{-1}\right)^T\cdot\left(\mathbf{F}^{-T}\right)^T\right)= \\ \quad=\dfrac{1}{2}\left(\mathbf{1}-\mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\right)=\mathbf{e}\,, \\ e_{ij}=e_{ji}\qquad i,j\in\{1,2,3\}\,.\end{cases}$$

Example 2.2 – *Obtain the material and spatial strain tensors for the motion in Example 2.1.*

Solution

The material strain tensor is

$$\mathbf{E}(\mathbf{X},t)=\frac{1}{2}\left(\mathbf{F}^T\cdot\mathbf{F}-\mathbf{1}\right)\overset{not}{\equiv}\frac{1}{2}\left(\begin{bmatrix}1&0&-A\\0&1&A\\-A&-A&1\end{bmatrix}\begin{bmatrix}1&0&-A\\0&1&-A\\-A&A&1\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)=$$

$$=\frac{1}{2}\begin{bmatrix}A^2&-A^2&-2A\\-A^2&A^2&0\\-2A&0&2A^2\end{bmatrix}$$

and the spatial strain tensor is

$$\mathbf{e}(\mathbf{X},t) = \frac{1}{2}\left(\mathbf{1} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1}\right) =$$

$$\stackrel{not}{\equiv} \frac{1}{2}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1+A^2 & A^2 & A \\ -A^2 & 1-A^2 & -A \\ A & A & 1 \end{bmatrix} \begin{bmatrix} 1+A^2 & -A^2 & A \\ A^2 & 1-A^2 & A \\ A & -A & 1 \end{bmatrix}\right) =$$

$$= \frac{1}{2}\begin{bmatrix} -3A^2-2A^4 & A^2+2A^4 & -2A-2A^3 \\ A^2+2A^4 & A^2-2A^4 & 2A^3 \\ -2A-2A^3 & 2A^3 & -2A^2 \end{bmatrix}.$$

Observe that $\mathbf{E} \neq \mathbf{e}$.

Remark 2.5. The material strain tensor $\mathbf{E}$ and the spatial strain tensor $\mathbf{e}$ are different tensors. *They are not the material and spatial descriptions of a same strain tensor*. Expressions (2.20) and (2.22),

$$(ds)^2 - (dS)^2 = 2d\mathbf{X} \cdot \mathbf{E} \cdot d\mathbf{X} = 2d\mathbf{x} \cdot \mathbf{e} \cdot d\mathbf{x} \ ,$$

clearly show this since each tensor is affected by a different vector ($d\mathbf{X}$ and $d\mathbf{x}$, respectively).

The *Green-Lagrange strain tensor* is *naturally* described in material description ($\mathbf{E}(\mathbf{X},t)$). In equation (2.20) it acts on element $d\mathbf{X}$ (defined in material configuration) and, hence, its denomination as *material strain tensor*. However, as all properties of the continuous medium, it may be described, if required, in spatial form ($\mathbf{E}(\mathbf{x},t)$) through the adequate substitution of the equation of motion.

The contrary occurs with the *Almansi strain tensor*: it is *naturally* described in spatial form and in equation (2.22) acts on the differential vector $d\mathbf{x}$ (defined in the spatial configuration) and, thus, its denomination as *spatial strain tensor*. It may also be described, if required, in material form ($\mathbf{e}(\mathbf{X},t)$).

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2.4.3 Strain Tensors in terms of the Displacement (Gradients)

Replacing expressions (2.12) and (2.15) into equations (2.21) and (2.23) yields the expressions of the strain tensors in terms of the material displacement gradient, $\mathbf{J}(\mathbf{X},t)$, and the spatial displacement gradient, $\mathbf{j}(\mathbf{x},t)$.

$$\boxed{\begin{aligned} &\mathbf{E} = \frac{1}{2}\Big((\mathbf{1}+\mathbf{J}^T)\cdot(\mathbf{1}+\mathbf{J}) - \mathbf{1}\Big) = \frac{1}{2}\left(\mathbf{J}+\mathbf{J}^T+\mathbf{J}^T\cdot\mathbf{J}\right) \\ &E_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j}+\frac{\partial U_j}{\partial X_i}+\frac{\partial U_k}{\partial X_i}\frac{\partial U_k}{\partial X_j}\right) \qquad i,j\in\{1,2,3\} \end{aligned}} \tag{2.24}$$

$$\boxed{\begin{aligned} &\mathbf{e} = \frac{1}{2}\Big(\mathbf{1} - (\mathbf{1}-\mathbf{j}^T)\cdot(\mathbf{1}-\mathbf{j})\Big) = \frac{1}{2}\left(\mathbf{j}+\mathbf{j}^T-\mathbf{j}^T\cdot\mathbf{j}\right) \\ &e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \qquad i,j\in\{1,2,3\} \end{aligned}} \tag{2.25}$$

2.5 Variation of Distances: Stretch and Unit Elongation

Consider now a particle P in the reference configuration and another particle Q, belonging to the differential neighborhood of P (see Figure 2.4). The corresponding positions in the present configuration are given by the points in space P' and Q' such that the distance between the two particles in the reference configuration, dS, is transformed into ds at the present time. The vectors $\mathbf{T}$ and $\mathbf{t}$ are the unit vectors in the directions $\overline{PQ}$ and $\overline{P'Q'}$, respectively.

Definition 2.3. The *stretch* or *stretch ratio* of a material point P (or a spatial point P') in the material direction $\mathbf{T}$ (or spatial direction $\mathbf{t}$) is the length of the *deformed* differential segment $\overline{P'Q'}$ per unit of length of the *original* differential segment $\overline{PQ}$.

The translation of the previous definition into mathematical language is

$$\boxed{\text{Stretch} \stackrel{def}{=} \lambda_T = \lambda_t = \frac{\overline{P'Q'}}{\overline{PQ}} = \frac{ds}{dS} \qquad (0<\lambda<\infty)} \,. \tag{2.26}$$

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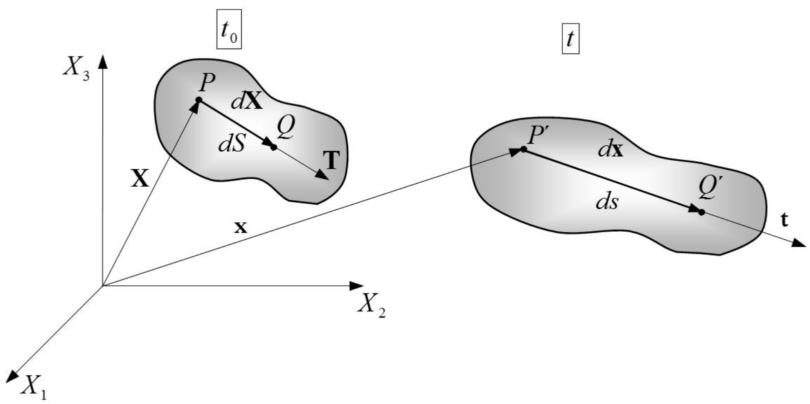

Figure 2.4: Differential segments and unit vectors in a continuous medium.

Definition 2.4. The *unit elongation*, *elongation ratio* or *extension* of a material point P (or a spatial point P') in the material direction $\mathbf{T}$ (or spatial direction $\mathbf{t}$)[5]is the increment of length of the *deformed* differential segment $\overline{P'Q'}$ per unit of length of the *original* differential segment $\overline{PQ}$.

The corresponding mathematical definition is

$$\boxed{\text{Unit elongation} \stackrel{def}{=} \varepsilon_T = \varepsilon_t = \frac{\Delta \overline{PQ}}{\overline{PQ}} = \frac{ds - dS}{dS}} \, . \qquad (2.27)$$

Equations (2.26) and (2.27) allow immediately relating the values of the unit elongation and the stretch for a same point and direction as follows.

$$\varepsilon = \frac{ds - dS}{dS} = \underbrace{\frac{ds}{dS}}_{\lambda} - 1 = \lambda - 1 \qquad (\Rightarrow -1 < \varepsilon < \infty) \qquad (2.28)$$

[5] Often, the subindices $(\bullet)_\mathbf{T}$ and $(\bullet)_\mathbf{t}$ will be dropped when referring to stretches or unit elongations. However, one must bear in mind that both stretches and unit elongations are always associated with a particular direction.

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Remark 2.6. The following deformations may take place:

- If $\lambda = 1$ $(\varepsilon = 0) \Rightarrow ds = dS$: The particles P and Q may have moved along time, but without increasing or decreasing the distance between them.
- If $\lambda > 1$ $(\varepsilon > 0) \Rightarrow ds > dS$: The distance between the particles P and Q has lengthened with the deformation of the medium.
- If $\lambda < 1$ $(\varepsilon < 0) \Rightarrow ds < dS$: The distance between the particles P and Q has shortened with the deformation of the medium.

2.5.1 Stretches, Unit Elongations and Strain Tensors

Consider equations (2.21) and (2.22) as well as the geometric expressions $d\mathbf{X} = \mathbf{T}\,dS$ and $d\mathbf{x} = \mathbf{t}\,ds$ (see Figure 2.4). Then,

$$\begin{cases} (ds)^2 - (dS)^2 = 2\,\underbrace{d\mathbf{X}}_{dS\,\mathbf{T}} \cdot \mathbf{E} \cdot \underbrace{d\mathbf{X}}_{dS\,\mathbf{T}} = 2\,(dS)^2\,\mathbf{T}\cdot\mathbf{E}\cdot\mathbf{T} \\ (ds)^2 - (dS)^2 = 2\,\underbrace{d\mathbf{x}}_{ds\,\mathbf{t}} \cdot \mathbf{e} \cdot \underbrace{d\mathbf{x}}_{ds\,\mathbf{t}} = 2\,(ds)^2\,\mathbf{t}\cdot\mathbf{e}\cdot\mathbf{t} \end{cases} \tag{2.29}$$

and dividing these expressions by $(dS)^2$ and $(ds)^2$, respectively, results in

$$\underbrace{\left(\frac{ds}{dS}\right)}_{\lambda}^{2} - 1 = \lambda^2 - 1 = 2\,\mathbf{T}\cdot\mathbf{E}\cdot\mathbf{T} \Rightarrow \boxed{\begin{array}{l} \lambda = \sqrt{1+2\mathbf{T}\cdot\mathbf{E}\cdot\mathbf{T}} \\ \varepsilon = \lambda - 1 = \sqrt{1+2\mathbf{T}\cdot\mathbf{E}\cdot\mathbf{T}} - 1 \end{array}} \tag{2.30}$$

$$1 - \underbrace{\left(\frac{dS}{ds}\right)}_{1/\lambda}^{2} = 1 - \left(\frac{1}{\lambda}\right)^2 = 2\,\mathbf{t}\cdot\mathbf{e}\cdot\mathbf{t} \Rightarrow \boxed{\begin{array}{l} \lambda = \dfrac{1}{\sqrt{1-2\mathbf{t}\cdot\mathbf{e}\cdot\mathbf{t}}} \\ \varepsilon = \lambda - 1 = \dfrac{1}{\sqrt{1-2\mathbf{t}\cdot\mathbf{e}\cdot\mathbf{t}}} - 1 \end{array}} \tag{2.31}$$

These equations allow calculating the unit elongation and stretch for a given direction (in material description, $\mathbf{T}$, or in spatial description, $\mathbf{t}$).

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Remark 2.7. The material and spatial strain tensors, $\mathbf{E}(\mathbf{X},t)$ and $\mathbf{e}(\mathbf{x},t)$, contain information on the stretches (and unit elongations) for any direction in a differential neighborhood of a given particle, as evidenced by (2.30) and (2.31).

Example 2.3 – *The spatial strain tensor for a given motion is*

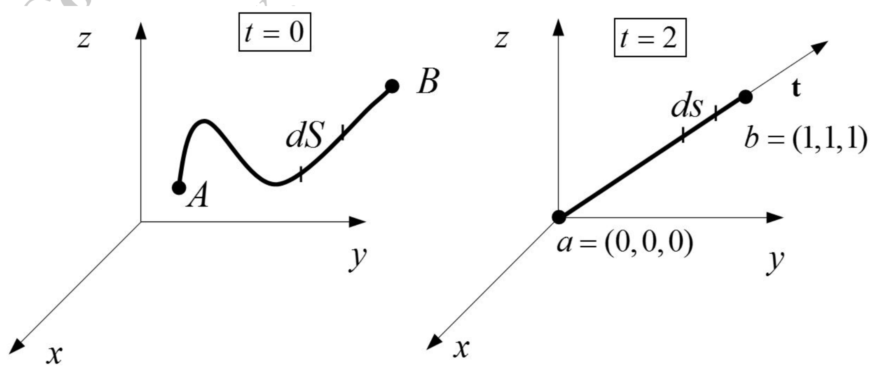
$$\mathbf{e}(\mathbf{x},t) \overset{not}{\equiv} \begin{bmatrix} 0 & 0 & -te^{tz} \\ 0 & 0 & 0 \\ -te^{tz} & 0 & t\left(2e^{tz}-e^{t}\right) \end{bmatrix} .$$

Calculate the length, at time $t=0$, of the segment that at time $t=2$ is rectilinear and joins points $a \equiv (0,0,0)$ and $b \equiv (1,1,1)$.

Solution

The shape and geometric position of the material segment at time $t=2$ is known. At time $t=0$ (reference time) the segment is not necessarily rectilinear and the positions of its extremes A and B (see figure below) are not known. To determine its length, (2.31) is applied for a unit vector in the direction of the spatial configuration $\mathbf{t}$,

$$\lambda = \frac{1}{\sqrt{1-2\,\mathbf{t}\cdot\mathbf{e}\cdot\mathbf{t}}} = \frac{ds}{dS} \quad \Longrightarrow \quad dS = \frac{1}{\lambda} ds \,.$$



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To obtain the stretch in the direction $\mathbf{t} \overset{not}{\equiv} [1,\ 1,\ 1]^T / \sqrt{3}$, the expression $\mathbf{t} \cdot \mathbf{e} \cdot \mathbf{t}$ is computed first as

$$\mathbf{t} \cdot \mathbf{e} \cdot \mathbf{t} \overset{not}{\equiv} \frac{1}{\sqrt{3}} [1,\ 1,\ 1] \begin{bmatrix} 0 & 0 & -t\mathrm{e}^{tz} \\ 0 & 0 & 0 \\ -t\mathrm{e}^{tz} & 0 & t\left(2\mathrm{e}^{tz} - \mathrm{e}^{t}\right) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}} = -\frac{1}{3} t\mathrm{e}^{t} \ .$$

Then, the corresponding stretch at time $t = 2$ is

$$\lambda = \frac{1}{\sqrt{1 + \frac{2}{3} t\mathrm{e}^{t}}} \quad \Longrightarrow \quad \lambda \Big|_{t=2} = \frac{1}{\sqrt{1 + \frac{4}{3}\mathrm{e}^{2}}} = \frac{\sqrt{3}}{\sqrt{3 + 4\mathrm{e}^{2}}} \ .$$

The length at time $t = 0$ of the segment AB is

$$l_{AB} = \int_A^B dS = \int_a^b \frac{1}{\lambda} ds = \frac{1}{\lambda} \underbrace{\int_a^b ds}_{l_{ab}} = \frac{1}{\lambda} l_{ab} = \frac{1}{\lambda} \sqrt{3}$$

and replacing the expression obtained above for the stretch at time $t = 2$ finally results in

$$l_{AB} = \sqrt{3 + 4\mathrm{e}^{2}} \ .$$

2.6 Variation of Angles

Consider a particle P and two additional particles Q and R, belonging to the differential neighborhood of P in the material configuration (see Figure 2.5), and the same particles occupying the spatial positions P', Q' and R'. The relationship between the angles that form the corresponding differential segments in the reference configuration (angle Θ) and the present configuration (angle θ) is to be considered next.

Applying (2.2) and (2.6) on the differential vectors that separate the particles,

$$\begin{cases} d\mathbf{x}^{(1)} = \mathbf{F} \cdot d\mathbf{X}^{(1)} \\ d\mathbf{x}^{(2)} = \mathbf{F} \cdot d\mathbf{X}^{(2)} \end{cases} \quad \Longrightarrow \quad \begin{cases} d\mathbf{X}^{(1)} = \mathbf{F}^{-1} \cdot d\mathbf{x}^{(1)} \\ d\mathbf{X}^{(2)} = \mathbf{F}^{-1} \cdot d\mathbf{x}^{(2)} \end{cases} \tag{2.32}$$

and using the definitions of the unit vectors $\mathbf{T}^{(1)}$, $\mathbf{T}^{(2)}$, $\mathbf{t}^{(1)}$ and $\mathbf{t}^{(2)}$ that establish the corresponding directions in Figure 2.5,

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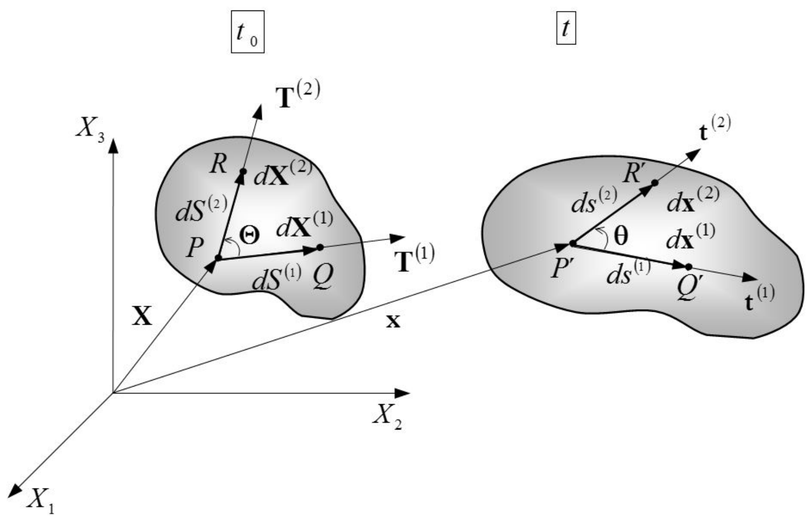

Figure 2.5: Angles between particles in a continuous medium.

$$\begin{cases} d\mathbf{X}^{(1)} = dS^{(1)}\mathbf{T}^{(1)} \\ d\mathbf{X}^{(2)} = dS^{(2)}\mathbf{T}^{(2)} \end{cases} \Longrightarrow \begin{cases} d\mathbf{x}^{(1)} = ds^{(1)}\mathbf{t}^{(1)} \ , \\ d\mathbf{x}^{(2)} = ds^{(2)}\mathbf{t}^{(2)} \ . \end{cases} \tag{2.33}$$

Finally, according to the definition in (2.26), the corresponding stretches are

$$\begin{cases} ds^{(1)} = \lambda^{(1)} dS^{(1)} \\ ds^{(2)} = \lambda^{(2)} dS^{(2)} \end{cases} \Longrightarrow \begin{cases} dS^{(1)} = \dfrac{1}{\lambda^{(1)}} ds^{(1)} \ , \\ dS^{(2)} = \dfrac{1}{\lambda^{(2)}} ds^{(2)} \ . \end{cases} \tag{2.34}$$

Expanding now the scalar product[6] of the vectors $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$,

$$\begin{aligned} & dS^{(1)} dS^{(2)} \cos\theta = \left| d\mathbf{x}^{(1)} \right| \left| d\mathbf{x}^{(2)} \right| \cos\theta = d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} \overset{not}{\equiv} \left[d\mathbf{x}^{(1)} \right]^T \left[d\mathbf{x}^{(2)} \right] = \\ & = \left[\mathbf{F} \cdot d\mathbf{X}^{(1)} \right]^T \left[\mathbf{F} \cdot d\mathbf{X}^{(2)} \right] \overset{not}{\equiv} d\mathbf{X}^{(1)} \cdot \left(\mathbf{F}^T \cdot \mathbf{F} \right) \cdot \ d\mathbf{X}^{(2)} = d\mathbf{X}^{(1)} \cdot (2\mathbf{E} + \mathbf{1}) \cdot \ d\mathbf{X}^{(2)} \\ & = dS^{(1)} \mathbf{T}^{(1)} \cdot (2\mathbf{E} + \mathbf{1}) \cdot \mathbf{T}^{(2)} dS^{(2)} = \frac{1}{\lambda^{(1)}} ds^{(1)} \mathbf{T}^{(1)} \cdot (2\mathbf{E} + \mathbf{1}) \cdot \mathbf{T}^{(2)} \frac{1}{\lambda^{(2)}} ds^{(2)} = \\ & = ds^{(1)} ds^{(2)} \frac{1}{\lambda^{(1)}} \frac{1}{\lambda^{(2)}} \mathbf{T}^{(1)} \cdot (2\mathbf{E} + \mathbf{1}) \cdot \mathbf{T}^{(2)}, \end{aligned} \tag{2.35}$$

[6] The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined in terms of the angle between them, θ, as $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cos\theta$.

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and, comparing the initial and final terms in (2.35), yields

$$\cos\theta = \frac{\mathbf{T}^{(1)} \cdot (\mathbf{1} + 2\mathbf{E}) \cdot \mathbf{T}^{(2)}}{\lambda^{(1)}\lambda^{(2)}} , \tag{2.36}$$

where the stretches $\lambda^{(1)}$ and $\lambda^{(2)}$ can be obtained by applying (2.30) to the directions $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(2)}$, resulting in

$$\cos\theta = \frac{\mathbf{T}^{(1)} \cdot (\mathbf{1} + 2\mathbf{E}) \cdot \mathbf{T}^{(2)}}{\sqrt{1 + 2\mathbf{T}^{(1)} \cdot \mathbf{E} \cdot \mathbf{T}^{(1)}}\sqrt{1 + 2\mathbf{T}^{(2)} \cdot \mathbf{E} \cdot \mathbf{T}^{(2)}}} . \tag{2.37}$$

In an analogous way, operating on the reference configuration, the angle Θ between the differential segments $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$ (in terms of $\mathbf{t}^{(1)}$, $\mathbf{t}^{(2)}$ and $\mathbf{e}$) is obtained,

$$\cos\Theta = \frac{\mathbf{t}^{(1)} \cdot (\mathbf{1} - 2\mathbf{e}) \cdot \mathbf{t}^{(2)}}{\sqrt{1 - 2\mathbf{t}^{(1)} \cdot \mathbf{e} \cdot \mathbf{t}^{(1)}}\sqrt{1 - 2\mathbf{t}^{(2)} \cdot \mathbf{e} \cdot \mathbf{t}^{(2)}}} . \tag{2.38}$$

Remark 2.8. Similarly to the discussion in Remark 2.7, the material and spatial strain tensors, $\mathbf{E}(\mathbf{X},t)$ and $\mathbf{e}(\mathbf{x},t)$, also contain information on the variation of the angles between differential segments in the differential neighborhood of a particle during the deformation process. These facts will be the basis for providing a physical interpretation of the components of the strain tensors in Section 2.7.

2.7 Physical Interpretation of the Strain Tensors

2.7.1 Material Strain Tensor

Consider a segment $\overline{PQ}$, oriented parallel to the X_1-axis in the reference configuration (see Figure 2.6). Before the deformation takes place, $\overline{PQ}$ has a known length $dS = dX$.

The length of $\overline{P'Q'}$ is sought. To this aim, consider the material strain tensor $\mathbf{E}$ given by its components,

$$\mathbf{E} \stackrel{not}{\equiv} \begin{bmatrix} E_{XX} & E_{XY} & E_{XZ} \\ E_{XY} & E_{YY} & E_{YZ} \\ E_{XZ} & E_{YZ} & E_{ZZ} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} . \tag{2.39}$$

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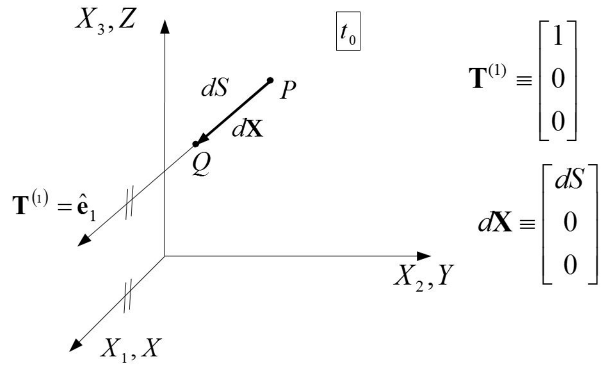

Figure 2.6: Differential segment in the reference configuration.

Consequently,

$$\mathbf{T}\cdot\mathbf{E}\cdot\mathbf{T} \overset{not}{\equiv} [\mathbf{T}]^T [\mathbf{E}] [\mathbf{T}] = [1,\ 0,\ 0] \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = E_{11}\ . \tag{2.40}$$

The stretch in the material direction X_1 is now obtained by replacing the value $\mathbf{T}\cdot\mathbf{E}\cdot\mathbf{T}$ into the expression for stretch (2.30), resulting in $\lambda_1 = \sqrt{1+2E_{11}}$. In an analogous manner, the segments oriented in the directions $X_2 \equiv Y$ and $X_3 \equiv Z$ are considered to obtain the values λ_2 and λ_3 as follows.

$$\boxed{\begin{array}{l} \lambda_1 = \sqrt{1+2E_{11}} = \sqrt{1+2E_{XX}} \ \Rightarrow\ \varepsilon_X = \lambda_X - 1 = \sqrt{1+2E_{XX}} - 1 \\ \lambda_2 = \sqrt{1+2E_{22}} = \sqrt{1+2E_{YY}} \ \Rightarrow\ \varepsilon_Y = \lambda_Y - 1 = \sqrt{1+2E_{YY}} - 1 \\ \lambda_3 = \sqrt{1+2E_{33}} = \sqrt{1+2E_{ZZ}} \ \Rightarrow\ \varepsilon_Z = \lambda_Z - 1 = \sqrt{1+2E_{ZZ}} - 1 \end{array}} \tag{2.41}$$

Remark 2.9. The components E_{XX}, E_{YY} and E_{ZZ} (or E_{11}, E_{22} and E_{33}) of the main diagonal of tensor $\mathbf{E}$ (denoted *longitudinal strains*) contain the information on stretch and unit elongations of the differential segments that were initially (in the reference configuration) oriented in the directions X, Y and Z, respectively.

- If $E_{XX} = 0 \ \Rightarrow\ \varepsilon_X = 0$: No unit elongation in direction X.
- If $E_{YY} = 0 \ \Rightarrow\ \varepsilon_Y = 0$: No unit elongation in direction Y.
- If $E_{ZZ} = 0 \ \Rightarrow\ \varepsilon_Z = 0$: No unit elongation in direction Z.

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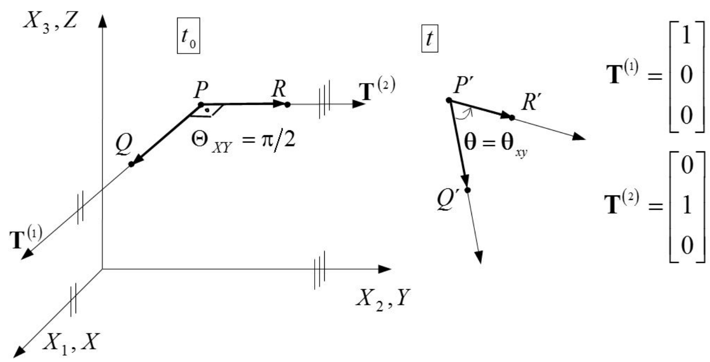

Figure 2.7: Angles between differential segments in the reference and present configurations.

Consider now the angle between segments $\overline{PQ}$ (parallel to the X_1-axis) and $\overline{PR}$ (parallel to the X_2-axis), where Q and R are two particles in the differential neighborhood of P in the material configuration and $P^{'}$, $Q^{'}$ and $R^{'}$ are the respective positions in the spatial configuration (see Figure 2.7). If the angle $(\Theta = \pi/2)$ between the segments in the reference configuration is known, the angle θ in the present configuration can be determined using (2.37) and taking into account their orthogonality ($\mathbf{T}^{(1)} \cdot \mathbf{T}^{(2)} = 0$) and the equalities $\mathbf{T}^{(1)} \cdot \mathbf{E} \cdot \mathbf{T}^{(1)} = E_{11}$, $\mathbf{T}^{(2)} \cdot \mathbf{E} \cdot \mathbf{T}^{(2)} = E_{22}$ and $\mathbf{T}^{(1)} \cdot \mathbf{E} \cdot \mathbf{T}^{(2)} = E_{12}$. That is,

$$\begin{aligned} \cos\theta &= \frac{\mathbf{T}^{(1)} \cdot (\mathbf{1} + 2\mathbf{E}) \cdot \mathbf{T}^{(2)}}{\sqrt{1 + 2\mathbf{T}^{(1)} \cdot \mathbf{E} \cdot \mathbf{T}^{(1)}}\sqrt{1 + 2\mathbf{T}^{(2)} \cdot \mathbf{E} \cdot \mathbf{T}^{(2)}}} \\ &= \frac{2E_{12}}{\sqrt{1 + 2E_{11}}\sqrt{1 + 2E_{22}}} \ , \end{aligned} \tag{2.42}$$

which is the same as

$$\theta \equiv \theta_{xy} = \frac{\pi}{2} - \arcsin \frac{2E_{XY}}{\sqrt{1 + 2E_{XX}}\sqrt{1 + 2E_{YY}}} \ . \tag{2.43}$$

The increment of the final angle with respect to its initial value results in

$$\varDelta\Theta_{XY} = \theta_{xy} - \underbrace{\Theta_{XY}}_{\pi/2} = -\arcsin \frac{2E_{XY}}{\sqrt{1 + 2E_{XX}}\sqrt{1 + 2E_{YY}}} \ . \tag{2.44}$$

Analogous results are obtained starting from pairs of segments that are oriented in different combinations of the coordinate axes, resulting in

$$\boxed{\begin{aligned} \Delta\Theta_{XY} &= -\arcsin \frac{2E_{XY}}{\sqrt{1+2E_{XX}}\sqrt{1+2E_{YY}}} \\ \Delta\Theta_{XZ} &= -\arcsin \frac{2E_{XZ}}{\sqrt{1+2E_{XX}}\sqrt{1+2E_{ZZ}}} \\ \Delta\Theta_{YZ} &= -\arcsin \frac{2E_{YZ}}{\sqrt{1+2E_{YY}}\sqrt{1+2E_{ZZ}}} \end{aligned}} \; . \tag{2.45}$$

Remark 2.10. The components E_{XY}, E_{XZ} and E_{YZ} (or E_{12}, E_{13} and E_{23}) of the tensor $\mathbf{E}$ (denoted *angular strains*) contain the information on variation of the angles between the differential segments that were initially (in the reference configuration) oriented in the directions X, Y and Z, respectively.

- If $E_{XY} = 0$: The deformation does not produce a variation in the angle between the two segments initially oriented in the directions X and Y.
- If $E_{XZ} = 0$: The deformation does not produce a variation in the angle between the two segments initially oriented in the directions X and Z.
- If $E_{YZ} = 0$: The deformation does not produce a variation in the angle between the two segments initially oriented in the directions Y and Z.

The physical interpretation of the components of the material strain tensor is shown in Figure 2.8 on an elemental parallelepiped in the neighborhood of a particle P with edges oriented in the direction of the coordinate axes.

2.7.2 Spatial Strain Tensor

Arguments similar to those of the previous subsection allow interpreting the spatial components of the strain tensor,

$$\mathbf{e} \stackrel{not}{\equiv} \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{xy} & e_{yy} & e_{yz} \\ e_{xz} & e_{yz} & e_{zz} \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{bmatrix} . \tag{2.46}$$

The components of the main diagonal (longitudinal strains) can be interpreted in terms of the stretches and unit elongations of the differential segments ori-

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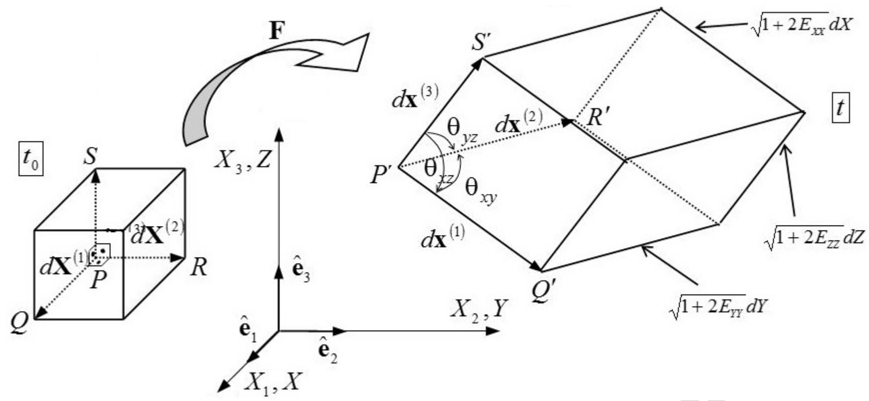

Figure 2.8: Physical interpretation of the material strain tensor.

ented in the direction of the coordinate axes in the present configuration,

$$\boxed{\begin{array}{l} \lambda_1 = \dfrac{1}{\sqrt{1-2e_{11}}} = \dfrac{1}{\sqrt{1-2e_{xx}}} \quad \Rightarrow \quad \varepsilon_x = \dfrac{1}{\sqrt{1-2e_{xx}}} - 1 \\ \lambda_2 = \dfrac{1}{\sqrt{1-2e_{22}}} = \dfrac{1}{\sqrt{1-2e_{yy}}} \quad \Rightarrow \quad \varepsilon_y = \dfrac{1}{\sqrt{1-2e_{yy}}} - 1 \\ \lambda_3 = \dfrac{1}{\sqrt{1-2e_{33}}} = \dfrac{1}{\sqrt{1-2e_{zz}}} \quad \Rightarrow \quad \varepsilon_z = \dfrac{1}{\sqrt{1-2e_{zz}}} - 1 \end{array}} \, , \qquad (2.47)$$

while the components outside the main diagonal (angular strains) contain information on the variation of the angles between the differential segments oriented in the direction of the coordinate axes in the present configuration,

$$\boxed{\begin{array}{l} \Delta\theta_{xy} = \dfrac{\pi}{2} - \Theta_{XY} = -\arcsin \dfrac{2e_{xy}}{\sqrt{1-2e_{xx}}\sqrt{1-2e_{yy}}} \\ \Delta\theta_{xz} = \dfrac{\pi}{2} - \Theta_{XZ} = -\arcsin \dfrac{2e_{xz}}{\sqrt{1-2e_{xx}}\sqrt{1-2e_{zz}}} \\ \Delta\theta_{yz} = \dfrac{\pi}{2} - \Theta_{YZ} = -\arcsin \dfrac{2e_{yz}}{\sqrt{1-2e_{yy}}\sqrt{1-2e_{zz}}} \end{array}} \, . \qquad (2.48)$$

Figure 2.9 summarizes the physical interpretation of the components of the spatial strain tensor.

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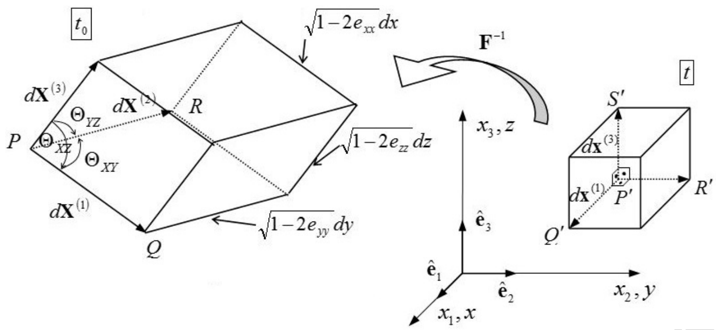

Figure 2.9: Physical interpretation of the spatial strain tensor.

2.8 Polar Decomposition

The *polar decomposition theorem* of tensor analysis establishes that, given a second-order tensor $\mathbf{F}$ such that $|\mathbf{F}| > 0$, there exist an orthogonal tensor $\mathbf{Q}$ [7] and two symmetric tensors $\mathbf{U}$ and $\mathbf{V}$ such that[8]

$$\left.\begin{array}{l} \mathbf{U} \overset{not}{=} \sqrt{\mathbf{F}^T \cdot \mathbf{F}} \\ \mathbf{V} \overset{not}{=} \sqrt{\mathbf{F} \cdot \mathbf{F}^T} \\ \mathbf{Q} = \mathbf{F} \cdot \mathbf{U}^{-1} = \mathbf{V}^{-1} \cdot \mathbf{F} \end{array}\right\} \implies \mathbf{F} = \mathbf{Q} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{Q}\,. \tag{2.49}$$

This decomposition is unique for each tensor $\mathbf{F}$ and is denominated *left polar decomposition* ($\mathbf{F} = \mathbf{Q} \cdot \mathbf{U}$) or *right polar decomposition* ($\mathbf{F} = \mathbf{V} \cdot \mathbf{Q}$). Tensors $\mathbf{U}$ and $\mathbf{V}$ are named right and left stretch tensors, respectively.

Considering now the deformation gradient tensor and the fundamental relation $d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$ defined in (2.2) as well as the polar decomposition given in (2.49), the following is obtained[9].

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} = (\mathbf{V} \cdot \mathbf{Q}) \cdot d\mathbf{X} = \overbrace{\mathbf{V} \cdot \overbrace{(\mathbf{Q} \cdot d\mathbf{X})}^{\text{rotation}}}^{\text{stretching}} \tag{2.50}$$

$$\boxed{\mathbf{F}(\bullet) \equiv \text{stretching} \overset{not}{\circ} \text{rotation}(\bullet)}$$

[7] A second-order tensor $\mathbf{Q}$ is orthogonal if $\mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{1}$ is verified.

[8] To obtain the square root of a tensor, first the tensor must be diagonalized, then the square root of the elements in the diagonal of the diagonalized component matrix are obtained and, finally, the diagonalization is undone.

[9] The notation ($\circ$) is used here to indicate the composition of two operations ξ and φ: $\mathbf{z} = \varphi \circ \xi(\mathbf{x})$.

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} = (\mathbf{Q} \cdot \mathbf{U}) \cdot d\mathbf{X} = \overbrace{\mathbf{Q} \cdot \overbrace{(\mathbf{U} \cdot d\mathbf{X})}^{\text{stretching}}}^{\text{rotation}} \tag{2.51}$$

$$\boxed{\mathbf{F}(\bullet) \equiv \text{rotation} \overset{not}{\circ} \text{stretching}(\bullet)}$$

Remark 2.11. An *orthogonal tensor* $\mathbf{Q}$ (such that $|\mathbf{Q}| = 1$) is named *rotation tensor* and the mapping $\mathbf{y} = \mathbf{Q} \cdot \mathbf{x}$ is denominated *rotation*. A rotation has the following properties:

- When applied on any vector $\mathbf{x}$, the result is another vector $\mathbf{y} = \mathbf{Q} \cdot \mathbf{x}$ with the same modulus,

$$\|\mathbf{y}\|^2 = \mathbf{y} \cdot \mathbf{y} \overset{not}{\equiv} [\mathbf{y}]^T \cdot [\mathbf{y}] = [\mathbf{Q} \cdot \mathbf{x}]^T \cdot [\mathbf{Q} \cdot \mathbf{x}] \overset{not}{\equiv} \mathbf{x} \cdot \underbrace{\mathbf{Q}^T \cdot \mathbf{Q}}_{\mathbf{1}} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2 \,.$$

- The result of multiplying (mapping) the orthogonal tensor $\mathbf{Q}$ to two vectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ with the same origin and that form an angle α between them, maintains the same angle between the images $\mathbf{y}^{(1)} = \mathbf{Q} \cdot \mathbf{x}^{(1)}$ and $\mathbf{y}^{(2)} = \mathbf{Q} \cdot \mathbf{x}^{(2)}$,

$$\frac{\mathbf{y}^{(1)} \cdot \mathbf{y}^{(2)}}{\|\mathbf{y}^{(1)}\| \, \|\mathbf{y}^{(2)}\|} = \frac{\mathbf{x}^{(1)} \cdot \mathbf{Q}^T \cdot \mathbf{Q} \cdot \mathbf{x}^{(2)}}{\|\mathbf{y}^{(1)}\| \|\mathbf{y}^{(2)}\|} = \frac{\mathbf{x}^{(1)} \cdot \mathbf{x}^{(2)}}{\|\mathbf{x}^{(1)}\| \, \|\mathbf{x}^{(2)}\|} = \cos\alpha \,.$$

In consequence, the mapping (rotation) $\mathbf{y} = \mathbf{Q} \cdot \mathbf{x}$ maintains the angles and distances.

Remark 2.12. Equations (2.50) establish that the relative motion in the neighborhood of the particle during the deformation process (characterized by tensor $\mathbf{F}$) can be understood as the *composition* of a *rotation* (characterized by the rotation tensor $\mathbf{Q}$, which maintains angles and distances) and a *stretching* or *deformation in itself* (which modifies angles and distances) characterized by the tensor $\mathbf{V}$ (see Figure 2.10).

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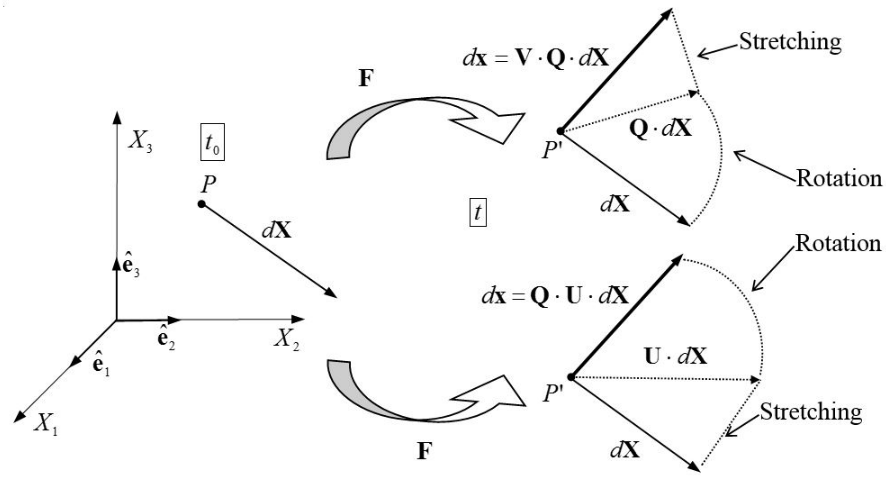

Figure 2.10: Polar decomposition.

Remark 2.13. Alternatively, equations (2.51) allow characterizing the relative motion in the neighborhood of a particle during the deformation process as the *superposition* of a *stretching* or *deformation in itself* (characterized by tensor $\mathbf{U}$) and a *rotation* (characterized by the rotation tensor $\mathbf{Q}$).

A rigid body motion is a particular case of deformation characterized by $\mathbf{U} = \mathbf{V} = \mathbf{1}$ and $\mathbf{Q} = \mathbf{F}$.

2.9 Volume Variation

Consider a particle P of the continuous medium in the reference configuration $(t = 0)$ which has a differential volume dV_0 associated with it (see Figure 2.11). This differential volume is characterized by the positions of another three particles Q, R and S belonging to the differential neighborhood of P, which are aligned with this particle in three arbitrary directions. The volume differential dV_t, associated with the same particle in the present configuration (at time t), will also be characterized by the spatial points $P^{'}$, $Q^{'}$, $R^{'}$ and $S^{'}$ corresponding to Figure 2.11 (the positions of which define a parallelepiped that is no longer oriented along the coordinate axes).

The relative position vectors between the particles in the material configuration are $d\mathbf{X}^{(1)}$, $d\mathbf{X}^{(2)}$ and $d\mathbf{X}^{(3)}$, and their counterparts in the spatial configuration are $d\mathbf{x}^{(1)} = \mathbf{F} \cdot d\mathbf{X}^{(1)}$, $d\mathbf{x}^{(2)} = \mathbf{F} \cdot d\mathbf{X}^{(2)}$ and $d\mathbf{x}^{(3)} = \mathbf{F} \cdot d\mathbf{X}^{(3)}$. Obviously, the relations

$$\begin{cases} d\mathbf{x}^{(i)} = \mathbf{F} \cdot d\mathbf{X}^{(i)} \\ dx_j^{(i)} = F_{jk}\, dX_k^{(i)} \qquad i, j, k \in \{1, 2, 3\} \end{cases} \tag{2.52}$$

are satisfied. Then, the volumes[10] associated with a particle in both configurations can be written as

$$dV_0 = \left(d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)} \right) \cdot d\mathbf{X}^{(3)} = \det \underbrace{\begin{bmatrix} dX_1^{(1)} & dX_2^{(1)} & dX_3^{(1)} \\ dX_1^{(2)} & dX_2^{(2)} & dX_3^{(2)} \\ dX_1^{(3)} & dX_2^{(3)} & dX_3^{(3)} \end{bmatrix}}_{[\mathbf{M}]} = |\mathbf{M}| \ ,$$

$$dV_t = \left(d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} \right) \cdot d\mathbf{x}^{(3)} = \det \underbrace{\begin{bmatrix} dx_1^{(1)} & dx_2^{(1)} & dx_3^{(1)} \\ dx_1^{(2)} & dx_2^{(2)} & dx_3^{(2)} \\ dx_1^{(3)} & dx_2^{(3)} & dx_3^{(3)} \end{bmatrix}}_{[\mathbf{m}]} = |\mathbf{m}| \ , \tag{2.53}$$

where $M_{ij} = dX_j^{(i)}$ and $m_{ij} = dx_j^{(i)}$. Considering these expressions,

$$m_{ij} = dx_j^{(i)} = F_{jk}\, dX_k^{(i)} = F_{jk}\, dM_{ik} = dM_{ik}\, F_{kj}^T \quad \Longrightarrow \quad \mathbf{m} = \mathbf{M} \cdot \mathbf{F}^T \tag{2.54}$$

is deduced and, consequently[11],

$$\left. \begin{array}{l} dV_t = |\mathbf{m}| = \left| \mathbf{M} \cdot \mathbf{F}^T \right| = |\mathbf{M}| \left| \mathbf{F}^T \right| = |\mathbf{F}| \underbrace{|\mathbf{M}|}_{dV_0} = |\mathbf{F}|\, dV_0 \\ dV_t = dV \left(\mathbf{x}(\mathbf{X},t), t \right) = |\mathbf{F}(\mathbf{X},t)|\, dV_0 (\mathbf{X},0) = |\mathbf{F}|_t\, dV_0 \end{array} \right\} \Longrightarrow \boxed{dV_t = |\mathbf{F}|_t\, dV_0} \tag{2.55}$$

[10] The volume of a parallelepiped is calculated as the scalar triple product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ of the concurrent edge-vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, which meet at any of the parallelepiped's vertices. Note that the scalar triple product is the determinant of the matrix constituted by the components of the above mentioned vectors arranged in rows.

[11] The expressions $|\mathbf{A} \cdot \mathbf{B}| = |\mathbf{A}| \cdot |\mathbf{B}|$ and $\left| \mathbf{A}^T \right| = |\mathbf{A}|$ are used here.

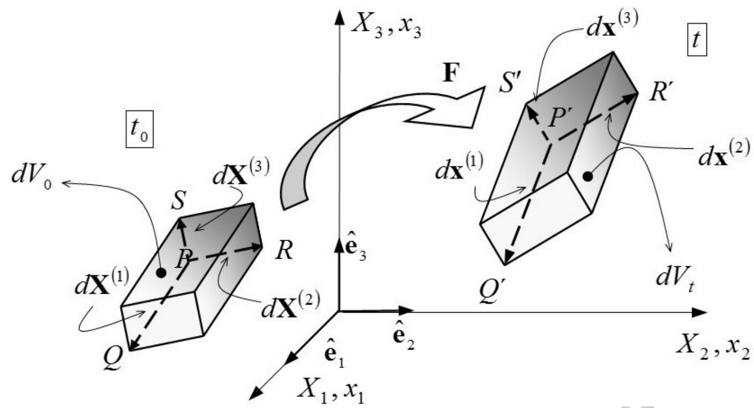

Figure 2.11: Variation of a volume differential element.

2.10 Area Variation

Consider an area differential dA associated with a particle P in the reference configuration and its variation along time. To define this area differential, consider two particles Q and R in the differential neighborhood of P, whose relative positions with respect to this particle are $d\mathbf{X}^{(1)}$ and $d\mathbf{X}^{(2)}$, respectively (see Figure 2.12). Consider also an arbitrary auxiliary particle S whose relative position vector is $d\mathbf{X}^{(3)}$. An *area differential vector* $d\mathbf{A} = dA\,\mathbf{N}$ associated with the *scalar differential area*, dA, is defined. The module of vector $d\mathbf{A}$ is dA and its direction is the same as that of the unit normal vector in the material configuration $\mathbf{N}$.

In the present configuration, at time t, the particle will occupy a point in space P' and will have an area differential da associated with it which, in turn, defines an area differential vector $d\mathbf{a} = da\,\mathbf{n}$, where $\mathbf{n}$ is the corresponding unit normal vector in the spatial configuration. Consider also the positions of the other particles Q', R' and S' and their relative position vectors $d\mathbf{x}^{(1)}$, $d\mathbf{x}^{(2)}$ and $d\mathbf{x}^{(3)}$.

The volumes dV_0 and dV_t of the corresponding parallelepipeds can be calculated as

$$\begin{aligned} dV_0 &= dH\,dA = \underbrace{d\mathbf{X}^{(3)}\cdot\mathbf{N}}_{dH}\,dA = d\mathbf{X}^{(3)}\cdot\underbrace{\mathbf{N}\,dA}_{d\mathbf{A}} = d\mathbf{A}\cdot d\mathbf{X}^{(3)} \\ dV_t &= dh\,da = \underbrace{d\mathbf{x}^{(3)}\cdot\mathbf{n}}_{dh}\,da = d\mathbf{x}^{(3)}\cdot\underbrace{\mathbf{n}\,da}_{d\mathbf{a}} = d\mathbf{a}\cdot d\mathbf{x}^{(3)} \end{aligned} \tag{2.56}$$

and, taking into account that $d\mathbf{x}^{(3)} = \mathbf{F}\cdot d\mathbf{X}^{(3)}$, as well as the expression for change in volume (2.55), results in

$$d\mathbf{a}\cdot\mathbf{F}\cdot d\mathbf{X}^{(3)} = d\mathbf{a}\cdot d\mathbf{x}^{(3)} = dV_t = |\mathbf{F}|\,dV_0 = |\mathbf{F}|\,d\mathbf{A}\cdot d\mathbf{X}^{(3)} \qquad \forall d\mathbf{X}^{(3)}\ . \tag{2.57}$$

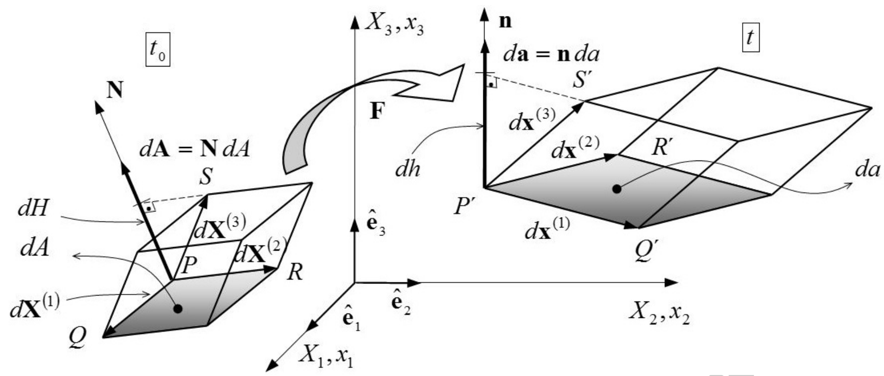

Figure 2.12: Variation of an area differential.

Comparing the first and last terms[12] in (2.57) and considering that the relative position of particle S can take any value (as can, therefore, vector $d\mathbf{X}^{(3)}$), finally yields

$$d\mathbf{a} \cdot \mathbf{F} = |\mathbf{F}|\, d\mathbf{A} \quad \Longrightarrow \quad \boxed{d\mathbf{a} = |\mathbf{F}|\, d\mathbf{A} \cdot \mathbf{F}^{-1}} . \tag{2.58}$$

To obtain the relation between the two *area differential scalars*, dA and da, expressions $d\mathbf{A} = \mathbf{N}\, dA$ and $d\mathbf{a} = \mathbf{n}\, da$ are replaced into (2.58) and the modules are taken, resulting in

$$da\, \mathbf{n} = |\mathbf{F}|\, \mathbf{N} \cdot d\mathbf{F}^{-1} dA \quad \Longrightarrow \quad da = |\mathbf{F}| \left\| \mathbf{N} \cdot d\mathbf{F}^{-1} \right\| dA\, . \tag{2.59}$$

2.11 Infinitesimal Strain

Infinitesimal strain theory (also denominated *small deformation theory*) is based on two simplifying hypotheses of the general theory (or *finite strain theory*) seen in the previous sections (see Figure 2.13).

> **Definition 2.5.** The simplifying hypotheses are:
>
> 1) *Displacements are very small* compared to the typical dimensions in the continuous medium: $\underline{\|\mathbf{u}\|} << \|\mathbf{X}\|$.
>
> 2) *Displacement gradients are very small* (infinitesimal).

[12] Here, the following tensor algebra theorem is taken into account: given two vectors $\mathbf{a}$ and $\mathbf{b}$, if the relation $\mathbf{a} \cdot \mathbf{x} = \mathbf{b} \cdot \mathbf{x}$ is satisfied for all values of $\mathbf{x}$, then $\mathbf{a} = \mathbf{b}$.

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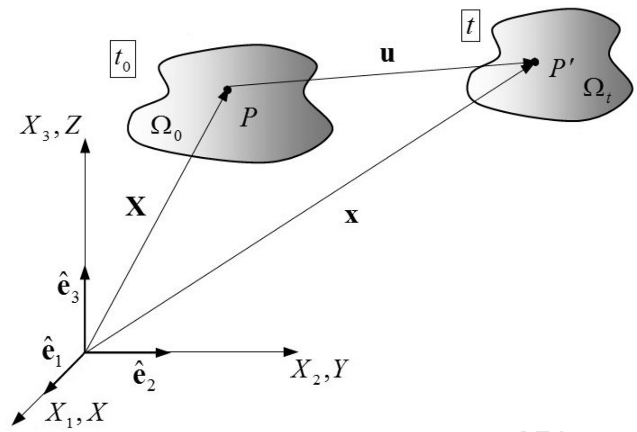

Figure 2.13: Infinitesimal strain in the continuous medium.

In accordance with the first hypothesis, the reference configuration Ω_0 and the present configuration Ω_t are very close together and are considered to be indistinguishable from one another. Consequently, the material and spatial coordinates coincide and discriminating between material and spatial descriptions no longer makes sense.

$$\begin{cases} \mathbf{x} = \mathbf{X} + \mathbf{u} \cong \mathbf{X} \\ x_i = X_i + u_i \cong X_i \end{cases} \Longrightarrow \begin{cases} \mathbf{U}(\mathbf{X},t) \overset{not}{=} \mathbf{u}(\mathbf{X},t) \equiv \mathbf{u}(\mathbf{x},t) \\ U_i(\mathbf{X},t) \overset{not}{=} u_i(\mathbf{X},t) \equiv u_i(\mathbf{x},t) \quad i \in \{1,2,3\} \end{cases} \tag{2.60}$$

The second hypothesis can be written in mathematical form as

$$\left| \frac{\partial u_i}{\partial x_j} \right| \ll 1 \qquad \forall\, i,j \in \{1,2,3\} \ . \tag{2.61}$$

2.11.1 Strain Tensors. Infinitesimal Strain Tensor

The material and spatial displacement gradient tensors coincide. Indeed, in view of (2.60),

$$\begin{cases} x_j = X_j \\ u_i(\mathbf{x},t) = U_i(\mathbf{X},t) \end{cases} \Longrightarrow \quad j_{ij} = \frac{\partial u_i}{\partial x_j} = \frac{\partial U_i}{\partial X_j} = J_{ij} \quad \Longrightarrow \quad \mathbf{j} = \mathbf{J} \tag{2.62}$$

and the material strain tensor results in

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$$
\begin{cases}
\mathbf{E} = \dfrac{1}{2}\left(\mathbf{J} + \mathbf{J}^T + \mathbf{J}^T \cdot \mathbf{J}\right) \cong \dfrac{1}{2}\left(\mathbf{J} + \mathbf{J}^T\right) \ , \\
E_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} + \underbrace{\dfrac{\partial u_k}{\partial x_i}\dfrac{\partial u_k}{\partial x_j}}_{\ll 1}\right) \cong \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) \ ,
\end{cases}
\tag{2.63}
$$

where the infinitesimal character of the second-order term $(\partial u_k \partial u_k / \partial x_j \partial x_i)$ has been taken into account. Operating in a similar manner with the spatial strain tensor,

$$
\begin{cases}
\mathbf{e} = \dfrac{1}{2}\left(\mathbf{j} + \mathbf{j}^T - \mathbf{j}^T \cdot \mathbf{j}\right) \cong \dfrac{1}{2}\left(\mathbf{j} + \mathbf{j}^T\right) = \dfrac{1}{2}\left(\mathbf{J} + \mathbf{J}^T\right) \ , \\
e_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i} - \underbrace{\dfrac{\partial u_k}{\partial x_i}\dfrac{\partial u_k}{\partial x_j}}_{\ll 1}\right) \cong \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) \ .
\end{cases}
\tag{2.64}
$$

Equations (2.63) and (2.64) allow defining the *infinitesimal strain tensor* (or small strain tensor) $\boldsymbol{\varepsilon}$ as[13]

$$
\boxed{
\begin{array}{ll}
\text{Infinitesimal strain tensor} &
\begin{cases}
\boldsymbol{\varepsilon} = \dfrac{1}{2}\left(\mathbf{J} + \mathbf{J}^T\right) \stackrel{not}{=} \nabla^s \mathbf{u} \\
\varepsilon_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) \qquad i, j \in \{1,2,3\}
\end{cases}
\end{array}
}
\tag{2.65}
$$

Remark 2.14. Under the infinitesimal strain hypothesis, the *material and spatial strain tensors coincide* and collapse into the *infinitesimal strain tensor*.

$$
\mathbf{E}(\mathbf{x},t) = \mathbf{e}(\mathbf{x},t) = \boldsymbol{\varepsilon}(\mathbf{x},t)
$$

Remark 2.15. The *infinitesimal strain tensor is symmetric*, as observed in its definition in (2.65).

$$
\boldsymbol{\varepsilon}^T = \frac{1}{2}\left(\mathbf{J} + \mathbf{J}^T\right)^T = \frac{1}{2}\left(\mathbf{J}^T + \mathbf{J}\right) = \boldsymbol{\varepsilon}
$$

[13] The *symmetric gradient* operator ∇^s is defined as $\nabla^s(\bullet) = ((\bullet) \otimes \nabla + \nabla \otimes (\bullet))/2$.

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Remark 2.16. The *components of the infinitesimal strain tensor* $\boldsymbol{\varepsilon}$ *are infinitesimal* $(\varepsilon_{ij} \ll 1)$. Proof is obvious from (2.65) and the condition that the components of $\mathbf{J} = \mathbf{j}$ are infinitesimal (see (2.61)).

Example 2.4 – *Determine under which conditions the motion in Example 2.1 constitutes an infinitesimal strain case and obtain the infinitesimal strain tensor for this case. Compare it with the result obtained from the spatial and material strain tensors in Example 2.2 taking into account the infinitesimal strain hypotheses.*

Solution

The equation of motion is given by

$$\begin{cases} x_1 = X_1 - AX_3 \\ x_2 = X_2 - AX_3 \\ x_3 = -AX_1 + AX_2 + X_3 \end{cases} ,$$

from which the displacement field is obtained

$$\mathbf{U}(\mathbf{X},t) = \mathbf{x} - \mathbf{X} \overset{not}{\equiv} \begin{bmatrix} U_1 = -AX_3 \\ U_2 = -AX_3 \\ U_3 = -AX_1 + AX_2 \end{bmatrix} .$$

It is obvious that, for the displacements to be infinitesimal, *A must be infinitesimal* $(A \ll 1)$. Now, to obtain the infinitesimal strain tensor, first the displacement gradient tensor $\mathbf{J}(\mathbf{X},t) = \mathbf{j}(\mathbf{x},t)$ must be computed,

$$\mathbf{J} = \mathbf{U} \otimes \nabla \overset{not}{\equiv} \begin{bmatrix} -AX_3 \\ -AX_3 \\ -AX_1 + AX_2 \end{bmatrix} \left[\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3} \right] = \begin{bmatrix} 0 & 0 & -A \\ 0 & 0 & -A \\ -A & A & 0 \end{bmatrix} .$$

Then, the infinitesimal strain tensor, in accordance to (2.65), is

$$\boldsymbol{\varepsilon} = \nabla^s \mathbf{U} \overset{not}{\equiv} \begin{bmatrix} 0 & 0 & -A \\ 0 & 0 & 0 \\ -A & 0 & 0 \end{bmatrix} .$$

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The material and spatial strain tensors obtained in Example 2.2 are, respectively,

$$\mathbf{E}(\mathbf{X},t) \stackrel{not}{\equiv} \frac{1}{2}\begin{bmatrix} A^2 & -A^2 & -2A \\ -A^2 & A^2 & 0 \\ -2A & 0 & 2A^2 \end{bmatrix} \quad \text{and}$$

$$\mathbf{e}(\mathbf{X},t) \stackrel{not}{\equiv} \frac{1}{2}\begin{bmatrix} -3A^2-2A^4 & A^2+2A^4 & -2A-2A^3 \\ A^2+2A^4 & A^2-2A^4 & 2A^3 \\ -2A-2A^3 & 2A^3 & -2A^2 \end{bmatrix}.$$

Neglecting the second-order and higher-order infinitesimal terms $\left(A^4 \ll A^3 \ll A^2 \ll A\right)$ results in

$$\mathbf{E} \stackrel{not}{\equiv} \begin{bmatrix} 0 & 0 & -A \\ 0 & 0 & -A \\ -A & A & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e} \stackrel{not}{\equiv} \begin{bmatrix} 0 & 0 & -A \\ 0 & 0 & -A \\ -A & A & 0 \end{bmatrix} \implies \mathbf{E} = \mathbf{e} = \boldsymbol{\varepsilon}\,,$$

which is in accordance with Remark 2.14.

2.11.2 Stretch. Unit Elongation

Considering the general expression (2.30) of the unit elongation in the direction $\mathbf{T} \cong \mathbf{t}$ $\left(\lambda_t = \sqrt{1+2\mathbf{t}\cdot\mathbf{E}\cdot\mathbf{t}}\right)$ and applying a Taylor series expansion[14] around 0 (taking into account that $\mathbf{E} = \boldsymbol{\varepsilon}$ is infinitesimal and, therefore, so is $x = \mathbf{t}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{t}$), yields

$$\boxed{\begin{aligned} &\lambda_t = \sqrt{1+\underbrace{2\mathbf{t}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{t}}_{x}} \cong 1+\mathbf{t}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{t} \\ &\varepsilon_t = \lambda_t - 1 = \mathbf{t}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{t} \end{aligned}} \tag{2.66}$$

2.11.3 Physical Interpretation of the Infinitesimal Strains

Consider the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ and its components in the coordinate system $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, shown in Figure 2.14,

$$\boldsymbol{\varepsilon} \stackrel{not}{\equiv} \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}. \tag{2.67}$$

[14] The Taylor series expansion of $\sqrt{1+x}$ around $x=0$ is $\sqrt{1+x} = 1 + x/2 + O\left(x^2\right)$.

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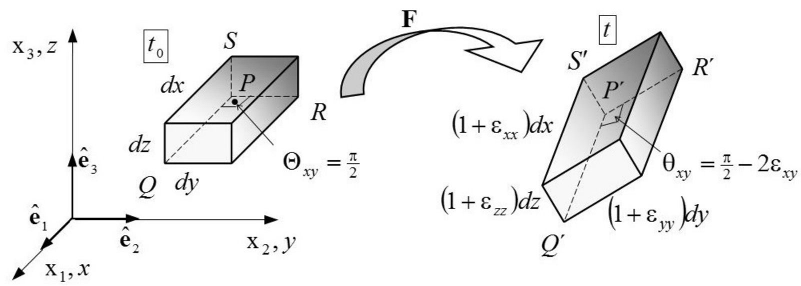

Figure 2.14: Physical interpretation of the infinitesimal strains.

Consider a differential segment $\overline{PQ}$ oriented in the reference configuration parallel to the coordinate axis $x_1 \equiv x$. The stretch λ_x and the unit elongation ε_x in this direction are, according to (2.66) with $\mathbf{t} = [1,\ 0,\ 0]^T$,

$$\lambda_x = 1 + \mathbf{t} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{t} = 1 + \varepsilon_{xx} \quad \Longrightarrow \quad \varepsilon_x = \lambda_x - 1 = \varepsilon_{xx} \ . \tag{2.68}$$

This allows assigning to the component $\varepsilon_{xx} \equiv \varepsilon_{11}$ the *physical meaning of unit elongation ε_x in the direction of the coordinate axis $x_1 \equiv x$*. A similar interpretation is deduced for the other components in the main diagonal of the tensor $\boldsymbol{\varepsilon}\,(\varepsilon_{xx},\ \varepsilon_{yy},\ \varepsilon_{zz})$,

$$\boxed{\varepsilon_{xx} = \varepsilon_x \ ; \quad \varepsilon_{yy} = \varepsilon_y \ ; \quad \varepsilon_{zz} = \varepsilon_z} \ . \tag{2.69}$$

Given now the components outside the main diagonal of $\boldsymbol{\varepsilon}$, consider the differential segments $\overline{PQ}$ and $\overline{PR}$ oriented in the reference configuration parallel to the coordinate directions x and y, respectively. Then, these two segments form an angle $\Theta_{xy} = \pi/2$ in this configuration. Applying (2.43), the increment in the corresponding angle results in[15]

$$\Delta\theta_{xy} = \theta_{xy} - \frac{\pi}{2} = -2 \arcsin \frac{\varepsilon_{xy}}{\underbrace{\sqrt{1+2\varepsilon_{xx}}}_{\simeq 1}\ \underbrace{\sqrt{1+2\varepsilon_{yy}}}_{\simeq 1}} \cong -2 \underbrace{\arcsin \varepsilon_{xy}}_{\simeq \varepsilon_{xy}} = -2\varepsilon_{xy} \ , \tag{2.70}$$

where the infinitesimal character of ε_{xx}, ε_{yy} and ε_{xy} has been taken into account. Consequently, ε_{xy} can be interpreted from (2.70) as *minus the semi-increment, produced by the strain, of the angle between the two differential segments initially oriented parallel to the coordinate directions x and y*. A similar interpretation is deduced for the other components ε_{xz} and ε_{yz},

$$\boxed{\varepsilon_{xy} = -\frac{1}{2}\Delta\theta_{xy} \quad ; \quad \varepsilon_{xz} = -\frac{1}{2}\Delta\theta_{xz} \quad ; \quad \varepsilon_{yz} = -\frac{1}{2}\Delta\theta_{yz}} \ . \tag{2.71}$$

[15] The Taylor series expansion of $\arcsin x$ around $x = 0$ is $\arcsin x = x + O\left(x^2\right)$.

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2.11.4 Engineering Strains. Vector of Engineering Strains

There is a strong tradition in engineering to use a particular denomination for the components of the infinitesimal strain tensor. This convention is named *engineering notation*, as opposed to the *scientific notation* generally used in continuum mechanics. Both notations are synthesized as follows.

$$
\boldsymbol{\varepsilon} \stackrel{not}{\equiv} \overbrace{\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \equiv \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix}}^{\text{scientific notation}} \equiv \overbrace{\begin{bmatrix} \varepsilon_{x} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{y} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{z} \end{bmatrix}}^{\text{engineering notation}} \tag{2.72}
$$

Remark 2.17. The components in the main diagonal of the strain tensor (named *longitudinal strains*) are denoted by $\varepsilon_{(\bullet)}$ and coincide with the *unit elongations in the directions of the coordinate axes*. Positive values of longitudinal strains $\left(\varepsilon_{(\bullet)} > 0\right)$ correspond to an increase in length of the corresponding differential segments in the reference configuration.

Remark 2.18. The components outside the main diagonal of the strain tensor are characterized by the values $\gamma_{(\bullet,\bullet)}$ (named *angular strains*) and can be interpreted as the *decrements of the corresponding angles oriented in the Cartesian directions of the reference configuration*. Positive values of angular strains $\left(\gamma_{(\bullet,\bullet)} > 0\right)$ indicate that the corresponding angles *close* with the deformation process.

In engineering, it is also frequent to exploit the symmetry of the infinitesimal strain tensor (see Remark 2.15) to work only with the six components of the tensor that are different, grouping them in the *vector of engineering strains*, which is defined as follows.

$$
\boldsymbol{\varepsilon} \in \mathbb{R}^6 \qquad \boldsymbol{\varepsilon} \stackrel{def}{=} \Big[\underbrace{\varepsilon_x\,,\ \varepsilon_y\,,\ \varepsilon_x}_{\substack{\text{longitudinal}\\ \text{strains}}}\,,\ \underbrace{\gamma_{xy}\,,\ \gamma_{xz}\,,\ \gamma_{yz}}_{\substack{\text{angular}\\ \text{strains}}} \Big]^T \tag{2.73}
$$

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2.11.5 Variation of the Angle between Two Differential Segments in Infinitesimal Strain

Consider any two differential segments, $\overline{PQ}$ and $\overline{PR}$, in the reference configuration and the angle Θ they define (see Figure 2.15). The angle formed by the corresponding deformed segments in the present configuration is $\theta = \Theta + \Delta\theta$. Applying (2.42) to this case results in

$$\cos\theta = \cos(\Theta + \Delta\theta) = \frac{\mathbf{T}^{(1)} \cdot (\mathbf{1} + 2\boldsymbol{\varepsilon}) \cdot \mathbf{T}^{(2)}}{\sqrt{1 + \underbrace{2\mathbf{T}^{(1)} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{T}^{(1)}}_{\ll 1}}\ \sqrt{1 + \underbrace{2\mathbf{T}^{(2)} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{T}^{(2)}}_{\ll 1}}}\ , \tag{2.74}$$

where $\mathbf{T}^{(1)}$ and $\mathbf{T}^{(2)}$ are the unit vectors in the directions of $\overline{PQ}$ and $\overline{PR}$ and, therefore, the relation $\mathbf{T}^{(1)} \cdot \mathbf{T}^{(2)} = \left\|\mathbf{T}^{(1)}\right\| \left\|\mathbf{T}^{(2)}\right\| \cos\Theta = \cos\Theta$ is fulfilled. Considering the infinitesimal character of the components of $\boldsymbol{\varepsilon}$ and $\Delta\theta$, the following holds true[16].

$$\begin{aligned}
\cos\theta &= \cos(\Theta + \Delta\theta) = \cos\Theta \cdot \underbrace{\cos\Delta\theta}_{\approx 1} - \sin\Theta \cdot \underbrace{\sin\Delta\theta}_{\approx \Delta\theta} = \\
&= \cos\Theta - \sin\Theta \cdot \Delta\theta = \frac{\overbrace{\mathbf{T}^{(1)} \cdot \mathbf{T}^{(2)}}^{= \cos\Theta} + 2\mathbf{T}^{(1)} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{T}^{(2)}}{\underbrace{\sqrt{1 + \mathbf{T}^{(1)} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{T}^{(1)}}}_{\approx 1}\ \underbrace{\sqrt{1 + \mathbf{T}^{(2)} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{T}^{(2)}}}_{\approx 1}} = \\
&= \cos\Theta + 2\mathbf{T}^{(1)} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{T}^{(2)}
\end{aligned} \tag{2.75}$$

Therefore, $\sin\Theta \cdot \Delta\theta = -2\mathbf{T}^{(1)} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{T}^{(2)}$ and

$$\boxed{\Delta\theta = -\frac{2\mathbf{T}^{(1)} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{T}^{(2)}}{\sin\Theta} = -\frac{2\mathbf{t}^{(1)} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{t}^{(2)}}{\sin\theta}}\ , \tag{2.76}$$

where the infinitesimal character of the strain has been taken into account and, thus, it follows that $\mathbf{T}^{(1)} \approx \mathbf{t}^{(1)}$, $\mathbf{T}^{(2)} \approx \mathbf{t}^{(2)}$ and $\Theta \approx \theta$.

2.11.6 Polar Decomposition

The polar decomposition of the deformation gradient tensor $\mathbf{F}$ is given by (2.49) for the general case of finite strain. In the case of infinitesimal strain, recall-

[16] The following Taylor series expansions around $x = 0$ are considered: $\sin x = x + O\left(x^2\right)$ and $\cos x = 1 + O\left(x^2\right)$

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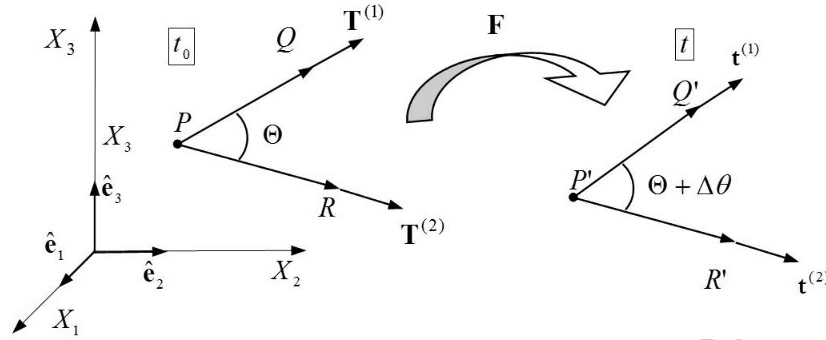

Figure 2.15: Variation of the angle between two differential segments in infinitesimal strain.

ing (2.12) and the infinitesimal character of the components of the tensor $\mathbf{J}$ (see (2.61)), tensor $\mathbf{U}$ in (2.49) can be written as[17]

$$\left.\begin{array}{l}\mathbf{U}=\sqrt{\mathbf{F}^T\cdot\mathbf{F}}=\sqrt{(\mathbf{1}+\mathbf{J}^T)\cdot(\mathbf{1}+\mathbf{J})}=\\ =\sqrt{\mathbf{1}+\mathbf{J}+\mathbf{J}^T+\underbrace{\mathbf{J}^T\cdot\mathbf{J}}_{\ll\mathbf{J}}}\approx\sqrt{\mathbf{1}+\mathbf{J}+\mathbf{J}^T}=\mathbf{1}+\underbrace{\frac{1}{2}\left(\mathbf{J}+\mathbf{J}^T\right)}_{\boldsymbol{\varepsilon}}\end{array}\right\}\Longrightarrow\boxed{\mathbf{U}=\mathbf{1}+\boldsymbol{\varepsilon}}\,. \tag{2.77}$$

In a similar manner, due to the infinitesimal character of the components of the tensor $\boldsymbol{\varepsilon}$ (see Remark 2.16), the inverse of tensor $\mathbf{U}$ results in[18]

$$\mathbf{U}^{-1}=(\mathbf{1}+\boldsymbol{\varepsilon})^{-1}=\mathbf{1}-\boldsymbol{\varepsilon}=\mathbf{1}-\frac{1}{2}\left(\mathbf{J}+\mathbf{J}^T\right)\,. \tag{2.78}$$

Therefore, the rotation tensor $\mathbf{Q}$ in (2.49) can be written as

$$\left.\begin{array}{l}\mathbf{Q}=\mathbf{F}\cdot\mathbf{U}^{-1}=(\mathbf{1}+\mathbf{J})\cdot\left(\mathbf{1}-\frac{1}{2}\left(\mathbf{J}+\mathbf{J}^T\right)\right)=\\ =\mathbf{1}+\mathbf{J}-\frac{1}{2}\left(\mathbf{J}+\mathbf{J}^T\right)-\underbrace{\frac{1}{2}\mathbf{J}\cdot\left(\mathbf{J}+\mathbf{J}^T\right)}_{\ll\mathbf{J}}=\mathbf{1}+\underbrace{\frac{1}{2}\left(\mathbf{J}-\mathbf{J}^T\right)}_{\boldsymbol{\Omega}}\end{array}\right\}\Longrightarrow\boxed{\mathbf{Q}=\mathbf{1}+\boldsymbol{\Omega}}\,. \tag{2.79}$$

[17] The Taylor series expansions of tensor $\sqrt{\mathbf{1}+\mathbf{x}}$ around $\mathbf{x}=0$ is $\sqrt{\mathbf{1}+\mathbf{x}}=\mathbf{1}+\mathbf{x}/2+O\left(\mathbf{x}^2\right)$.
[18] The Taylor series expansions of tensor $(\mathbf{1}+\mathbf{x})^{-1}$ around $\mathbf{x}=\mathbf{0}$ is $(\mathbf{1}+\mathbf{x})^{-1}=\mathbf{1}-\mathbf{x}+O\left(\mathbf{x}^2\right)$.

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Equation (2.79) defines the *infinitesimal rotation tensor* $\boldsymbol{\Omega}$ [19] as follows.

$$\text{Infinitesimal rotation tensor} \quad \begin{cases} \boldsymbol{\Omega} \stackrel{def}{=} \dfrac{1}{2}\left(\mathbf{J}-\mathbf{J}^T\right) = \dfrac{1}{2}\left(\mathbf{u}\otimes\nabla - \nabla\otimes\mathbf{u}\right) \stackrel{def}{=} \nabla^a\mathbf{u} \\ \Omega_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} - \dfrac{\partial u_j}{\partial x_i}\right) \ll 1 \qquad i,j \in \{1,2,3\} \end{cases} \tag{2.80}$$

Remark 2.19. The tensor $\boldsymbol{\Omega}$ is *antisymmetric*. Indeed,

$$\begin{cases} \boldsymbol{\Omega}^T = \dfrac{1}{2}\left(\mathbf{J}-\mathbf{J}^T\right)^T = \dfrac{1}{2}\left(\mathbf{J}^T-\mathbf{J}\right) = -\boldsymbol{\Omega} \\ \Omega_{ji} = -\Omega_{ij} \qquad i,j\in\{1,2,3\} \end{cases}.$$

Consequently, the terms in the main diagonal of $\boldsymbol{\Omega}$ are zero, and its matrix of components has the structure

$$[\boldsymbol{\Omega}] = \begin{bmatrix} 0 & \Omega_{12} & -\Omega_{31} \\ -\Omega_{12} & 0 & \Omega_{23} \\ \Omega_{31} & -\Omega_{23} & 0 \end{bmatrix}.$$

In a small rotation context, tensor $\boldsymbol{\Omega}$ *characterizes the rotation* ($\mathbf{Q} = \mathbf{1} + \boldsymbol{\Omega}$) and, thus, the denomination of infinitesimal rotation tensor. Since it is an antisymmetric tensor, it is defined solely by three different components (Ω_{23}, Ω_{31}, Ω_{12}), which form the *infinitesimal rotation vector* $\boldsymbol{\theta}$ [20],

$$\text{Infinitesimal rotation vector:} \quad \boldsymbol{\theta} \stackrel{not}{\equiv} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -\Omega_{23} \\ -\Omega_{31} \\ -\Omega_{12} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \dfrac{\partial u_3}{\partial x_2} - \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_1}{\partial x_3} - \dfrac{\partial u_3}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2} \end{bmatrix} \stackrel{def}{=} \frac{1}{2}\nabla\times\mathbf{u}. \tag{2.81}$$

[19] The *antisymmetric gradient* operator ∇^a is defined as $\nabla^a(\bullet) = [(\bullet)\otimes\nabla - \nabla\otimes(\bullet)]/2$.
[20] The operator *rotational of* $(\bullet)$ is denoted as $\nabla\times(\bullet)$.

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Expressions (2.12), (2.65) and (2.79) allow writing

$$\mathbf{F} = \mathbf{1} + \mathbf{J} + \underbrace{\frac{1}{2}\left(\mathbf{J}+\mathbf{J}^T\right)}_{\boldsymbol{\varepsilon}} + \underbrace{\frac{1}{2}\left(\mathbf{J}-\mathbf{J}^T\right)}_{\boldsymbol{\Omega}} \Longrightarrow \boxed{\mathbf{F} = \mathbf{1} + \boldsymbol{\varepsilon} + \boldsymbol{\Omega}} . \tag{2.82}$$

Remark 2.20. The results of *performing a dot product of the infinitesimal rotation tensor* $\boldsymbol{\Omega}$ and *performing a cross product of the infinitesimal rotation vector* $\boldsymbol{\theta}$ with any vector $\mathbf{r} \equiv [r_1, r_2, r_3]^T$ (see Figure 2.16) coincide. Indeed,

$$\boldsymbol{\Omega}\cdot\mathbf{r} \stackrel{not}{\equiv} \begin{bmatrix} 0 & \Omega_{12} & -\Omega_{31} \\ -\Omega_{12} & 0 & \Omega_{23} \\ \Omega_{31} & -\Omega_{23} & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \Omega_{12}\, r_2 - \Omega_{31}\, r_3 \\ -\Omega_{12}\, r_1 + \Omega_{23}\, r_3 \\ \Omega_{31}\, r_1 - \Omega_{23}\, r_2 \end{bmatrix} ,$$

$$\begin{aligned} \boldsymbol{\theta}\times\mathbf{r} \stackrel{not}{\equiv} & \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \theta_1 & \theta_2 & \theta_3 \\ r_1 & r_2 & r_3 \end{vmatrix} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ -\Omega_{23} & -\Omega_{31} & -\Omega_{12} \\ r_1 & r_2 & r_3 \end{vmatrix} = \\ = & \begin{bmatrix} \Omega_{12}\, r_2 - \Omega_{31}\, r_3 \\ -\Omega_{12}\, r_1 + \Omega_{23}\, r_3 \\ \Omega_{31}\, r_1 - \Omega_{23}\, r_2 \end{bmatrix} . \end{aligned}$$

Consequently, vector $\boldsymbol{\Omega}\cdot\mathbf{r} = \boldsymbol{\theta}\times\mathbf{r}$ has the following characteristics:

- It is orthogonal to vector $\mathbf{r}$ (because it is the result of a vector product in which $\mathbf{r}$ is involved).
- Its module is infinitesimal (because $\boldsymbol{\theta}$ is infinitesimal).
- Vector $\mathbf{r} + \boldsymbol{\Omega}\cdot\mathbf{r} = \mathbf{r} + \boldsymbol{\theta}\times\mathbf{r}$ can be considered, except for higher-order infinitesimals, as the result of applying *a rotation* $\boldsymbol{\theta}$ on vector $\mathbf{r}$.

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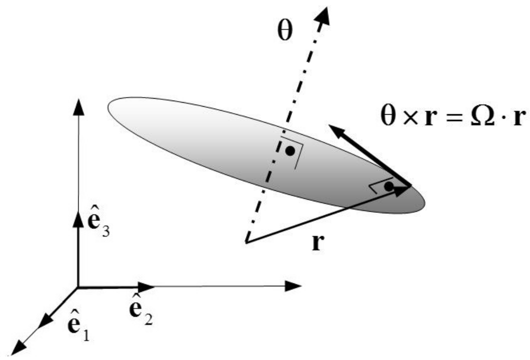

Figure 2.16: Product of the infinitesimal rotation vector and tensor on a vector **r**.

Consider now a differential segment $d\mathbf{X}$ in the neighborhood of a particle P in the reference configuration (see Figure 2.17). In accordance with (2.82), the stretching transforms this vector into vector $d\mathbf{x}$ as follows.

$$
d\mathbf{x} = \mathbf{F}\cdot d\mathbf{X} = (\mathbf{1}+\boldsymbol{\varepsilon}+\boldsymbol{\Omega})\cdot d\mathbf{X} = \overbrace{\boldsymbol{\varepsilon}\cdot d\mathbf{X}}^{\text{stretching}} + \overbrace{(\mathbf{1}+\boldsymbol{\Omega})\cdot d\mathbf{X}}^{\text{rotation}}
$$
$$
\boxed{\mathbf{F}(\bullet) \equiv \text{stretching}(\bullet) + \text{rotation}(\bullet)} \tag{2.83}
$$

Remark 2.21. Under infinitesimal strain hypotheses, the expression in (2.83) characterizes the relative motion of a particle, in the differential neighborhood of this particle, as the following sum:

a) A *stretching* or *deformation in itself*, characterized by the infinitesimal strain tensor $\boldsymbol{\varepsilon}$.

b) A *rotation* characterized by the infinitesimal rotation tensor $\boldsymbol{\Omega}$ which, in the infinitesimal strain context, maintains angles and distances.

The *superposition* (stretching ∘ rotation) of the general finite strain case (see Remark 2.12) degenerates, for the infinitesimal strain case, into a simple *addition* (stretching + rotation).

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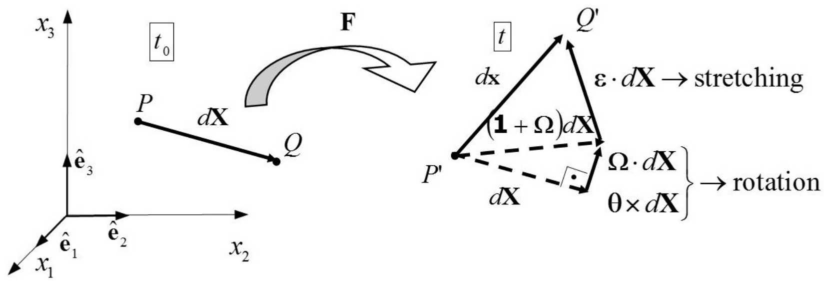

Figure 2.17: Stretching and rotation in infinitesimal strain.

2.12 Volumetric Strain

> **Definition 2.6.** The *volumetric strain* is the increment produced by the deformation of the volume associated with a particle, per unit of volume in the reference configuration.

This definition can be mathematically expressed as (see Figure 2.18)

$$\boxed{\text{Volumetric strain: } e(\mathbf{X},t) \stackrel{def}{=} \frac{dV(\mathbf{X},t) - dV(\mathbf{X},0)}{dV(\mathbf{X},0)} \stackrel{not}{=} \frac{dV_t - dV_0}{dV_0}} \,. \tag{2.84}$$

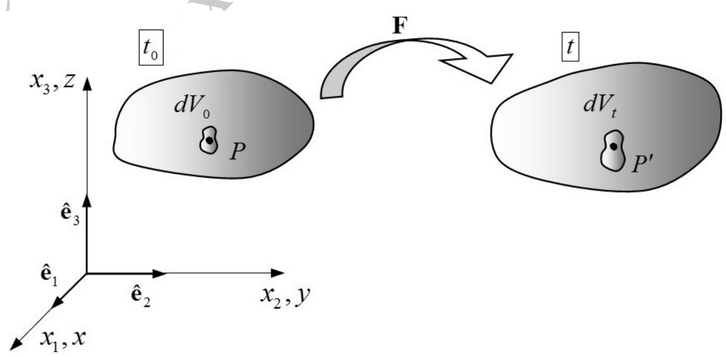

Figure 2.18: Volumetric strain.

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Equation (2.55) allows expressing, in turn, the volumetric strain as follows:

• *Finite strain*

$$e = \frac{dV_t - dV_0}{dV_0} = \frac{|\mathbf{F}|_t \, dV_0 - dV_0}{dV_0} \quad \Longrightarrow \quad \boxed{e = |\mathbf{F}| - 1} \tag{2.85}$$

• *Infinitesimal strain*

Considering (2.49) and recalling that $\mathbf{Q}$ is an orthogonal tensor ($|\mathbf{Q}| = 1$), yields

$$|\mathbf{F}| = |\mathbf{Q} \cdot \mathbf{U}| = |\mathbf{Q}| \, |\mathbf{U}| = |\mathbf{U}| = |\mathbf{1} + \boldsymbol{\varepsilon}| = \det \begin{bmatrix} 1+\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & 1+\varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & 1+\varepsilon_{zz} \end{bmatrix} , \tag{2.86}$$

where (2.77) has been considered. Taking into account that the components of $\boldsymbol{\varepsilon}$ are infinitesimal, and neglecting in the expression of its determinant the second-order and higher-order infinitesimal terms, results in

$$|\mathbf{F}| = \det \begin{bmatrix} 1+\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & 1+\varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & 1+\varepsilon_{zz} \end{bmatrix} = 1 + \underbrace{\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}}_{\mathrm{Tr}(\boldsymbol{\varepsilon})} + O\left(\varepsilon^2\right) \approx 1 + \mathrm{Tr}(\boldsymbol{\varepsilon}) \ . \tag{2.87}$$

Then, introducing (2.87) into (2.85) yields, for the infinitesimal strain case

$$\left. \begin{aligned} & dV_t = (1 + \mathrm{Tr}(\boldsymbol{\varepsilon}))\, dV_0 \\ & e = \frac{dV_t - dV_0}{dV_0} = |\mathbf{F}| - 1 \end{aligned} \right\} \quad \Longrightarrow \quad \boxed{e = \mathrm{Tr}(\boldsymbol{\varepsilon})} \ . \tag{2.88}$$

2.13 Strain Rate

In the previous sections of this chapter, the concept of *strain* has been studied, defined as the variation of the relative position (angles and distances) of the particles in the neighborhood of a given particle. In the following sections, the rate at which this relative position changes will be considered by introducing the concept of *strain rate* as a measure of the variation in the relative position between particles per unit of time.

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2.13.1 Velocity Gradient Tensor

Consider the configuration corresponding to a time t, two particles of the continuous medium P and Q that occupy the spatial points P' and Q' at said instant of time (see Figure 2.19), their velocities $\mathbf{v}_P = \mathbf{v}(\mathbf{x},t)$ and $\mathbf{v}_Q = \mathbf{v}(\mathbf{x}+d\mathbf{x},t)$, and their relative velocity,

$$d\mathbf{v}(\mathbf{x},t) = \mathbf{v}_Q - \mathbf{v}_P = \mathbf{v}(\mathbf{x}+d\mathbf{x},t) - \mathbf{v}(\mathbf{x},t) \ . \tag{2.89}$$

Then,

$$\begin{cases} d\mathbf{v} = \dfrac{\partial \mathbf{v}}{\partial \mathbf{x}} \cdot d\mathbf{x} = \boldsymbol{l} \cdot d\mathbf{x} \\ d\mathrm{v}_i = \dfrac{\partial \mathrm{v}_i}{\partial x_j} dx_j = l_{ij}\, dx_j \qquad i,j \in \{1,2,3\} \end{cases} , \tag{2.90}$$

where the *spatial velocity gradient tensor* $\boldsymbol{l}(\mathbf{x},t)$ has been introduced.

$$\boxed{\text{Spatial velocity gradient tensor} \quad \begin{cases} \boldsymbol{l}(\mathbf{x},t) \stackrel{def}{=} \dfrac{\partial \mathbf{v}(\mathbf{x},t)}{\partial \mathbf{x}} \\ \boldsymbol{l} = \mathbf{v} \otimes \nabla \\ l_{ij} = \dfrac{\partial \mathrm{v}_i}{\partial x_j} \qquad i,j \in \{1,2,3\} \end{cases}} \tag{2.91}$$

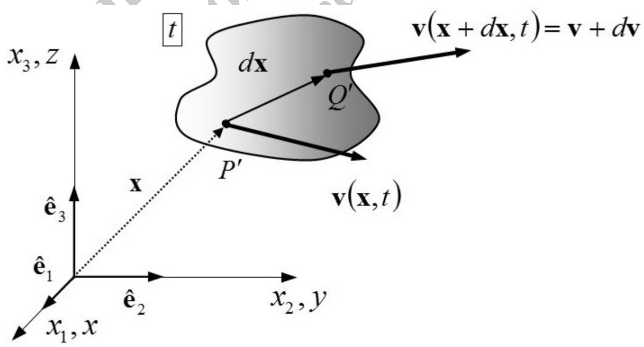

Figure 2.19: Velocities of two particles in the continuous medium.

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2.13.2 Strain Rate and Spin Tensors

The velocity gradient tensor can be split into a symmetric and an antisymmetric part[21],

$$\boldsymbol{l} = \mathbf{d} + \mathbf{w} \ , \tag{2.92}$$

where $\mathbf{d}$ is a symmetric tensor denominated *strain rate tensor*,

$$\text{Strain rate tensor} \begin{cases} \mathbf{d} \stackrel{def}{=} sym\,(\boldsymbol{l}) = \dfrac{1}{2}\left(\boldsymbol{l} + \boldsymbol{l}^T\right) = \dfrac{1}{2}\left(\mathbf{v}\otimes\nabla + \nabla\otimes\mathbf{v}\right) \stackrel{not}{=} \nabla^s \mathbf{v} \\ d_{ij} = \dfrac{1}{2}\left(\dfrac{\partial \mathrm{v}_i}{\partial x_j} + \dfrac{\partial \mathrm{v}_j}{\partial x_i}\right) \qquad i,j \in \{1,2,3\} \\ [\mathbf{d}] = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & d_{23} \\ d_{13} & d_{23} & d_{33} \end{bmatrix} \end{cases} \tag{2.93}$$

and $\mathbf{w}$ is an antisymmetric tensor denominated *rotation rate tensor* or *spin tensor*, whose expression is

$$\text{Rotation rate (spin) tensor} \begin{cases} \mathbf{w} \stackrel{def}{=} skew\,(\boldsymbol{l}) = \dfrac{1}{2}\left(\boldsymbol{l} - \boldsymbol{l}^T\right) = \dfrac{1}{2}\left(\mathbf{v}\otimes\nabla - \nabla\otimes\mathbf{v}\right) \stackrel{not}{=} \nabla^a \mathbf{v} \\ w_{ij} = \dfrac{1}{2}\left(\dfrac{\partial \mathrm{v}_i}{\partial x_j} - \dfrac{\partial \mathrm{v}_j}{\partial x_i}\right) \qquad i,j \in \{1,2,3\} \\ [\mathbf{w}] = \begin{bmatrix} 0 & w_{12} & -w_{31} \\ -w_{12} & 0 & w_{23} \\ w_{31} & -w_{23} & 0 \end{bmatrix} \end{cases} \tag{2.94}$$

2.13.3 Physical Interpretation of the Strain Rate Tensor

Consider a differential segment defined by the particles P and Q of Figure 2.20 and the variation of their squared length along time,

$$\begin{aligned} \frac{d}{dt} ds^2 &= \frac{d}{dt}\,(d\mathbf{x}\cdot d\mathbf{x}) = \frac{d}{dt}\,(d\mathbf{x})\cdot d\mathbf{x} + d\mathbf{x}\cdot\frac{d}{dt}\,(d\mathbf{x}) = \\ &= d\left(\frac{d\mathbf{x}}{dt}\right)\cdot\, d\mathbf{x} + d\mathbf{x}\cdot d\left(\frac{d\mathbf{x}}{dt}\right) = d\mathbf{v}\cdot d\mathbf{x} + d\mathbf{x}\cdot d\mathbf{v} \ , \end{aligned} \tag{2.95}$$

[21] Every second-order tensor $\mathbf{a}$ can be decomposed into the sum of its symmetric part ($sym\,(\mathbf{a})$) and its antisymmetric or skew-symmetric part ($skew\,(\mathbf{a})$) in the form:
$\mathbf{a} = sym\,(\mathbf{a}) + skew\,(\mathbf{a})$ with $sym\,(\mathbf{a}) = (\mathbf{a}+\mathbf{a}^T)/2$ and $skew\,(\mathbf{a}) = (\mathbf{a}-\mathbf{a}^T)/2$.

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and using relations (2.90) and (2.93), the expression

$$\frac{d}{dt}ds^2 = \left(d\mathbf{x}\cdot \boldsymbol{l}^T\right)\cdot d\mathbf{x} + d\mathbf{x}\cdot(\boldsymbol{l}\cdot d\mathbf{x}) = d\mathbf{x}\cdot\left(\boldsymbol{l}^T+\boldsymbol{l}\right)\cdot d\mathbf{x} = 2\,d\mathbf{x}\cdot\mathbf{d}\cdot d\mathbf{x} \tag{2.96}$$

is obtained. Differentiating now (2.20) with respect to time and taking into account (2.96) yields

$$\begin{aligned} 2\,d\mathbf{x}\cdot\mathbf{d}\cdot d\mathbf{x} &= \frac{d}{dt}ds^2(t) = \frac{d}{dt}\left(ds^2(t) - dS^2\right) = \\ &= \frac{d}{dt}\left(2\,d\mathbf{X}\cdot\mathbf{E}(\mathbf{X},t)\cdot d\mathbf{X}\right) = 2\,d\mathbf{X}\cdot\frac{d\mathbf{E}}{dt}\cdot\, d\mathbf{X} = 2\,d\mathbf{X}\cdot\dot{\mathbf{E}}\cdot d\mathbf{X}\,. \end{aligned} \tag{2.97}$$

Replacing (2.2) into (2.97) results in[22]

$$\begin{aligned} & d\mathbf{X}\cdot\dot{\mathbf{E}}\cdot d\mathbf{X} = d\mathbf{x}\cdot\mathbf{d}\cdot d\mathbf{x} \overset{not}{\equiv} [d\mathbf{x}]^T[\mathbf{d}][d\mathbf{x}] = [d\mathbf{X}]^T\left[\mathbf{F}^T\cdot\mathbf{d}\cdot\mathbf{F}\right][d\mathbf{X}] \\ & \Longrightarrow \quad d\mathbf{X}\cdot\left(\mathbf{F}^T\cdot\mathbf{d}\cdot\mathbf{F}-\dot{\mathbf{E}}\right)\cdot d\mathbf{X} = 0 \quad \forall\, d\mathbf{X} \quad \Longrightarrow \quad \mathbf{F}^T\cdot\mathbf{d}\cdot\mathbf{F}-\dot{\mathbf{E}} = \mathbf{0} \\ & \boxed{\dot{\mathbf{E}} = \mathbf{F}^T\cdot\mathbf{d}\cdot\mathbf{F}}\,. \end{aligned} \tag{2.98}$$

Remark 2.22. Equation (2.98) shows the existing relationship between the strain rate tensor $\mathbf{d}(\mathbf{x},t)$ and the material derivative of the material strain tensor $\dot{\mathbf{E}}(\mathbf{X},t)$, providing a physical interpretation (and justifying the denomination) of tensor $\mathbf{d}(\mathbf{x},t)$. However, the same equation reveals that tensors $\mathbf{d}(\mathbf{x},t)$ and $\dot{\mathbf{E}}(\mathbf{X},t)$ are not exactly the same. Both tensors will coincide in the following cases:

- In the reference configuration: $t = t_0 \;\Rightarrow\; \mathbf{F}\big|_{t=t_0} = \mathbf{1}$.
- In infinitesimal strain theory: $\mathbf{x}\approx\mathbf{X} \;\Rightarrow\; \mathbf{F} = \dfrac{\partial\mathbf{x}}{\partial\mathbf{X}} \approx \mathbf{1}$.

[22] Here, the following tensor algebra theorem is used: given a second-order tensor $\mathbf{A}$, if $\mathbf{x}\cdot\mathbf{A}\cdot\mathbf{x} = 0$ is verified for all vectors $\mathbf{x}\neq\mathbf{0}$, then $\mathbf{A}\equiv\mathbf{0}$.

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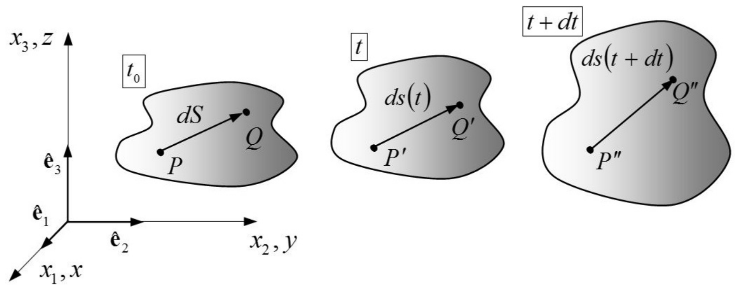

Figure 2.20: Differential segment between particles of the continuous medium along time.

2.13.4 Physical Interpretation of the Rotation Rate Tensor

Taking into account the antisymmetric character of $\mathbf{w}$ (which implies it can be defined using only three different components), the vector

$$\boldsymbol{\omega} = \frac{1}{2} rot(\mathbf{v}) = \frac{1}{2} \nabla \times \mathbf{v} \overset{not}{\equiv} \frac{1}{2} \begin{bmatrix} -\left(\dfrac{\partial v_2}{\partial x_3} - \dfrac{\partial v_3}{\partial x_2}\right) \\ -\left(\dfrac{\partial v_3}{\partial x_1} - \dfrac{\partial v_1}{\partial x_3}\right) \\ -\left(\dfrac{\partial v_1}{\partial x_2} - \dfrac{\partial v_2}{\partial x_1}\right) \end{bmatrix} = \begin{bmatrix} -w_{23} \\ -w_{31} \\ -w_{12} \end{bmatrix} \tag{2.99}$$

is extracted from (2.94). Vector $2\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is named *vorticiy vector* [23]. It can be proven (in an analogous manner to Remark 2.20) that the equality

$$\boldsymbol{\omega} \times \mathbf{r} = \mathbf{w} \cdot \mathbf{r} \qquad \forall\, \mathbf{r} \tag{2.100}$$

is satisfied. Therefore, it is possible to characterize $\boldsymbol{\omega}$ as the angular velocity of a rotation motion, and $\boldsymbol{\omega} \times \mathbf{r} = \mathbf{w} \cdot \mathbf{r}$ as the rotation velocity of the point that has $\mathbf{r}$ as the position vector with respect to the rotation center (see Figure 2.21). Then, considering (2.90) and (2.92),

$$d\mathbf{v} = \boldsymbol{l} \cdot d\mathbf{x} = (\mathbf{d} + \mathbf{w}) \cdot d\mathbf{x} = \underbrace{\mathbf{d} \cdot d\mathbf{x}}_{\substack{\text{stretch}\\\text{velocity}}} + \underbrace{\mathbf{w} \cdot d\mathbf{x}}_{\substack{\text{rotation}\\\text{velocity}}} \;, \tag{2.101}$$

which allows describing the relative velocity $d\mathbf{v}$ of the particles in the neighborhood of a given particle P (see Figure 2.22) as the *sum* of a *relative stretch*

[23] Observe the similarity in the structure of tensors $\boldsymbol{\Omega}$ and $\boldsymbol{\theta}$ in Section 2.11.6 and of tensors $\mathbf{w}$ and $\boldsymbol{\omega}$ seen here.

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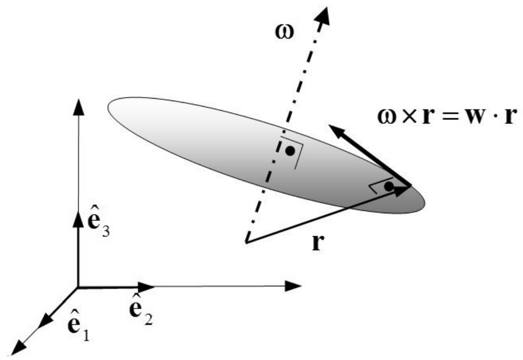

Figure 2.21: Vorticity vector.

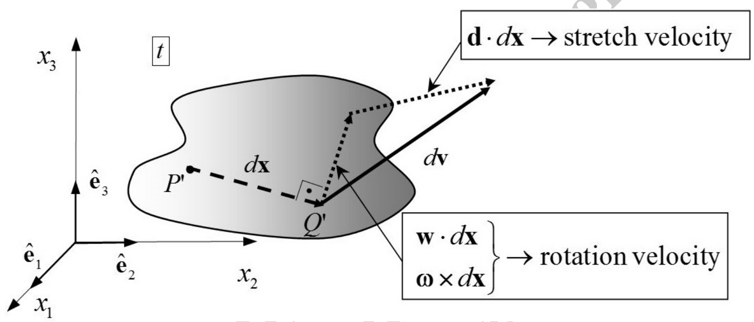

Figure 2.22: Stretch and rotation velocities.

velocity (characterized by the strain rate tensor $\mathbf{d}$) and a *relative rotation velocity* (characterized by the spin tensor $\mathbf{w}$ or the vorticity vector $2\boldsymbol{\omega}$).

2.14 Material Time Derivatives of Strain and Other Magnitude Tensors

2.14.1 Deformation Gradient Tensor and its Inverse Tensor

Differentiating the expression of $\mathbf{F}$ in (2.3) with respect to time[24],

$$F_{ij} = \frac{\partial x_i(\mathbf{X},t)}{\partial X_j} \quad \Longrightarrow \quad \frac{dF_{ij}}{dt} = \frac{\partial}{\partial t}\frac{\partial x_i(\mathbf{X},t)}{\partial X_j} = \frac{\partial}{\partial X_j}\underbrace{\frac{\partial x_i(\mathbf{X},t)}{\partial t}}_{\mathrm{v}_i} \quad \Longrightarrow \tag{2.102}$$

[24] The Schwartz Theorem (equality of mixed partial derivatives) guarantees that for a function $\Phi(x_1, x_2 \dots x_n)$ that is continuous and has continuous derivatives, $\partial^2\Phi/(\partial x_i \partial x_j) = \partial^2\Phi/(\partial x_j \partial x_i)\ \ \forall i,j$ is satisfied.

$$\frac{dF_{ij}}{dt}=\frac{\partial \mathrm{v}_i(\mathbf{X},t)}{\partial X_j}=\underbrace{\frac{\partial \mathrm{v}_i(\mathbf{x}(\mathbf{X},t))}{\partial x_k}}_{l_{ik}}\underbrace{\frac{\partial x_k}{\partial X_j}}_{F_{kj}}=l_{ik}\,F_{kj} \quad\Longrightarrow$$

$$\boxed{\begin{array}{l}\dfrac{d\mathbf{F}}{dt}\overset{not}{=}\dot{\mathbf{F}}=\boldsymbol{l}\cdot\mathbf{F}\\[2mm] \dfrac{dF_{ij}}{dt}=\dot{F}_{ij}=l_{ik}\,F_{kj} \qquad i,j\in\{1,2,3\}\end{array}} \tag{2.102 (cont.)}$$

where (2.91) has been taken into account for the velocity gradient tensor $\boldsymbol{l}$. To obtain the material time derivative of tensor $\mathbf{F}^{-1}$, the time derivative of the identity $\mathbf{F}\cdot\mathbf{F}^{-1}=\mathbf{1}$ is performed[25].

$$\mathbf{F}\cdot\mathbf{F}^{-1}=\mathbf{1} \quad\Longrightarrow\quad \frac{d}{dt}\left(\mathbf{F}\cdot\mathbf{F}^{-1}\right)=\frac{d\mathbf{F}}{dt}\cdot\mathbf{F}^{-1}+\mathbf{F}\cdot\frac{d\left(\mathbf{F}^{-1}\right)}{dt}=\mathbf{0}$$

$$\Longrightarrow\quad \frac{d\left(\mathbf{F}^{-1}\right)}{dt}=-\mathbf{F}^{-1}\cdot\underbrace{\dot{\mathbf{F}}}_{\boldsymbol{l}\cdot\mathbf{F}}\cdot\mathbf{F}^{-1}=-\mathbf{F}^{-1}\cdot\boldsymbol{l}\cdot\underbrace{\mathbf{F}\cdot\mathbf{F}^{-1}}_{\mathbf{1}}=-\mathbf{F}^{-1}\cdot\boldsymbol{l} \quad\Longrightarrow$$

$$\boxed{\begin{array}{l}\dfrac{d\left(\mathbf{F}^{-1}\right)}{dt}=-\mathbf{F}^{-1}\cdot\boldsymbol{l}\\[2mm] \dfrac{dF^{-1}_{ij}}{dt}=-F^{-1}_{ik}\,l_{kj} \qquad i,j\in\{1,2,3\}\end{array}} \tag{2.103}$$

2.14.2 Material and Spatial Strain Tensors

From (2.21), (2.102) and (2.93), it follows[26]

$$\mathbf{E}=\frac{1}{2}\left(\mathbf{F}^T\cdot\mathbf{F}-\mathbf{1}\right) \Longrightarrow \frac{d\mathbf{E}}{dt}=\dot{\mathbf{E}}=\frac{1}{2}\left(\dot{\mathbf{F}}^T\cdot\mathbf{F}+\mathbf{F}^T\cdot\dot{\mathbf{F}}\right)=$$

$$=\frac{1}{2}\left(\mathbf{F}^T\cdot\boldsymbol{l}^T\cdot\mathbf{F}+\mathbf{F}^T\cdot\boldsymbol{l}\cdot\mathbf{F}\right)=\frac{1}{2}\,\mathbf{F}^T\cdot\underbrace{\left(\boldsymbol{l}+\boldsymbol{l}^T\right)}_{2\mathbf{d}}\cdot\mathbf{F}=\mathbf{F}^T\cdot\mathbf{d}\cdot\mathbf{F}$$

$$\Longrightarrow\quad \dot{\mathbf{E}}=\mathbf{F}^T\cdot\mathbf{d}\cdot\mathbf{F}\,. \tag{2.104}$$

[25] The material time derivative of the inverse tensor $d\left(\mathbf{F}^{-1}\right)/dt$ must not be confused with the inverse of the material derivative of the tensor: $\left(\dot{\mathbf{F}}\right)^{-1}$. These two tensors are completely different tensors.

[26] Observe that the result is the same as the one obtained in (2.98) using an alternative procedure.

Using (2.23) and (2.103) for the spatial strain tensor $\mathbf{e}$ yields

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{1} - \mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\right) \Rightarrow \frac{d\mathbf{e}}{dt} = \dot{\mathbf{e}} = -\frac{1}{2}\left(\frac{d}{dt}\left(\mathbf{F}^{-T}\right)\cdot\mathbf{F}^{-1} + \mathbf{F}^{-T}\cdot\frac{d}{dt}\left(\mathbf{F}^{-1}\right)\right) =$$

$$= \frac{1}{2}\left(\boldsymbol{l}^T\cdot\mathbf{F}^{-T}\cdot\mathbf{F}^{-1} + \mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\cdot\boldsymbol{l}\right)$$

$$\Longrightarrow \quad \dot{\mathbf{e}} = \frac{1}{2}\left(\boldsymbol{l}^T\cdot\mathbf{F}^{-T}\cdot\mathbf{F}^{-1} + \mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\cdot\boldsymbol{l}\right) . \tag{2.105}$$

2.14.3 Volume and Area Differentials

The volume differential $dV(\mathbf{X},t)$ associated with a certain particle P varies along time (see Figure 2.23) and, in consequence, it makes sense to calculate its material derivative. Differentiating (2.55) for a volume differential results in

$$dV(\mathbf{X},t) = |\mathbf{F}(\mathbf{X},t)|\; dV_0(\mathbf{X}) \quad \Longrightarrow \quad \frac{d}{dt}dV(t) = \frac{d|\mathbf{F}|}{dt}dV_0 . \tag{2.106}$$

Therefore, the material derivative of the determinant of the deformation gradient tensor $|\mathbf{F}|$ is[27]

$$\frac{d|\mathbf{F}|}{dt} = \frac{d|\mathbf{F}|}{dF_{ij}}\frac{dF_{ij}}{dt} = |\mathbf{F}|\,F_{ji}^{-1}\underbrace{\frac{dF_{ij}}{dt}}_{l_{ik}F_{kj}} = |\mathbf{F}|\,F_{ji}^{-1}l_{ik}F_{kj} = |\mathbf{F}|\underbrace{F_{kj}F_{ji}^{-1}}_{\left[\mathbf{F}\cdot\mathbf{F}^{-1}\right]_{ki}=\delta_{ki}} l_{ik} =$$

$$= |\mathbf{F}|\,\delta_{ki}\,l_{ik} = |\mathbf{F}|\,l_{ii} = |\mathbf{F}|\frac{\partial v_i}{\partial x_i} = |\mathbf{F}|\,\nabla\cdot\mathbf{v} \quad \Longrightarrow \quad \boxed{\frac{d|\mathbf{F}|}{dt} = |\mathbf{F}|\,\nabla\cdot\mathbf{v}}\,, \tag{2.107}$$

where (2.102) and (2.91) have been considered. Introducing (2.107) into (2.106) and taking into account (2.55) finally results in

$$\frac{d}{dt}(dV) = (\nabla\cdot\mathbf{v})\,|\mathbf{F}|\,dV_0 = (\nabla\cdot\mathbf{v})\,dV . \tag{2.108}$$

Operating in a similar manner yields the material derivative of the area differential associated with a certain particle P and a given direction $\mathbf{n}$ (see Figure 2.24). The area differential vector associated with a particle in the reference configuration, $d\mathbf{A}(\mathbf{X}) = dA\,\mathbf{N}$, and in the present configuration, $d\mathbf{a}(\mathbf{x},t) = da\,\mathbf{n}$, are related through $d\mathbf{a} = |\mathbf{F}|\cdot d\mathbf{A}\cdot\mathbf{F}^{-1}$ (see (2.59)) and, differentiating this ex-

[27] The derivative of the determinant of a tensor $\mathbf{A}$ with respect to the same tensor can be written in compact notation as $d|\mathbf{A}|/d\mathbf{A} = |\mathbf{A}|\cdot\mathbf{A}^{-T}$ or, in index notation, as $d|\mathbf{A}|/dA_{ij} = |\mathbf{A}|\cdot A_{ji}^{-1}$.

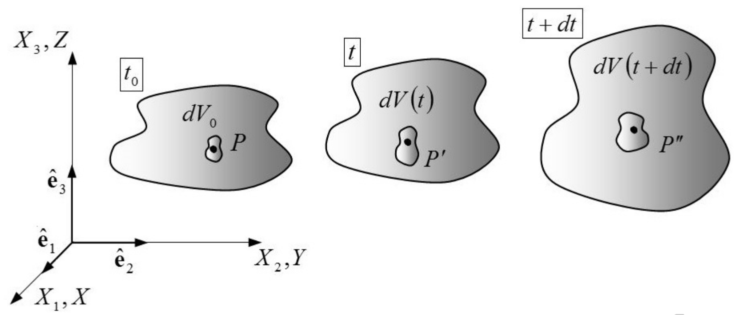

Figure 2.23: Variation of the volume differential.

pression, results in

$$\frac{d}{dt}(d\mathbf{a}) = \frac{d}{dt}\left(|\mathbf{F}| \cdot d\mathbf{A} \cdot \mathbf{F}^{-1}\right) = \underbrace{\frac{d|\mathbf{F}|}{dt}}_{|\mathbf{F}|\nabla\cdot\mathbf{v}} d\mathbf{A} \cdot \mathbf{F}^{-1} + |\mathbf{F}| \cdot d\mathbf{A} \underbrace{\frac{d}{dt}\left(\mathbf{F}^{-1}\right)}_{-\mathbf{F}^{-1}\cdot \boldsymbol{l}} =$$

$$= (\nabla\cdot\mathbf{v}) \underbrace{|\mathbf{F}|\, d\mathbf{A} \cdot \mathbf{F}^{-1}}_{d\mathbf{a}} - \underbrace{|\mathbf{F}|\, d\mathbf{A} \cdot \mathbf{F}^{-1}}_{d\mathbf{a}} \cdot \boldsymbol{l} \quad \Longrightarrow$$

$$\boxed{\frac{d}{dt}(d\mathbf{a}) = (\nabla\cdot\mathbf{v})\, d\mathbf{a} - d\mathbf{a}\cdot\boldsymbol{l} = d\mathbf{a}\cdot\left((\nabla\cdot\mathbf{v})\mathbf{1} - \boldsymbol{l}\right)}\,, \tag{2.109}$$

where (2.103) and (2.107) have been considered.

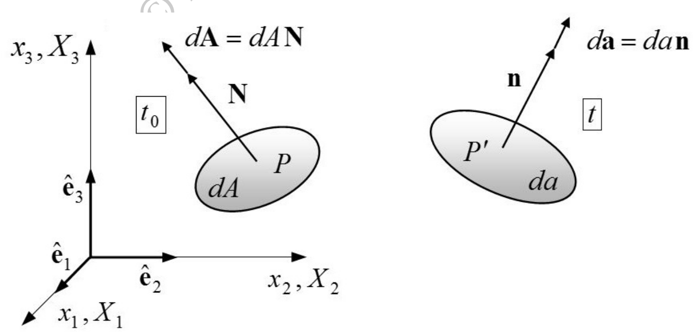

Figure 2.24: Variation of the area differential.

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2.15 Motion and Strains in Cylindrical and Spherical Coordinates

The expressions and equations obtained in intrinsic or compact notation are independent of the coordinate system considered. However, the expressions of the components depend on the coordinate system used. In addition to the *Cartesian coordinate system*, which has been used in the previous sections, two orthogonal curvilinear coordinate systems will be considered here: *cylindrical coordinates* and *spherical coordinates*.

Remark 2.23. An orthogonal curvilinear coordinate system (generically referred to as $\{a,b,c\}$), is characterized by its physical unit basis $\{\hat{\mathbf{e}}_a,\hat{\mathbf{e}}_b,\hat{\mathbf{e}}_c\}$ ($\|\hat{\mathbf{e}}_a\| = \|\hat{\mathbf{e}}_b\| = \|\hat{\mathbf{e}}_c\| = 1$), whose components are orthogonal to each other ($\hat{\mathbf{e}}_a \cdot \hat{\mathbf{e}}_b = \hat{\mathbf{e}}_a \cdot \hat{\mathbf{e}}_c = \hat{\mathbf{e}}_b \cdot \hat{\mathbf{e}}_c = 0$), as is also the case in a Cartesian system. The fundamental difference is that the orientation of the curvilinear basis changes at each point in space ($\hat{\mathbf{e}}_m \equiv \hat{\mathbf{e}}_m(\mathbf{x}) \quad m \in \{a,b,c\}$). Therefore, for the purposes here, an orthogonal curvilinear coordinate system can be considered as a *mobile Cartesian coordinate system* $\{x',y',z'\}$ associated with a curvilinear basis $\{\hat{\mathbf{e}}_a,\hat{\mathbf{e}}_b,\hat{\mathbf{e}}_c\}$ (see Figure 2.25).

Remark 2.24. The components, of a certain magnitude of vectorial character ($\mathbf{v}$) or tensorial character ($\mathbf{T}$) in an orthogonal curvilinear coordinate system $\{a,b,c\}$, can be obtained as the *corresponding components in the local Cartesian system* $\{x',y',z'\}$:

$$\mathbf{v} \stackrel{not}{\equiv} \begin{bmatrix} \mathrm{v}_a \\ \mathrm{v}_b \\ \mathrm{v}_c \end{bmatrix} \equiv \begin{bmatrix} \mathrm{v}_{x'} \\ \mathrm{v}_{y'} \\ \mathrm{v}_{z'} \end{bmatrix} \qquad \mathbf{T} \stackrel{not}{\equiv} \begin{bmatrix} T_{aa} & T_{ab} & T_{ac} \\ T_{ba} & T_{bb} & T_{bc} \\ T_{ca} & T_{cb} & T_{cc} \end{bmatrix} \equiv \begin{bmatrix} T_{x'x'} & T_{x'y'} & T_{x'z'} \\ T_{y'x'} & T_{y'y'} & T_{y'z'} \\ T_{z'x'} & T_{z'y'} & T_{z'z'} \end{bmatrix}$$

Remark 2.25. The curvilinear components of the differential operators (the ∇ operator and its derivatives) *are not the same* as their counterparts in the local coordinate system $\{x',y',z'\}$. They *must be defined specifically for each case*. Their value for cylindrical and spherical coordinates is provided in the corresponding section.

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2.15.1 Cylindrical Coordinates

The position of a certain point in space can be defined by its cylindrical coordinates $\{r, \theta, z\}$ (see Figure 2.25). The figure also shows the physical orthonormal basis $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z\}$. This basis changes at each point in space according to

$$\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \hat{\mathbf{e}}_\theta \qquad \text{and} \qquad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r \ . \tag{2.110}$$

Figure 2.26 shows the corresponding differential element. The expressions in cylindrical coordinates of some of the elements treated in this chapter are:

- **Nabla operator, ∇**

$$\nabla = \frac{\partial}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{\partial}{\partial z}\hat{\mathbf{e}}_z \quad \Longrightarrow \quad \nabla \overset{not}{\equiv} \left[\frac{\partial}{\partial r}, \ \frac{1}{r}\frac{\partial}{\partial \theta}, \ \frac{\partial}{\partial z}\right]^T \tag{2.111}$$

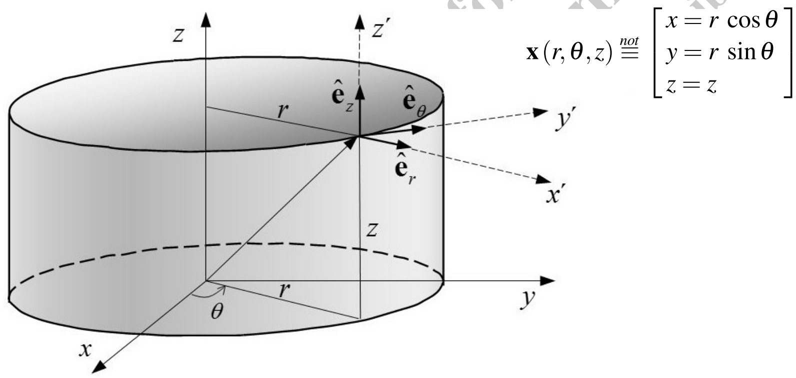

Figure 2.25: Cylindrical coordinates.

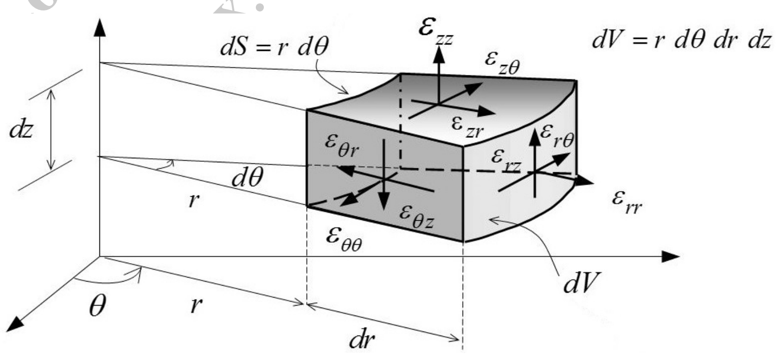

Figure 2.26: Differential element in cylindrical coordinates.

• Displacement vector, u, and velocity vector, v

$$\mathbf{u} = u_r \hat{\mathbf{e}}_r + u_\theta \hat{\mathbf{e}}_\theta + u_z \hat{\mathbf{e}}_z \quad \Longrightarrow \quad \mathbf{u} \stackrel{not}{\equiv} [u_r\ ,\ u_\theta\ ,\ u_z]^T \tag{2.112}$$

$$\mathbf{v} = \mathrm{v}_r \hat{\mathbf{e}}_r + \mathrm{v}_\theta \hat{\mathbf{e}}_\theta + \mathrm{v}_z \hat{\mathbf{e}}_z \quad \Longrightarrow \quad \mathbf{v} \stackrel{not}{\equiv} [\mathrm{v}_r\ ,\ \mathrm{v}_\theta\ ,\ \mathrm{v}_z]^T \tag{2.113}$$

• Infinitesimal strain tensor, $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left((\mathbf{u}\otimes\nabla) + (\mathbf{u}\otimes\nabla)^T\right) \stackrel{not}{\equiv} \begin{bmatrix} \varepsilon_{x'x'} & \varepsilon_{x'y'} & \varepsilon_{x'z'} \\ \varepsilon_{x'y'} & \varepsilon_{y'y'} & \varepsilon_{y'z'} \\ \varepsilon_{x'z'} & \varepsilon_{y'z'} & \varepsilon_{z'z'} \end{bmatrix} = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\ \varepsilon_{rz} & \varepsilon_{\theta z} & \varepsilon_{zz} \end{bmatrix}$$

$$\begin{aligned}
&\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \\
&\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \qquad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\
&\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)
\end{aligned} \tag{2.114}$$

The components of $\boldsymbol{\varepsilon}$ are presented on the corresponding differential element in Figure (2.26).

• Strain rate tensor, d

$$\mathbf{d} = \frac{1}{2}\left((\mathbf{v}\otimes\nabla) + (\mathbf{v}\otimes\nabla)^T\right) \stackrel{not}{\equiv} \begin{bmatrix} d_{x'x'} & d_{x'y'} & d_{x'z'} \\ d_{x'y'} & d_{y'y'} & d_{y'z'} \\ d_{x'z'} & d_{y'z'} & d_{z'z'} \end{bmatrix} = \begin{bmatrix} d_{rr} & d_{r\theta} & d_{rz} \\ d_{r\theta} & d_{\theta\theta} & d_{\theta z} \\ d_{rz} & d_{\theta z} & d_{zz} \end{bmatrix}$$

$$\begin{aligned}
&d_{rr} = \frac{\partial \mathrm{v}_r}{\partial r} \qquad d_{\theta\theta} = \frac{1}{r}\frac{\partial \mathrm{v}_\theta}{\partial \theta} + \frac{\mathrm{v}_r}{r} \qquad d_{zz} = \frac{\partial \mathrm{v}_z}{\partial z} \\
&d_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial \mathrm{v}_r}{\partial \theta} + \frac{\partial \mathrm{v}_\theta}{\partial r} - \frac{\mathrm{v}_\theta}{r}\right) \qquad d_{rz} = \frac{1}{2}\left(\frac{\partial \mathrm{v}_r}{\partial z} + \frac{\partial \mathrm{v}_z}{\partial r}\right) \\
&d_{\theta z} = \frac{1}{2}\left(\frac{\partial \mathrm{v}_\theta}{\partial z} + \frac{1}{r}\frac{\partial \mathrm{v}_z}{\partial \theta}\right)
\end{aligned} \tag{2.115}$$

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2.15.2 Spherical Coordinates

A point in space is defined by its spherical coordinates $\{r, \theta, \phi\}$. The physical orthonormal basis $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi\}$ is presented in Figure 2.27. This basis changes at each point in space according to

$$\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \hat{\mathbf{e}}_\theta \ , \qquad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = -\hat{\mathbf{e}}_r \qquad \text{and} \qquad \frac{\partial \hat{\mathbf{e}}_\phi}{\partial \theta} = \mathbf{0} \ . \tag{2.116}$$

The expressions in spherical coordinates of some of the elements treated in this chapter are:

• **Nabla operator, ∇**

$$\nabla = \frac{\partial}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\mathbf{e}}_\phi \quad \Longrightarrow \quad \nabla \overset{not}{\equiv} \left[\frac{\partial}{\partial r} \ , \ \frac{1}{r}\frac{\partial}{\partial \theta} \ , \ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right]^T \tag{2.117}$$

• **Displacement vector, u, and velocity vector, v**

$$\mathbf{u} = u_r\hat{\mathbf{e}}_r + u_\theta\hat{\mathbf{e}}_\theta + u_\phi\hat{\mathbf{e}}_\phi \quad \Longrightarrow \quad \mathbf{u} \overset{not}{\equiv} \left[u_r \ , \ u_\theta \ , \ u_\phi\right]^T \tag{2.118}$$

$$\mathbf{v} = \mathrm{v}_r\hat{\mathbf{e}}_r + \mathrm{v}_\theta\hat{\mathbf{e}}_\theta + \mathrm{v}_\phi\hat{\mathbf{e}}_\phi \quad \Longrightarrow \quad \mathbf{v} \overset{not}{\equiv} \left[\mathrm{v}_r \ , \ \mathrm{v}_\theta \ , \ \mathrm{v}_\phi\right]^T \tag{2.119}$$

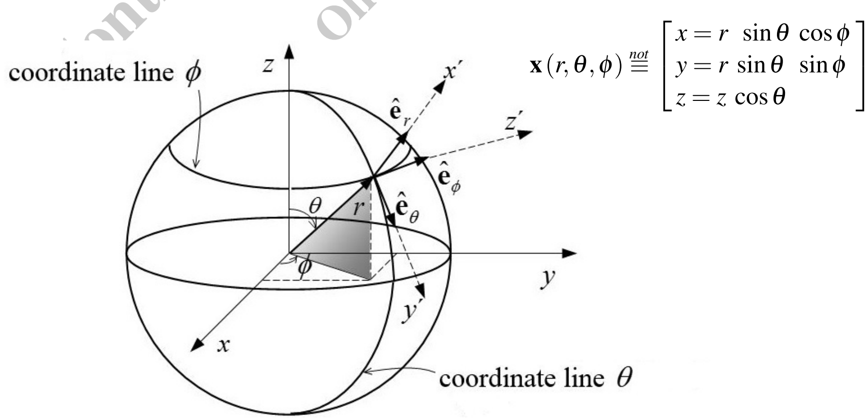

Figure 2.27: Spherical coordinates.

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• Infinitesimal strain tensor, $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left((\mathbf{u}\otimes\nabla)+(\mathbf{u}\otimes\nabla)^T\right) \overset{not}{\equiv} \begin{bmatrix} \varepsilon_{x'x'} & \varepsilon_{x'y'} & \varepsilon_{x'z'} \\ \varepsilon_{x'y'} & \varepsilon_{y'y'} & \varepsilon_{y'z'} \\ \varepsilon_{x'z'} & \varepsilon_{y'z'} & \varepsilon_{z'z'} \end{bmatrix} = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{r\phi} \\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} & \varepsilon_{\theta\phi} \\ \varepsilon_{r\phi} & \varepsilon_{\theta\phi} & \varepsilon_{\phi\phi} \end{bmatrix}$$

$$\begin{aligned}
&\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \\
&\varepsilon_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta}{r}\cot\phi + \frac{u_r}{r} \\
&\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \qquad \varepsilon_{r\phi} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right) \\
&\varepsilon_{\theta\phi} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi}{r}\cot\phi\right)
\end{aligned} \tag{2.120}$$

The components of $\boldsymbol{\varepsilon}$ are presented on the corresponding differential element in Figure 2.28.

• Strain rate tensor, d

$$\mathbf{d} = \frac{1}{2}\left((\mathbf{v}\otimes\nabla)+(\mathbf{v}\otimes\nabla)^T\right) \overset{not}{\equiv} \begin{bmatrix} d_{x'x'} & d_{x'y'} & d_{x'z'} \\ d_{x'y'} & d_{y'y'} & d_{y'z'} \\ d_{x'z'} & d_{y'z'} & d_{z'z'} \end{bmatrix} = \begin{bmatrix} d_{rr} & d_{r\theta} & d_{r\phi} \\ d_{r\theta} & d_{\theta\theta} & d_{\theta\phi} \\ d_{r\phi} & d_{\theta\phi} & d_{\phi\phi} \end{bmatrix}$$

$$\begin{aligned}
&d_{rr} = \frac{\partial \mathrm{v}_r}{\partial r} \qquad d_{\theta\theta} = \frac{1}{r}\frac{\partial \mathrm{v}_\theta}{\partial \theta} + \frac{\mathrm{v}_r}{r} \\
&d_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial \mathrm{v}_\phi}{\partial \phi} + \frac{\mathrm{v}_\theta}{r}\cot\phi + \frac{\mathrm{v}_r}{r} \\
&d_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial \mathrm{v}_r}{\partial \theta} + \frac{\partial \mathrm{v}_\theta}{\partial r} - \frac{\mathrm{v}_\theta}{r}\right) \qquad d_{r\phi} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial \mathrm{v}_r}{\partial \phi} + \frac{\partial \mathrm{v}_\phi}{\partial r} - \frac{\mathrm{v}_\phi}{r}\right) \\
&d_{\theta\phi} = \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial \mathrm{v}_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial \mathrm{v}_\phi}{\partial \theta} - \frac{\mathrm{v}_\phi}{r}\cot\phi\right)
\end{aligned} \tag{2.121}$$

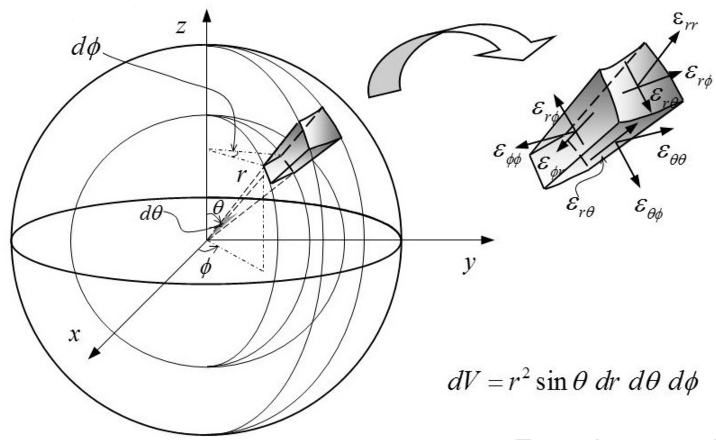


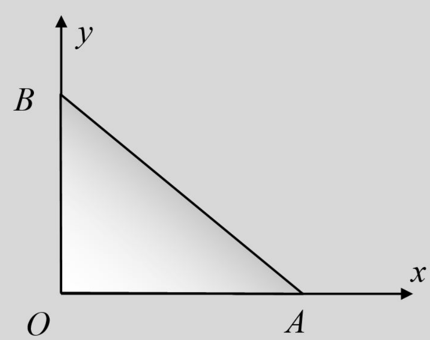
Figure 2.28: Differential element in spherical coordinates.

PROBLEMS

Problem 2.1 – *A deformation that takes place in a continuous medium has the following consequences on the triangle shown in the figure below:*

1. *The segment* $\overline{OA}$ *increases its initial length in* $(1+p)$.
2. *The angle AOB decreases in q radians its initial value.*
3. *The area increases its initial value in* $(1+r)$.
4. $p,q,r,s \ll 1$.

The deformation is uniform and the z-axis is one of the principal directions of the deformation gradient tensor, which is symmetric. In addition, the stretch in this direction is known to be $\lambda_z = 1+s$. *Obtain the infinitesimal strain tensor.*



Solution

A uniform deformation implies that the *deformation gradient tensor* $(\mathbf{F})$ does not depend on the spatial variables. Consequently, the *strain tensor* $(\mathbf{E})$ and the stretches (λ) do not depend on them either. Also, note that the problem is to be solved under infinitesimal strain theory.

The initial and final lengths of a segment parallel to the x-axis are related as follows.

$$\left.\begin{aligned}
&\overline{OA}_{final} = \int_O^A \lambda_x \, dX = \lambda_x \int_O^A dX = \lambda_x \overline{OA}_{initial} \\
&\overline{OA}_{final} = (1+p)\,\overline{OA}_{initial}
\end{aligned}\right\} \Longrightarrow \quad \lambda_x = 1+p$$

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Also, an initial right angle (the angle between the x- and y-axes) is related to its corresponding final angle after the deformation through

$$\left.\begin{array}{l} \text{initial angle} = \dfrac{\pi}{2} \\[2ex] \text{final angle} = \dfrac{\pi}{2} + \Delta\Phi_{xy} \end{array}\right\} \Longrightarrow \Delta\Phi_{xy} = -\gamma_{xy} = -2\varepsilon_{xy} = -q \Longrightarrow \varepsilon_{xy} = \frac{q}{2} \,.$$

In addition, $\mathbf{F}$ is symmetric and the z-axis is a principal direction, therefore

$$\mathbf{F} \stackrel{not}{\equiv} \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{12} & F_{22} & 0 \\ 0 & 0 & F_{33} \end{bmatrix} \stackrel{not}{\equiv} \mathbf{1} + \mathbf{J} \stackrel{not}{\equiv} \begin{bmatrix} 1 + \dfrac{\partial u_x}{\partial x} & \dfrac{\partial u_x}{\partial y} & \dfrac{\partial u_x}{\partial z} \\ \dfrac{\partial u_y}{\partial x} & 1 + \dfrac{\partial u_y}{\partial y} & \dfrac{\partial u_y}{\partial z} \\ \dfrac{\partial u_z}{\partial x} & \dfrac{\partial u_z}{\partial y} & 1 + \dfrac{\partial u_z}{\partial z} \end{bmatrix} ,$$

which reveals the nature of the components of the displacement vector,

$$\left\{\begin{array}{lll} \dfrac{\partial u_x}{\partial z} = \dfrac{\partial u_y}{\partial z} = 0 & \Longrightarrow & \left\{\begin{array}{l} u_x(x,y) \,, \\ u_y(x,y) \,, \end{array}\right. \\[2ex] \dfrac{\partial u_z}{\partial x} = \dfrac{\partial u_z}{\partial y} = 0 & \Longrightarrow & u_z(z) \,. \end{array}\right.$$

Then, the following components of the strain tensor can be computed.

$$\begin{array}{lll} \varepsilon_{xz} = \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial z} + \dfrac{\partial u_z}{\partial x}\right) = 0 & \Longrightarrow & \varepsilon_{xz} = 0 \\[2ex] \varepsilon_{xz} = \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial z} + \dfrac{\partial u_z}{\partial x}\right) = 0 & \Longrightarrow & \varepsilon_{xz} = 0 \\[2ex] \left.\begin{array}{l} \varepsilon_{zz} = \dfrac{\partial u_z}{\partial z} = \lambda_z - 1 \\ \lambda_z = 1 + s \end{array}\right\} & \Longrightarrow & \varepsilon_{zz} = s \end{array}$$

In infinitesimal strain theory, $\mathbf{F} = \mathbf{1} + \boldsymbol{\varepsilon} + \boldsymbol{\Omega}$, where $\Omega_{33} = 0$ since the infinitesimal rotation tensor is antisymmetric. Thus, $F_{zz} = 1 + \varepsilon_{zz}$ results in $F_{zz} = 1 + s$. Now, the relation between the initial and final areas is $d\mathbf{A} = |\mathbf{F}|\, d\mathbf{A}_0 \cdot \mathbf{F}^{-1}$, where the inverse tensor of $\mathbf{F}$ is calculated using the notation

$$\mathbf{F} \stackrel{not}{\equiv} \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & 1+s \end{bmatrix} \quad \text{with} \quad \mathbf{B}^{-1} \stackrel{not}{\equiv} \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix} ,$$

which yields the inverse tensor of $\mathbf{F}$,

$$\mathbf{F}^{-1} \stackrel{not}{\equiv} \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & \dfrac{1}{1+s} \end{bmatrix} .$$

The area differential vector is defined as

$$d\mathbf{A}_0 \stackrel{not}{\equiv} \begin{bmatrix} 0 \\ 0 \\ dA_0 \end{bmatrix} \quad \Longrightarrow \quad d\mathbf{A}_0 \cdot \mathbf{F}^{-1} \stackrel{not}{\equiv} \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{1+s} dA_0 \end{bmatrix} .$$

Then, taking into account that $|\mathbf{F}| = \mathrm{Tr}(\boldsymbol{\varepsilon}) + 1$, and neglecting second-order terms results in

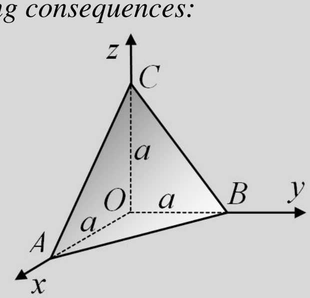
$$\left. \begin{array}{l} dA = (1+r)\, dA_0 \\ dA = (1+p+s+\varepsilon_{yy})\, \dfrac{1}{1+s} dA_0 \end{array} \right\} \quad \Longrightarrow \quad \varepsilon_{yy} = r - p \, .$$

Finally, since the *strain tensor* is symmetric,

$$\boxed{\boldsymbol{\varepsilon} \stackrel{not}{\equiv} \begin{bmatrix} p & \dfrac{q}{2} & 0 \\ \dfrac{q}{2} & r-p & 0 \\ 0 & 0 & s \end{bmatrix}} .$$

Problem 2.2 – *A uniform deformation* $(\mathbf{F} = \mathbf{F}(t))$ *is produced on the tetrahedron shown in the figure below, with the following consequences:*

1. *Points O, A and B do not move.*
2. *The volume of the solid becomes p times its initial volume.*
3. *The length of segment* $\overline{AC}$ *becomes* $p/\sqrt{2}$ *times its initial length.*
4. *The final angle AOC has a value of* $45°$.



Then,

a) *Justify why the infinitesimal strain theory cannot be used here.*
b) *Determine the deformation gradient tensor, the possible values of p and the displacement field in its material and spatial forms.*
c) *Draw the deformed solid.*

Solution

a) The angle AOC changes from 90° to 45° therefore, it is obvious that the deformation involved is not infinitesimal. In addition, under infinitesimal strain theory $\Delta\Phi \ll 1$ is satisfied and, in this problem, $\Delta\Phi = \pi/4 \approx 0.7854$.

Observation: strains are dimensionless; in engineering, small strains are usually considered when these are of order $10^{-3} - 10^{-4}$.

b) The conditions in the statement of the problem must be imposed one by one:

1. Considering that $\mathbf{F}(\mathbf{X},t) = \mathbf{F}(t)$ and knowing that $d\mathbf{x} = \mathbf{F}\cdot d\mathbf{X}$, the latter can be integrated as

$$\mathbf{x} = \int d\mathbf{x} = \int \mathbf{F}d\mathbf{X} = \mathbf{F}\int d\mathbf{X} = \mathbf{F}(t)\cdot\mathbf{X} + \mathbf{C}(t)$$

$$\text{with} \quad \mathbf{F} \overset{not}{\equiv} \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \quad \text{and} \quad \mathbf{C} \overset{not}{\equiv} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix},$$

which results in 12 unknowns. Imposing now the conditions in the statement,

Point O does not move:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = [\mathbf{F}] \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \mathbf{C} \quad \Longrightarrow \quad \mathbf{C} \overset{not}{\equiv} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Point A does not move:

$$\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = [\mathbf{F}] \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a\,F_{11} \\ a\,F_{21} \\ a\,F_{31} \end{bmatrix} \quad \Longrightarrow \quad \begin{cases} F_{11} = 1 \\ F_{21} = 0 \\ F_{31} = 0 \end{cases}$$

Point B does not move:

$$\begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} = [\mathbf{F}] \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} = \begin{bmatrix} aF_{12} \\ aF_{22} \\ aF_{32} \end{bmatrix} \quad \Longrightarrow \quad \begin{cases} F_{12} = 0 \\ F_{22} = 1 \\ F_{32} = 0 \end{cases}$$

Grouping all the information obtained results in

$$\mathbf{F} \overset{not}{\equiv} \begin{bmatrix} 1 & 0 & F_{13} \\ 0 & 1 & F_{23} \\ 0 & 0 & F_{33} \end{bmatrix} .$$

2. The condition in the statement imposes that $V_{final} = pV_{initial}$.

Expression $dV_f = |\mathbf{F}|\, dV_0$ allows to locally relate the differential volumes at different instants of time. In this case, $\mathbf{F}$ is constant for each fixed t, thus, the expression can be integrated and the determinant of $\mathbf{F}$ can be moved outside the integral,

$$V_f = \int_V dV_f = \int_{V_0} |\mathbf{F}|\, dV_0 = |\mathbf{F}| \int_{V_0} dV_0 = |\mathbf{F}|\, V_0 .$$

Therefore, $|\mathbf{F}| = F_{33} = p$ must be imposed.

3. The condition in the statement imposes that $l_{AC,final} = \frac{p}{\sqrt{2}} l_{AC,initial}$.

Since $\mathbf{F}$ is constant, the transformation is linear, that is, it transforms straight lines into straight lines. Hence, $\overline{AC}$ in the deformed configuration must also be a rectilinear segment. Then,

$$\mathbf{x}_C = \mathbf{F} \cdot \mathbf{X}_C \overset{not}{\equiv} \begin{bmatrix} 1 & 0 & F_{13} \\ 0 & 1 & F_{23} \\ 0 & 0 & F_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} = \begin{bmatrix} aF_{13} \\ aF_{23} \\ ap \end{bmatrix} \quad \text{and}$$

$$\begin{aligned} l_{AC,final} = l_{A'C'} &= \big| [aF_{13},\, aF_{23},\, ap] - [a,\, 0,\, 0] \big| = \big| [a(F_{13}-1)\,,\, aF_{23},\, ap] \big| = \\ &= \sqrt{(a(F_{13}-1))^2 + (aF_{23})^2 + (ap)^2} = a\sqrt{(F_{13}-1)^2 + F_{23}^2 + p^2} = \\ &= \frac{p}{\sqrt{2}}\, l_{AC} = \frac{p}{\sqrt{2}} \sqrt{2}\, a = pa . \end{aligned}$$

Therefore,

$$\sqrt{(F_{13}-1)^2 + F_{23}^2 + p^2} = p \;\Rightarrow\; (F_{13}-1)^2 + F_{23}^2 = 0 \;\Rightarrow\; F_{13} = 1;\; F_{23} = 0$$

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and the deformation gradient tensor results in

$$\mathbf{F} \stackrel{not}{\equiv} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{bmatrix} ,$$

such that only the value of p remains to be found.

4. The condition in the statement imposes that $AOC_{final} = 45^\circ = \pi/4$. Considering $d\mathbf{X}^{(1)} \stackrel{not}{\equiv} [1,\ 0,\ 0]$ and $d\mathbf{X}^{(2)} \stackrel{not}{\equiv} [0,\ 0,\ 1]$, the corresponding vectors in the spatial configuration are computed as

$$d\mathbf{x}^{(1)} = \mathbf{F} \cdot d\mathbf{X}^{(1)} \stackrel{not}{\equiv} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ,$$

$$d\mathbf{x}^{(2)} = \mathbf{F} \cdot d\mathbf{X}^{(2)} \stackrel{not}{\equiv} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ p \end{bmatrix} .$$

Then,

$$\cos\left(AOC_{final}\right) = \cos 45^\circ = \frac{d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)}}{\left|d\mathbf{x}^{(1)}\right| \left|d\mathbf{x}^{(2)}\right|} = \frac{\sqrt{2}}{2}$$

is imposed, with

$$\left|d\mathbf{x}^{(1)}\right| = 1 \quad , \quad \left|d\mathbf{x}^{(2)}\right| = \sqrt{1+p^2} \qquad \text{and} \qquad d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = 1$$

such that

$$\frac{1}{\sqrt{1+p^2}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \qquad \Longrightarrow \qquad p = \pm 1 .$$

But $|\mathbf{F}| = p > 0$, and, consequently, $p = 1$. Then, the *deformation gradient tensor* is

$$\boxed{\mathbf{F} \stackrel{not}{\equiv} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} .$$

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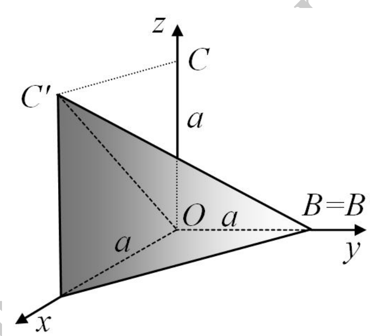
The equation of motion is determined by means of $\mathbf{x} = \mathbf{F} \cdot \mathbf{X}$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X+Z \\ Y \\ Z \end{bmatrix},$$

which allows determining the displacement field in material and spatial descriptions as

$$\boxed{\mathbf{U}(\mathbf{X},t) = \mathbf{x} - \mathbf{X} \stackrel{not}{\equiv} \begin{bmatrix} Z \\ 0 \\ 0 \end{bmatrix}} \qquad \text{and} \qquad \boxed{\mathbf{u}(\mathbf{x},t) \stackrel{not}{\equiv} \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix}} .$$

c) The graphical representation of the deformed tetrahedron is:



Problem 2.3 *– A uniform deformation is applied on the solid shown in the figure below. Determine:*

a) The general expression of the material description of the displacement field $\mathbf{U}(\mathbf{X},t)$ *in terms of the material displacement gradient tensor* $\mathbf{J}$.

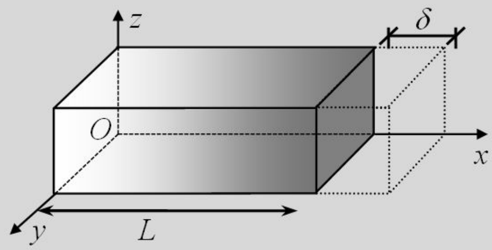
b) The expression of $\mathbf{U}(\mathbf{X},t)$ *when, in addition, the following boundary conditions are satisfied:*

$$\begin{aligned} & U_Y = U_Z = 0 \quad , \qquad \forall\, X, Y, Z \\ & U_X\big|_{X=0} = 0 \quad , \qquad \forall\, X, Y \\ & U_X\big|_{X=L} = \delta \end{aligned}$$

c) The possible values (positive and negative) that δ may take. Justify the answer obtained.

d) The material and spatial strain tensors and the infinitesimal strain tensor.

e) Plot the curves $E_{XX}-\delta/L$, $e_{xx}-\delta/L$ and $\varepsilon_x-\delta/L$ for all possible values of δ, indicating every significant value.



Solution

a) A uniform deformation implies that $\mathbf{F}(\mathbf{X},t)=\mathbf{F}(t)\,,\ \forall t,\mathbf{X}$. The deformation gradient tensor is related to the material displacement gradient tensor through the expression $\mathbf{F}=\mathbf{1}+\mathbf{J}$. Therefore, if $\mathbf{F}=\mathbf{F}(t)$, then $\mathbf{J}=\mathbf{J}(t)$. Taking into account the definition of $\mathbf{J}$ and integrating its expression results in

$$\mathbf{J}=\frac{\partial\mathbf{U}(\mathbf{X},t)}{\partial\mathbf{X}} \quad\Longrightarrow\quad d\mathbf{U}=\mathbf{J}\,d\mathbf{X} \quad\Longrightarrow\quad \int d\mathbf{U}=\int\mathbf{J}\,d\mathbf{X}$$

$$\Longrightarrow\quad \int d\mathbf{U}=\mathbf{J}\int d\mathbf{X} \quad\Longrightarrow\quad \mathbf{U}=\mathbf{J}\cdot\mathbf{X}+\mathbf{C}(t)\ .$$

where $\mathbf{C}(t)$ is an integration constant. Then, the general expression of the material description of the displacement field is

$$\boxed{\mathbf{U}(\mathbf{X},t)=\mathbf{J}(t)\cdot\mathbf{X}+\mathbf{C}(t)}\ .$$

b) Using the previous result and applying the boundary conditions given in the statement of the problem will yield the values of $\mathbf{J}$ and $\mathbf{C}$.
Boundary conditions:

$U_Y=U_Z=0\ ,\quad \forall\,X,Y,Z \quad\Rightarrow\quad$ Points only move in the X-direction.

$U_X\big|_{X=0}=0\ ,\quad \forall\,Y,Z \quad\Rightarrow\quad$ The YZ plane at the origin is fixed.

$U_X\big|_{X=L}=\delta\ ,\quad \forall\,Y,Z \quad\Rightarrow\quad$ This plane moves in a uniform manner in the X-direction.

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If the result obtained in *a)* is written in component form, the equations and conclusions that can be reached will be understood better.

$$\begin{aligned} U_X &= J_{11}X + J_{12}Y + J_{13}Z + C_1 \\ U_Y &= J_{21}X + J_{22}Y + J_{23}Z + C_2 \\ U_Z &= J_{31}X + J_{32}Y + J_{33}Z + C_3 \end{aligned}$$

From the first boundary condition:

$$\begin{aligned} U_Y = 0\,, \quad \forall\, X,Y,Z \quad &\Longrightarrow \quad J_{21} = J_{22} = J_{23} = C_2 = 0 \\ U_Z = 0\,, \quad \forall\, X,Y,Z \quad &\Longrightarrow \quad J_{31} = J_{32} = J_{33} = C_3 = 0 \end{aligned}$$

From the second boundary condition:

$$U_X\big|_{X=0} = 0\,, \quad \forall\, Y,Z \quad \Longrightarrow \quad J_{12} = J_{13} = C_1 = 0$$

From the third boundary condition:

$$U_X\big|_{X=L} = \delta\,, \quad \forall\, Y,Z \quad \Longrightarrow \quad J_{11}L = \delta \quad \Rightarrow \quad J_{11} = \frac{\delta}{L}$$

Finally,

$$\mathbf{J} \stackrel{not}{\equiv} \begin{bmatrix} \dfrac{\delta}{L} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \qquad \mathbf{C} \stackrel{not}{\equiv} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \boxed{\mathbf{U}(\mathbf{X}) = \mathbf{J}\cdot\mathbf{X} + \mathbf{C} \stackrel{not}{\equiv} \begin{bmatrix} \dfrac{\delta}{L}X \\ 0 \\ 0 \end{bmatrix}}\,.$$

c) In order to justify all the possible positive and negative values that δ may take, the condition $|\mathbf{F}| > 0$ must be imposed. Therefore, the determinant of $\mathbf{F}$ must be computed,

$$\mathbf{F} = \mathbf{1} + \mathbf{J} \stackrel{not}{\equiv} \begin{bmatrix} 1 + \dfrac{\delta}{L} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad |\mathbf{F}| = 1 + \frac{\delta}{L} > 0 \quad \Longrightarrow \quad \boxed{\delta > -L}\,.$$

d) To obtain the spatial and material strain tensors as well as the infinitesimal strain tensor, their respective definitions must be taken into account.

$$\text{Spatial strain tensor:} \qquad \mathbf{e} = \frac{1}{2}\left(\mathbf{1} - \mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\right)$$

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$$\text{Material strain tensor:} \qquad \mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{1}\right)$$

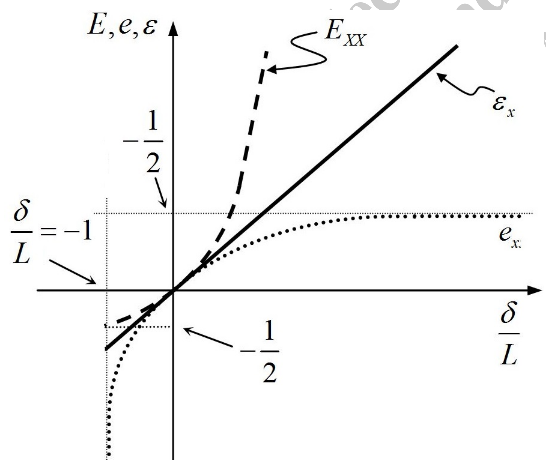
$$\text{Infinitesimal strain tensor:} \qquad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{J}^T \cdot \mathbf{J}\right)$$

Applying these definitions using the values of **F** and **J** calculated in *b)* and *c)*, the corresponding expressions are obtained.

$$\mathbf{e} \overset{not}{\equiv} \begin{bmatrix} e_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{with} \quad e_{xx} = \left(\frac{\delta}{L} + \frac{1}{2}\frac{\delta^2}{L^2}\right) \Big/ \left(1 + \frac{\delta}{L}\right)^2$$

$$\mathbf{E} \overset{not}{\equiv} \begin{bmatrix} E_{XX} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{with} \quad E_{XX} = \frac{\delta}{L} + \frac{1}{2}\frac{\delta^2}{L^2} \quad ; \quad \boldsymbol{\varepsilon} \overset{not}{\equiv} \begin{bmatrix} \dfrac{\delta}{L} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

e) Plotting the curves $E_{XX} - \delta/L$, $e_{xx} - \delta/L$ and $\varepsilon_x - \delta/L$ together yields:



Here,

- E_{XX} is a second-order parabola that contains the origin and has its minimum at $\delta/L = -1$, i.e., for $E_{XX} = -1/2$.
- ε_x is the identity straight line (45° slope and contains the origin).
- e_{xx} has two asymptotes, a vertical one at $\delta/L = -1$ and a horizontal at $e_{xx} = 1/2$.

It can be concluded, then, that for small δ/L strains the three functions have a very similar behavior and the same slope at the origin. That is, the same result will be obtained with any of the definitions of strain tensor. However, outside this domain (large or finite strains) the three curves are clearly different.

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EXERCISES

2.1 – *Consider the velocity fields*

$$\mathbf{v}_1 \stackrel{not}{\equiv} \left[\frac{x}{1+t}, \quad \frac{2y}{1+t}, \quad \frac{3z}{1+t} \right]^T \qquad \textit{and} \qquad \mathbf{v}_2 \stackrel{not}{\equiv} \left[\frac{X}{1+t}, \quad \frac{2Y}{1+t}, \quad \frac{3Z}{1+t} \right]^T .$$

Determine:

a) The material description of $\mathbf{v}_1$ *and the spatial description of* $\mathbf{v}_2$ *(consider* $t=0$ *is the reference configuration).*

b) The density distribution in both cases (consider ρ_0 *is the initial density).*

c) The material and spatial descriptions of the displacement field as well as the material (Green-Lagrange) and spatial (Almansi) strain tensors for the velocity field $\mathbf{v}_1$.

d) Repeat c) for configurations close to the reference configuration $(t \to 0)$.

e) Prove that the two strain tensors coincide for the conditions stated in d).

2.2 – *The equation of motion in a continuous medium is*

$$x = X + Yt \ , \quad y = Y \ , \quad z = Z \ .$$

Obtain the length at time $t=2$ *of the segment of material line that at time* $t=1$ *is defined in parametric form as*

$$x(\alpha) = 0 \ , \quad y(\alpha) = \alpha^2 \ , \quad z(\alpha) = \alpha \qquad 0 \le \alpha \le 1 \ .$$

2.3 – *Consider the material strain tensor*

$$\mathbf{E} \stackrel{not}{\equiv} \begin{bmatrix} 0 & t e^{tX} & 0 \\ t e^{tX} & 0 & 0 \\ 0 & 0 & t e^{tY} \end{bmatrix} .$$

Obtain the length at time $t=1$ *of the segment that at time* $t=0$ *(reference configuration) is straight and joins the points* $(1,1,1)$ *and* $(2,2,2)$.

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2.4 – *The equation of motion of a continuous medium is*

$$x = X \ , \quad y = Y \ , \quad z = Z - Xt \ .$$

Calculate the angle formed at time $t = 0$ by the differential segments that at time $t = t$ are parallel to the x- and z-axes.

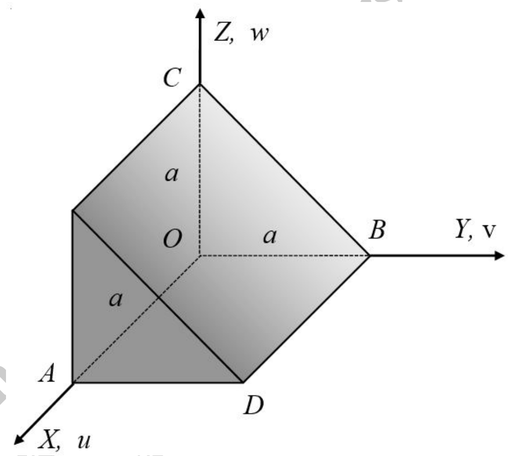
2.5 – *The following information is known in relation to a certain displacement field given in material description,* $\mathbf{U}(X,Y,Z)$*:*

1) It is lineal in X, Y, Z.

2) It is antisymmetric with respect to plane $Y = 0$, that is, the following is satisfied:

$$\mathbf{U}(X,Y,Z) = -\mathbf{U}(X,-Y,Z) \qquad \forall\, X,Y,Z$$

3) Under said displacement field, the volume of the element in the figure does not change, its angle AOB remains constant, the segment $\overline{OB}$ becomes $\sqrt{2}$ times its initial length and the vertical component of the displacement at point B is positive $(w_B > 0)$.



Determine:

a) The most general expression of the given displacement field, such that conditions 1) and 2) are satisfied.

b) The expression of $\mathbf{U}$ *when, in addition, condition 3) is satisfied. Obtain the deformation gradient tensor and the material strain tensor. Draw the deformed shape of the element in the figure, indicating the most significant values.*

c) The directions (defined by their unit vectors $\mathbf{T}$*) for which the deformation is reduced to a stretch (there is no rotation).*

NOTE: Finite strains must be considered (not infinitesimal ones).

2.6 – *The solid in the figure undergoes a uniform deformation such that points A, B and C do not move. Assuming an infinitesimal strain framework,*

a) Express the displacement field in terms of "generic" values of the stretches and rotations.

b) Identify the null components of the strain tensor and express the rotation vector in terms of the stretches.

In addition, the following is known:

1) Segment $\overline{AE}$ becomes $(1+p)$ times its initial length.

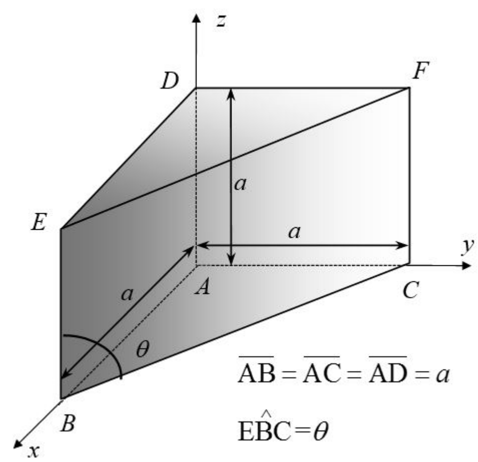
2) The volume becomes $(1+q)$ times its initial value.

3) The angle θ increases its value in r (given in radians).

Under these conditions, determine:

c) The strain tensor, the rotation vector and the displacement field in terms of p, q and r.

NOTE: The values of p, q and r are small and its second-order infinitesimal terms can be neglected.



2.7 – *The solid in the figure undergoes a uniform deformation with the following consequences:*

1) The x- and z-axes are both material lines. Point A does not move.

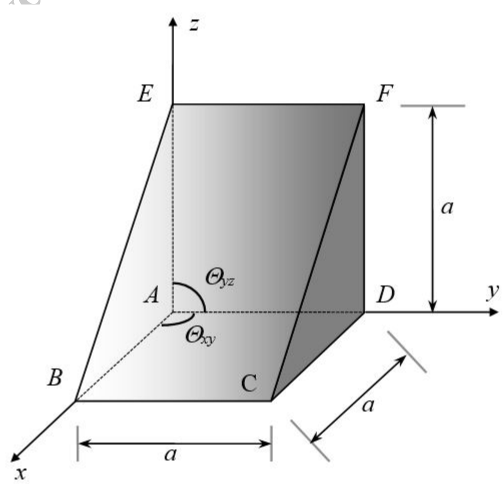
2) The volume of the solid remains constant.

3) The angle θ_{xy} remains constant.

4) The angle θ_{yz} increases in r radians.

5) The segment $\overline{AF}$ becomes $(1+p)$ times its initial length.

6) The area of the triangle ABE becomes $(1+q)$ its initial value.



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Then,

a) Express the displacement field in terms of "generic" values of the stretches and rotations.

b) Identify the null components of the strain tensor and express the rotation vector in terms of the stretches.

c) Determine the strain tensor, the rotation vector and the displacement field in terms of p, q and r.

NOTE: The values of p, q and r are small and its second-order infinitesimal terms can be neglected.

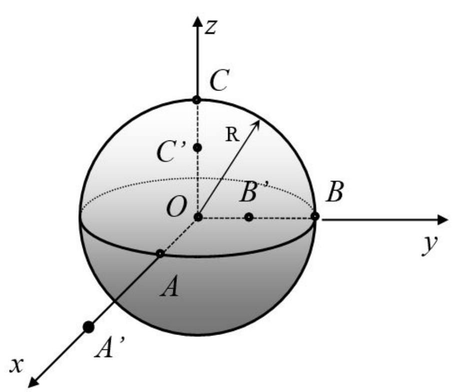
2.8 – *The sphere in the figure undergoes a uniform deformation* $(\mathbf{F} = const.)$ *such that points A, B and C move to positions* A', B' *and* C', *respectively. Point O does not move. Determine:*

a) The deformation gradient tensor in terms of p and q.

b) The equation of the deformed external surface of the sphere. Indicate which type of surface it is and draw it.

c) The material and spatial strain tensors. Obtain the value of p in terms of q when the material is assumed to be incompressible.

d) Repeat c) using infinitesimal strain theory. Prove that when p and q are small, the results of c) and d) coincide.



$$\overline{AA'} = p$$
$$\overline{BB'} = q$$
$$\overline{CC'} = q$$
$$p > 0$$
$$q > 0$$

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Chapter 3
Compatibility Equations

3.1 Introduction

Given a sufficiently regular displacement field $\mathbf{U}(\mathbf{X},t)$, it is always possible to find the corresponding strain field (for example, the Green-Lagrange strain field) by differentiating this strain field with respect to its coordinates (in this case, the material ones)[1],

$$E_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} + \frac{\partial U_k}{\partial X_i}\frac{\partial U_k}{\partial X_j}\right) \stackrel{not}{=} \frac{1}{2}\left(U_{i,j} + U_{j,i} + U_{k,i}\, U_{k,j}\right) \qquad i,j \in \{1,2,3\}. \tag{3.1}$$

In the infinitesimal strain case, given a displacement field $\mathbf{u}(\mathbf{x},t)$, the strain field

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \stackrel{not}{=} \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad i,j \in \{1,2,3\} \tag{3.2}$$

is obtained.

The question can be formulated in reverse, that is, given a strain field $\boldsymbol{\varepsilon}(\mathbf{x},t)$, is it possible to find a displacement field $\mathbf{u}(\mathbf{x},t)$ such that $\boldsymbol{\varepsilon}(\mathbf{x},t)$ is its infinitesimal strain tensor? This is not always possible and the answer provides the so-called *compatibility equations*.

Expression (3.2) constitutes a system of 6 (due to symmetry) partial differential equations (PDEs) with 3 unknowns: $u_1(\mathbf{x},t)$, $u_2(\mathbf{x},t)$, $u_3(\mathbf{x},t)$. This system is overdetermined because there exist more conditions than unknowns, and it may not have a solution.

Therefore, for a second-order symmetric tensor $\boldsymbol{\varepsilon}(\mathbf{x},t)$ to correspond to a strain tensor (and, thus, be integrable and there exist a displacement field from which it comes) it is necessary that this tensor verifies certain conditions. These conditions are denominated compatibility conditions or equations and guarantee

[1] Here, the simplified notation $\partial U_i/\partial X_j \stackrel{not}{=} U_{i,j}$ is used.

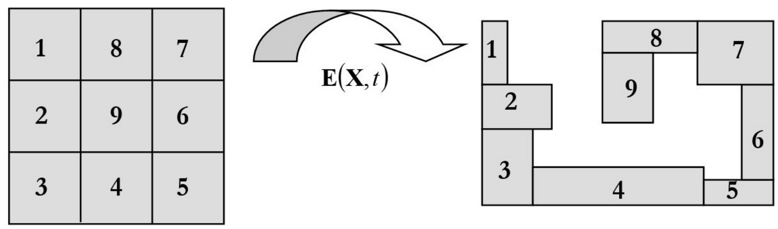

Figure 3.1: Non-compatible strain field.

the *continuity* of the continuous medium during the deformation process (see Figure 3.1).

Definition 3.1. The *compatibility conditions* are conditions that a second-order tensor must satisfy in order to be a strain tensor and, therefore, for there to exist a displacement field from which it comes.

Remark 3.1. Note that, to define a strain tensor, the 6 components of a symmetric tensor cannot be written arbitrarily. These must satisfy the compatibility conditions.

Remark 3.2. Given a displacement field, one can always obtain, through differentiation, an associated strain field that automatically satisfies the compatibility conditions. Therefore, in this case, there is no sense in verifying that the compatibility conditions are satisfied.

3.2 Preliminary Example: Compatibility Equations of a Potential Vector Field

A given vector field $\mathbf{v}(\mathbf{x},t)$ is a *potential field* if there exists a scalar function $\phi(\mathbf{x},t)$ (named potential function) such that its gradient is $\mathbf{v}(\mathbf{x},t)$,

$$\begin{cases} \mathbf{v}(\mathbf{x},t) = \nabla\phi(\mathbf{x},t) \ , \\ \mathrm{v}_i(\mathbf{x},t) = \dfrac{\partial\phi(\mathbf{x},t)}{\partial x_i} \qquad i \in \{1,2,3\} \ . \end{cases} \tag{3.3}$$

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Therefore, given a scalar (continuous) function $\phi(\mathbf{x},t)$, it is always possible to define a *potential* vector field $\mathbf{v}(\mathbf{x},t)$ such that the scalar function is its potential, as defined in (3.3).

Now, the reverse question is posed: given a vector field $\mathbf{v}(\mathbf{x},t)$, does there exist a scalar function $\phi(\mathbf{x},t)$ such that $\nabla\phi(\mathbf{x},t) = \mathbf{v}(\mathbf{x},t)$? This is written in component form as

$$\begin{aligned} \mathrm{v}_x = \frac{\partial\phi}{\partial x} \quad &\Longrightarrow \quad \mathrm{v}_x - \frac{\partial\phi}{\partial x} = 0\,, \\ \mathrm{v}_y = \frac{\partial\phi}{\partial y} \quad &\Longrightarrow \quad \mathrm{v}_y - \frac{\partial\phi}{\partial y} = 0\,, \\ \mathrm{v}_z = \frac{\partial\phi}{\partial z} \quad &\Longrightarrow \quad \mathrm{v}_z - \frac{\partial\phi}{\partial z} = 0\,, \end{aligned} \tag{3.4}$$

which corresponds to a system of PDEs with 3 equations and 1 unknown $(\phi(\mathbf{x},t))$, thus, the system is overdetermined and may not have a solution.

Differentiating once (3.4) with respect to (x,y,z) yields

$$\begin{array}{lll} \dfrac{\partial\mathrm{v}_x}{\partial x} = \dfrac{\partial^2\phi}{\partial x^2}\,, & \dfrac{\partial\mathrm{v}_x}{\partial y} = \dfrac{\partial^2\phi}{\partial x\partial y}\,, & \dfrac{\partial\mathrm{v}_x}{\partial z} = \dfrac{\partial^2\phi}{\partial x\partial z}\,, \\ \dfrac{\partial\mathrm{v}_y}{\partial x} = \dfrac{\partial^2\phi}{\partial y\partial x}\,, & \dfrac{\partial\mathrm{v}_y}{\partial y} = \dfrac{\partial^2\phi}{\partial y^2}\,, & \dfrac{\partial\mathrm{v}_y}{\partial z} = \dfrac{\partial^2\phi}{\partial y\partial z}\,, \\ \dfrac{\partial\mathrm{v}_z}{\partial x} = \dfrac{\partial^2\phi}{\partial z\partial x}\,, & \dfrac{\partial\mathrm{v}_z}{\partial y} = \dfrac{\partial^2\phi}{\partial z\partial y}\,, & \dfrac{\partial\mathrm{v}_z}{\partial z} = \dfrac{\partial^2\phi}{\partial z^2}\,, \end{array} \tag{3.5}$$

which represents a system of *9 equations*. Considering the equality of mixed partial derivatives, it is observed that *6 different functions* (second derivatives) of the unknown ϕ are involved in these 9 equations,

$$\frac{\partial^2\phi}{\partial x^2}\,,\quad \frac{\partial^2\phi}{\partial y^2}\,,\quad \frac{\partial^2\phi}{\partial z^2}\,,\quad \frac{\partial^2\phi}{\partial x\partial y}\,,\quad \frac{\partial^2\varphi}{\partial x\partial z} \quad \text{and} \quad \frac{\partial^2\phi}{\partial y\partial z}\,. \tag{3.6}$$

So, they can be removed from the original system (3.5) and 3 relations, named compatibility conditions, can be established between the first partial derivatives of the components of $\mathbf{v}(\mathbf{x},t)$.

Hence, for there to exist a scalar function $\phi(\mathbf{x},t)$ such that $\nabla\phi(\mathbf{x},t) = \mathbf{v}(\mathbf{x},t)$, the given vector field $\mathbf{v}(\mathbf{x},t)$ must satisfy the following compatibility conditions.

$$\left.\begin{array}{l} \dfrac{\partial\mathrm{v}_y}{\partial x} - \dfrac{\partial\mathrm{v}_x}{\partial y} = 0 \overset{def}{=} S_z \\ \dfrac{\partial\mathrm{v}_x}{\partial z} - \dfrac{\partial\mathrm{v}_z}{\partial x} = 0 \overset{def}{=} S_y \\ \dfrac{\partial\mathrm{v}_z}{\partial y} - \dfrac{\partial\mathrm{v}_y}{\partial z} = 0 \overset{def}{=} S_x \end{array}\right\} \text{ where } \mathbf{S} \overset{not}{\equiv} \begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix} \equiv \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathrm{v}_x & \mathrm{v}_y & \mathrm{v}_z \end{vmatrix} \overset{not}{\equiv} rot\ \mathbf{v} \overset{not}{=} \nabla\times\mathbf{v} \tag{3.7}$$

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In consequence, from (3.7), the compatibility equations can be written as

$$\boxed{\text{Compatibility equations of a potential vector field} \quad \begin{cases} \nabla\times\mathbf{v}=\mathbf{0} \\ \dfrac{\partial \mathrm{v}_i}{\partial x_j}-\dfrac{\partial \mathrm{v}_j}{\partial x_i}=0 \qquad i,j\in\{1,2,3\} \end{cases}} \tag{3.8}$$

Remark 3.3. The 3 compatibility equations (3.7) or (3.8) are not independent of one another and a functional relation can be established between them. Indeed, applying the condition that the divergence of the rotational of a vector field is null[2], $\nabla\cdot(\nabla\times\mathbf{v})=0$.

3.3 Compatibility Conditions for Infinitesimal Strains

Consider the infinitesimal strain field $\boldsymbol{\varepsilon}(\mathbf{x},t)$ with components

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\overset{not}{=}\frac{1}{2}\left(u_{i,j}+u_{j,i}\right) \qquad i,j\in\{1,2,3\}\ , \tag{3.9}$$

which may be written in matrix form as

$$[\boldsymbol{\varepsilon}]=\begin{bmatrix}\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz}\end{bmatrix}=\begin{bmatrix}\dfrac{\partial u_x}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial y}+\dfrac{\partial u_y}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_x}{\partial z}+\dfrac{\partial u_z}{\partial x}\right)\\ \times & \dfrac{\partial u_y}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial u_y}{\partial z}+\dfrac{\partial u_z}{\partial y}\right)\\ (symm) & \times & \dfrac{\partial u_z}{\partial z}\end{bmatrix}. \tag{3.10}$$

Due to the symmetry in (3.10), only 6 different equations are obtained,

$$\begin{aligned}
&\varepsilon_{xx}-\frac{\partial u_x}{\partial x}=0\ , && \varepsilon_{xy}-\frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right)=0\ ,\\
&\varepsilon_{yy}-\frac{\partial u_y}{\partial y}=0\ , && \varepsilon_{xz}-\frac{1}{2}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)=0\ ,\\
&\varepsilon_{zz}-\frac{\partial u_z}{\partial z}=0\ , && \varepsilon_{yz}-\frac{1}{2}\left(\frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y}\right)=0\ .
\end{aligned} \tag{3.11}$$

[2] A theorem of differential geometry states that the divergence of the rotational of *any* field is null, $\nabla\cdot[\nabla\times(\bullet)]=0$.

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Equation (3.11) is a system of 6 PDEs with 3 unknowns, which are the components of the displacement vector $\mathbf{u}(\mathbf{x},t) \stackrel{not}{\equiv} [u_x,\, u_y,\, u_z]^T$. In general, this problem will not have a solution unless certain compatibility conditions are satisfied. To obtain these conditions, the equations in (3.11) are differentiated twice with respect to their spatial coordinates,

$$
\begin{array}{c}
\dfrac{\partial^2 \left(\varepsilon_{xx} - \dfrac{\partial u_x}{\partial x} \right)}{\partial x^2,\ \partial y^2,\ \partial z^2,\ \partial xy,\ \partial xz,\ \partial yz} = 6 \text{ equations} \\
\vdots \qquad\qquad\qquad \vdots \\
\dfrac{\partial^2 \left(\varepsilon_{yz} - \dfrac{1}{2} \left(\dfrac{\partial u_y}{\partial z} + \dfrac{\partial u_z}{\partial y} \right) \right)}{\partial x^2,\ \partial y^2,\ \partial z^2,\ \partial xy,\ \partial xz,\ \partial yz} = 6 \text{ equations} \, ,
\end{array}
\tag{3.12}
$$

providing a total of 36 equations,

$$
\underbrace{
\begin{array}{l}
\dfrac{\partial^2 \varepsilon_{xx}}{\partial x^2} = \dfrac{\partial^3 u_x}{\partial x^3} \\
\dfrac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \dfrac{\partial^3 u_x}{\partial x \partial y^2} \\
\dfrac{\partial^2 \varepsilon_{xx}}{\partial z^2} = \dfrac{\partial^3 u_x}{\partial x \partial z^2} \\
\dfrac{\partial^2 \varepsilon_{xx}}{\partial x \partial y} = \dfrac{\partial^3 u_x}{\partial x^2 \partial y} \\
\dfrac{\partial^2 \varepsilon_{xx}}{\partial x \partial z} = \dfrac{\partial^3 u_x}{\partial x^2 \partial z} \\
\dfrac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \dfrac{\partial^3 u_x}{\partial x \partial y \partial z}
\end{array}
}_{(18 \text{ eqns for } \varepsilon_{xx},\, \varepsilon_{yy},\, \varepsilon_{zz})}
\quad \cdots \quad
\underbrace{
\begin{array}{l}
\dfrac{\partial^2 \varepsilon_{yz}}{\partial x^2} = \dfrac{1}{2} \left(\dfrac{\partial^3 u_y}{\partial z \partial x^2} + \dfrac{\partial^3 u_z}{\partial y \partial x^2} \right) \\
\dfrac{\partial^2 \varepsilon_{yz}}{\partial y^2} = \dfrac{1}{2} \left(\dfrac{\partial^3 u_y}{\partial z \partial y^2} + \dfrac{\partial^3 u_z}{\partial y^3} \right) \\
\dfrac{\partial^2 \varepsilon_{yz}}{\partial z^2} = \dfrac{1}{2} \left(\dfrac{\partial^3 u_y}{\partial z^3} + \dfrac{\partial^3 u_z}{\partial y \partial z^2} \right) \\
\dfrac{\partial^2 \varepsilon_{yz}}{\partial x \partial y} = \dfrac{1}{2} \left(\dfrac{\partial^3 u_y}{\partial z \partial x \partial y} + \dfrac{\partial^3 u_z}{\partial y^2 \partial x} \right) \\
\dfrac{\partial^2 \varepsilon_{yz}}{\partial x \partial z} = \dfrac{1}{2} \left(\dfrac{\partial^3 u_y}{\partial z^2 \partial x} + \dfrac{\partial^3 u_z}{\partial y \partial x \partial z} \right) \\
\dfrac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} = \dfrac{1}{2} \left(\dfrac{\partial^3 u_y}{\partial z^2 \partial y} + \dfrac{\partial^3 u_z}{\partial y^2 \partial z} \right)
\end{array}
}_{(18 \text{ eqns for } \varepsilon_{xy},\, \varepsilon_{xz},\, \varepsilon_{yz})}
\tag{3.13}
$$

All the possible *third derivatives* of each component of the displacements u_x, u_y and u_z are involved in these 36 equations. Thus, there are 30 different derivatives,

$$
\begin{array}{l}
\dfrac{\partial^3 u_x}{\partial x^3,\ \partial x^2 y,\ \partial x^2 z,\ \partial y^3,\ \partial y^2 x,\ \partial y^2 z,\ \partial z^3,\ \partial z^2 x,\ \partial z^2 y,\ \partial xyz} = 10 \text{ derivatives} \, , \\
\dfrac{\partial^3 u_y}{\partial x^3,\ \partial x^2 y,\ \partial x^2 z,\ \partial y^3,\ \partial y^2 x,\ \partial y^2 z,\ \partial z^3,\ \partial z^2 x,\ \partial z^2 y,\ \partial xyz} = 10 \text{ derivatives} \, , \\
\dfrac{\partial^3 u_z}{\partial x^3,\ \partial x^2 y,\ \partial x^2 z,\ \partial y^3,\ \partial y^2 x,\ \partial y^2 z,\ \partial z^3,\ \partial z^2 x,\ \partial z^2 y,\ \partial xyz} = 10 \text{ derivatives} \, ,
\end{array}
\tag{3.14}
$$

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which constitute the 30 unknowns in the system of 36 equations

$$f_n\Bigg(\underbrace{\frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l}}_{30},\ \frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l}\Bigg) \qquad n \in \{1,2\,...\,36\} \tag{3.15}$$

defined in (3.13). Therefore, the 30 unknowns, which are the *displacement derivatives* $\partial^3 u_i/(\partial x_j \partial x_k \partial x_l)$, can be eliminated from this system and 6 equations are obtained. In these equations, the third derivatives mentioned above do not appear, but there will be *21 second derivatives of the strain tensor* $\partial^2 \varepsilon_{ij}/(\partial x_k \partial x_l)$. After the corresponding algebraic operations, the resulting equations are

$$\text{Compatibility equations}\left\{\begin{array}{l}
S_{xx} \overset{def}{=} \dfrac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \dfrac{\partial^2 \varepsilon_{zz}}{\partial y^2} - 2\dfrac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} = 0 \\[2ex]
S_{yy} \overset{def}{=} \dfrac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \dfrac{\partial^2 \varepsilon_{xx}}{\partial z^2} - 2\dfrac{\partial^2 \varepsilon_{xz}}{\partial x \partial z} = 0 \\[2ex]
S_{zz} \overset{def}{=} \dfrac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \dfrac{\partial^2 \varepsilon_{yy}}{\partial x^2} - 2\dfrac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0 \\[2ex]
S_{xy} \overset{def}{=} -\dfrac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} + \dfrac{\partial}{\partial z}\left(\dfrac{\partial \varepsilon_{yz}}{\partial x} + \dfrac{\partial \varepsilon_{xz}}{\partial y} - \dfrac{\partial \varepsilon_{xy}}{\partial z}\right) = 0 \\[2ex]
S_{xz} \overset{def}{=} -\dfrac{\partial^2 \varepsilon_{yy}}{\partial x \partial z} + \dfrac{\partial}{\partial y}\left(\dfrac{\partial \varepsilon_{yz}}{\partial x} - \dfrac{\partial \varepsilon_{xz}}{\partial y} + \dfrac{\partial \varepsilon_{xy}}{\partial z}\right) = 0 \\[2ex]
S_{yz} \overset{def}{=} -\dfrac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} + \dfrac{\partial}{\partial x}\left(-\dfrac{\partial \varepsilon_{yz}}{\partial x} + \dfrac{\partial \varepsilon_{xz}}{\partial y} + \dfrac{\partial \varepsilon_{xy}}{\partial z}\right) = 0
\end{array}\right. \tag{3.16}$$

which constitute the *compatibility equations for the infinitesimal strain tensor* $\boldsymbol{\varepsilon}$. The compact expression corresponding to the 6 equations in (3.16) is

$$\text{Compatibility equations for the infinitesimal strain tensor}\ \Big\{\ \mathbf{S} = \nabla \times (\boldsymbol{\varepsilon} \times \nabla) = \mathbf{0} \tag{3.17}$$

Another way of expressing the compatibility conditions (3.16) is in terms of the three-index operator named *permutation operator* (e_{ijk}). In this case, the compatibility equations can be written as

$$\boxed{S_{mn} = e_{mjq} e_{nir} \varepsilon_{ij,qr} = 0}\ . \tag{3.18}$$

Remark 3.4. The 6 equations (3.16) are not functionally independent and, taking again into account the fact that the divergence of the rotational of a field is intrinsically null, the following functional relations can be established between them.

$$\nabla \cdot \mathbf{S} = \nabla \cdot (\nabla \times (\boldsymbol{\varepsilon} \times \nabla)) = \mathbf{0} \quad \Longrightarrow \quad \begin{cases} \dfrac{\partial S_{xx}}{\partial x} + \dfrac{\partial S_{xy}}{\partial y} + \dfrac{\partial S_{xz}}{\partial z} = 0 \\ \dfrac{\partial S_{xy}}{\partial x} + \dfrac{\partial S_{yy}}{\partial y} + \dfrac{\partial S_{yz}}{\partial z} = 0 \\ \dfrac{\partial S_{xz}}{\partial x} + \dfrac{\partial S_{yz}}{\partial y} + \dfrac{\partial S_{zz}}{\partial z} = 0 \end{cases}$$

Remark 3.5. The three-index operator denominated *permutation operator* is given by

$$e_{ijk} = \begin{cases} 0 \ \rightarrow & \text{if an index is repeated,} \\ & i = j \ \text{ or } \ i = k \ \text{ or } \ j = k \\ 1 \ \rightarrow & \text{positive (clockwise) direction of the indexes,} \\ & i, j, k \in \{123, 231, 312\} \\ -1 \ \rightarrow & \text{negative (counterclockwise) direction of the indexes,} \\ & i, j, k \in \{132, 321, 213\} \end{cases}$$

This definition is summarized in graphic form in Figure 3.2.

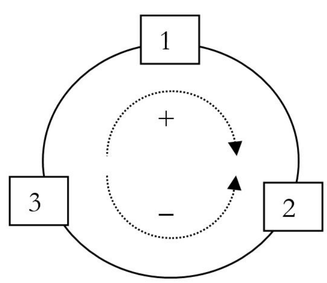

Figure 3.2: Definition of the permutation operator, e_{ijk}.

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Finally, another possible expression of the compatibility conditions is

$$\boxed{\varepsilon_{ij,kl}+\varepsilon_{kl,ij}-\varepsilon_{ik,jl}-\varepsilon_{jl,ik}=0 \qquad i,j,k,l\ \in\{1,2,3\}}\ . \tag{3.19}$$

Remark 3.6. Since the compatibility equations (3.16) only involve the second spatial derivatives of the components of the strain tensor $\boldsymbol{\varepsilon}(\mathbf{x},t)$, *every strain tensor* that is *linear* (first-order polynomial) with respect to the *spatial variables* will be *compatible* and, therefore, integrable. As a particular case, *every uniform strain tensor* $\boldsymbol{\varepsilon}(t)$ *is integrable*.

3.4 Integration of the Infinitesimal Strain Field

3.4.1 Preliminary Equations

Consider the rotation tensor $\boldsymbol{\Omega}(\mathbf{x},t)$ for the infinitesimal strain case (see Chapter 2 , Section 2.11.6),

$$\begin{cases} \boldsymbol{\Omega}=\dfrac{1}{2}\left(\mathbf{u}\otimes\nabla-\nabla\otimes\mathbf{u}\right)\,, \\ \Omega_{ij}=\dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j}-\dfrac{\partial u_j}{\partial x_i}\right) \qquad i,j\in\{1,2,3\}\ . \end{cases} \tag{3.20}$$

and the infinitesimal rotation vector $\boldsymbol{\theta}(\mathbf{x},t)$, associated with said rotation tensor, defined as[3]

$$\boldsymbol{\theta}=\frac{1}{2}rot\ \mathbf{u}=\frac{1}{2}\nabla\times\mathbf{u}\stackrel{not}{\equiv}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix}=\begin{bmatrix}-\Omega_{23}\\ -\Omega_{31}\\ -\Omega_{12}\end{bmatrix}=\begin{bmatrix}-\Omega_{yz}\\ -\Omega_{zx}\\ -\Omega_{xy}\end{bmatrix}. \tag{3.21}$$

Differentiating the infinitesimal rotation tensor in (3.20) with respect to a coordinate x_k yields

$$\Omega_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right) \quad\Longrightarrow\quad \frac{\partial\Omega_{ij}}{\partial x_k}=\frac{1}{2}\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right). \tag{3.22}$$

---

[3] The tensor $\boldsymbol{\Omega}$ is skew-symmetric, i.e., $\boldsymbol{\Omega}\stackrel{not}{\equiv}\begin{bmatrix}0 & \Omega_{12} & -\Omega_{31}\\ -\Omega_{12} & 0 & \Omega_{23}\\ \Omega_{31} & -\Omega_{23} & 0\end{bmatrix}$.

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Adding and subtracting in (3.22) the term $\partial^2 u_k / (2\,\partial x_i \partial x_j)$ and rearranging the expression obtained results in

$$
\begin{aligned}
\frac{\partial \Omega_{ij}}{\partial x_k} &= \frac{1}{2}\frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\frac{\partial^2 u_k}{\partial x_i \partial x_j} - \frac{1}{2}\frac{\partial^2 u_k}{\partial x_i \partial x_j} = \\
&= \frac{\partial}{\partial x_j}\underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)}_{\varepsilon_{ik}} - \frac{\partial}{\partial x_i}\underbrace{\frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)}_{\varepsilon_{jk}} = \frac{\partial \varepsilon_{ik}}{\partial x_j} - \frac{\partial \varepsilon_{jk}}{\partial x_i}\,.
\end{aligned}
\tag{3.23}
$$

This expression can now be used to calculate the Cartesian derivatives of the components of the infinitesimal rotation vector, $\boldsymbol{\theta}(\mathbf{x},t)$, given in (3.21), as follows.

$$
\boxed{\nabla\theta_1 \begin{cases}
\dfrac{\partial \theta_1}{\partial x} = -\dfrac{\partial \Omega_{yz}}{\partial x} = \dfrac{\partial \varepsilon_{xz}}{\partial y} - \dfrac{\partial \varepsilon_{xy}}{\partial z} \\[2ex]
\dfrac{\partial \theta_1}{\partial y} = -\dfrac{\partial \Omega_{yz}}{\partial y} = \dfrac{\partial \varepsilon_{yz}}{\partial y} - \dfrac{\partial \varepsilon_{yy}}{\partial z} \\[2ex]
\dfrac{\partial \theta_1}{\partial z} = -\dfrac{\partial \Omega_{yz}}{\partial z} = \dfrac{\partial \varepsilon_{zz}}{\partial y} - \dfrac{\partial \varepsilon_{zy}}{\partial z}
\end{cases}}
\tag{3.24}
$$

$$
\boxed{\nabla\theta_2 \begin{cases}
\dfrac{\partial \theta_2}{\partial x} = -\dfrac{\partial \Omega_{zx}}{\partial x} = \dfrac{\partial \varepsilon_{xx}}{\partial z} - \dfrac{\partial \varepsilon_{xz}}{\partial x} \\[2ex]
\dfrac{\partial \theta_2}{\partial y} = -\dfrac{\partial \Omega_{zx}}{\partial y} = \dfrac{\partial \varepsilon_{xy}}{\partial z} - \dfrac{\partial \varepsilon_{yz}}{\partial x} \\[2ex]
\dfrac{\partial \theta_2}{\partial z} = -\dfrac{\partial \Omega_{zx}}{\partial z} = \dfrac{\partial \varepsilon_{xz}}{\partial z} - \dfrac{\partial \varepsilon_{zz}}{\partial x}
\end{cases}}
\tag{3.25}
$$

$$
\boxed{\nabla\theta_3 \begin{cases}
\dfrac{\partial \theta_3}{\partial x} = -\dfrac{\partial \Omega_{xy}}{\partial x} = \dfrac{\partial \varepsilon_{xy}}{\partial x} - \dfrac{\partial \varepsilon_{xx}}{\partial y} \\[2ex]
\dfrac{\partial \theta_3}{\partial y} = -\dfrac{\partial \Omega_{xy}}{\partial y} = \dfrac{\partial \varepsilon_{yy}}{\partial x} - \dfrac{\partial \varepsilon_{xy}}{\partial y} \\[2ex]
\dfrac{\partial \theta_3}{\partial z} = -\dfrac{\partial \Omega_{xy}}{\partial z} = \dfrac{\partial \varepsilon_{yz}}{\partial x} - \dfrac{\partial \varepsilon_{xz}}{\partial y}
\end{cases}}
\tag{3.26}
$$

Assume the value of the infinitesimal rotation vector $\boldsymbol{\theta}(\mathbf{x},t)$ is known and, through it by means of (3.21), the value of the infinitesimal rotation tensor

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$\boldsymbol{\Omega}(\mathbf{x},t)$ is also known. Then, the displacement gradient tensor $\mathbf{J}(\mathbf{x},t)$ (see Chapter 2, Section 2.11.6) becomes

$$\left\{\begin{array}{l} \mathbf{J} = \dfrac{\partial \mathbf{u}(\mathbf{x},t)}{\partial x} = \boldsymbol{\varepsilon} + \boldsymbol{\Omega} \\ J_{ij} = \dfrac{\partial u_i}{\partial x_j} = \underbrace{\dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right)}_{\varepsilon_{ij}} + \underbrace{\dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} - \dfrac{\partial u_j}{\partial x_i}\right)}_{\Omega_{ij}} = \varepsilon_{ij} + \Omega_{ij} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i,j \in \{1,2,3\} \ . \end{array}\right. \tag{3.27}$$

Finally, writing in explicit form the different components in (3.27) and taking into account (3.21), the following is obtained[4].

$$\boxed{\begin{array}{llll} & j=1 & j=2 & j=3 \\ i=1: & \dfrac{\partial u_x}{\partial x} = \varepsilon_{xx} & \dfrac{\partial u_x}{\partial y} = \varepsilon_{xy} - \theta_3 & \dfrac{\partial u_x}{\partial z} = \varepsilon_{xz} + \theta_2 \\ i=2: & \dfrac{\partial u_y}{\partial x} = \varepsilon_{xy} + \theta_3 & \dfrac{\partial u_y}{\partial y} = \varepsilon_{yy} & \dfrac{\partial u_y}{\partial z} = \varepsilon_{yz} - \theta_1 \\ i=3: & \dfrac{\partial u_z}{\partial x} = \varepsilon_{xz} - \theta_2 & \dfrac{\partial u_z}{\partial y} = \varepsilon_{yz} + \theta_1 & \dfrac{\partial u_z}{\partial z} = \varepsilon_{zz} \end{array}} \tag{3.28}$$

3.4.2 Integration of the Strain Field

Consider $\boldsymbol{\varepsilon}(\mathbf{x},t)$ is the infinitesimal strain field one wants to integrate. This operation is performed in two steps:

1) Using (3.24) through (3.26), the *infinitesimal rotation vector* $\boldsymbol{\theta}(\mathbf{x},t)$ *is integrated*. The integration, with respect to space, of the infinitesimal rotation vector in (3.24) through (3.26) leads to a solution of the type

$$\theta_i = \tilde{\theta}_i(x,y,z,t) + c_i(t) \qquad i \in \{1,2,3\} \ , \tag{3.29}$$

where the integration constants $c_i(t)$, which, in general, may be a function of time, can be determined if the value (or the evolution along time) of the infinitesimal rotation vector at some point of the medium is known.

2) Once the infinitesimal strain tensor $\boldsymbol{\varepsilon}(\mathbf{x},t)$ and the infinitesimal rotation vector $\boldsymbol{\theta}(\mathbf{x},t)$ are known, *the displacement field* $\mathbf{u}(\mathbf{x},t)$ *is integrated*. The system of first-order PDEs defined in (3.28) is used, resulting in

$$u_i = \tilde{u}_i(x,y,z,t) + c'_i(t) \qquad i \in \{1,2,3\} \ . \tag{3.30}$$

[4] According to (3.21), $\boldsymbol{\Omega} \overset{not}{\equiv} \begin{bmatrix} 0 & \Omega_{12} & -\Omega_{31} \\ -\Omega_{12} & 0 & \Omega_{23} \\ \Omega_{31} & -\Omega_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix}$.

Again, the integration constants $c'_i(t)$ that appear, which, in general, will be a function of time, are determined when the value (or the evolution along time) of the displacements at some point of space is known.

Remark 3.7. The integration processes in steps 1) and 2) involve integrating systems of first-order PDEs. If the compatibility equations in (3.16) are satisfied, these systems will be integrable (without leading to contradictions in their integration process) and will finally allow obtaining the displacement field.

Remark 3.8. The presence of the integration constants in (3.29) and (3.30) shows that an integrable strain tensor, $\boldsymbol{\varepsilon}(\mathbf{x},t)$, determines the motion of each instant of time except for a rotation $\mathbf{c}(t) \stackrel{not}{=} \hat{\boldsymbol{\theta}}(t)$ and a translation $\mathbf{c}'(t) \stackrel{not}{=} \hat{\mathbf{u}}(t)$.

$$\boldsymbol{\varepsilon}(\mathbf{x},t) \begin{cases} \boldsymbol{\theta}(\mathbf{x},t) = \tilde{\boldsymbol{\theta}}(\mathbf{x},t) + \hat{\boldsymbol{\theta}}(t) \\ \mathbf{u}(\mathbf{x},t) = \tilde{\mathbf{u}}(\mathbf{x},t) + \hat{\mathbf{u}}(t) \end{cases}$$

From these uniform rotation $\hat{\boldsymbol{\theta}}(t)$ and translation $\hat{\mathbf{u}}(t)$ the displacement field

$$\boxed{\mathbf{u}^*(\mathbf{x},t) = \hat{\boldsymbol{\Omega}}(t)\,\mathbf{x} + \hat{\mathbf{u}}(t)} \quad \Longrightarrow \quad \mathbf{u}^* \otimes \nabla = \hat{\boldsymbol{\Omega}}$$

can be defined, which corresponds to a *rigid body motion*[5]. Indeed, the strain associated with this displacement is null,

$$\boldsymbol{\varepsilon}^*(\mathbf{x},t) = \nabla^s \mathbf{u}^* = \frac{1}{2}(\mathbf{u}^* \otimes \nabla + \nabla \otimes \mathbf{u}^*) = \frac{1}{2}\big(\hat{\boldsymbol{\Omega}} + \underbrace{\hat{\boldsymbol{\Omega}}^T}_{-\hat{\boldsymbol{\Omega}}}\big) = \mathbf{0}\,,$$

as corresponds to the concept of rigid body (without deformation). Consequently, it is concluded that *every compatible strain field determines the displacements of the continuous medium except for a rigid body motion*, which must be determined by means of the appropriate boundary conditions.

[5] The rigid body rotation tensor $\hat{\boldsymbol{\Omega}}(t)$ (antisymmetric) is defined based on the rotation vector $\hat{\boldsymbol{\theta}}(t)$ as $\hat{\boldsymbol{\Omega}} \stackrel{not}{\equiv} \begin{bmatrix} 0 & \hat{\Omega}_{12} & -\hat{\Omega}_{31} \\ -\hat{\Omega}_{12} & 0 & \hat{\Omega}_{23} \\ \hat{\Omega}_{31} & -\hat{\Omega}_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\hat{\theta}_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix}$.

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Example 3.1 – *A certain motion is defined by the infinitesimal strain tensor*

$$\boldsymbol{\varepsilon}(\mathbf{x},t) \overset{not}{\equiv} \begin{bmatrix} 8x & -\dfrac{y}{2} & \dfrac{3}{2}x^2z \\ -\dfrac{y}{2} & x & 0 \\ \dfrac{3}{2}x^2z & 0 & x^3 \end{bmatrix} .$$

Obtain the corresponding displacement vector $\mathbf{u}(\mathbf{x},t)$ *and the infinitesimal rotation tensor* $\boldsymbol{\Omega}(\mathbf{x},t)$ *taking into account that* $\mathbf{u}(\mathbf{x},t)\big|_{\mathbf{x}=[0,0,0]^T} \overset{not}{\equiv} [3t,\ 0,\ 0]^T$ *and* $\boldsymbol{\Omega}(\mathbf{x},t)\big|_{\mathbf{x}=[0,0,0]^T} = \mathbf{0}$.

Solution

Infinitesimal rotation vector

Posing the systems of equations defined in (3.24) through (3.26) results in

$$\begin{array}{llll}
\dfrac{\partial\theta_1}{\partial x} = 0 \quad ; & \dfrac{\partial\theta_1}{\partial y} = 0 \ ; & \dfrac{\partial\theta_1}{\partial z} = 0 & \Rightarrow\ \theta_1 = C_1(t)\ , \\
\dfrac{\partial\theta_2}{\partial x} = -3xz \ ; & \dfrac{\partial\theta_2}{\partial y} = 0 \ ; & \dfrac{\partial\theta_2}{\partial z} = -\dfrac{3}{2}x^2 & \Rightarrow\ \theta_2 = -\dfrac{3}{2}x^2z + C_2(t)\ , \\
\dfrac{\partial\theta_3}{\partial x} = 0 \quad ; & \dfrac{\partial\theta_3}{\partial y} = \dfrac{3}{2} \ ; & \dfrac{\partial\theta_3}{\partial z} = 0 & \Rightarrow\ \theta_3 = \dfrac{3}{2}y + C_3(t)\ .
\end{array}$$

The integration constants $C_i(t)$ are determined by imposing that $\boldsymbol{\Omega}(\mathbf{x},t)\big|_{\mathbf{x}=(0,0,0)^T} = \mathbf{0}$ (and, therefore, the infinitesimal rotation vector $\boldsymbol{\theta}(\mathbf{x},t)\big|_{\mathbf{x}=(0,0,0)^T} = \mathbf{0}$), that is,

$$C_1(t) = C_2(t) = C_3(t) = 0 \quad \Longrightarrow \quad \boldsymbol{\theta}(\mathbf{x}) \overset{not}{\equiv} \left[0\ ,\ -\frac{3}{2}x^2z\ ,\ \frac{3}{2}y \right]^T$$

and the infinitesimal rotation tensor is

$$\boldsymbol{\Omega}(\mathbf{x}) \overset{not}{\equiv} \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{3}{2}y & -\dfrac{3}{2}x^2z \\ \dfrac{3}{2}y & 0 & 0 \\ \dfrac{3}{2}x^2z & 0 & 0 \end{bmatrix} .$$

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Displacement vector

Posing, and integrating, the systems of equations in (3.28) produces

$$
\begin{aligned}
&\frac{\partial u_1}{\partial x}=8x \quad ; \quad \frac{\partial u_1}{\partial y}=-2y \; ; \quad \frac{\partial u_1}{\partial z}=0 \quad \Rightarrow\; u_1=4x^2-y^2+C_1'(t)\ ,\\
&\frac{\partial u_2}{\partial x}=y \quad ; \quad \frac{\partial u_2}{\partial y}=x \quad ; \quad \frac{\partial u_2}{\partial z}=0 \quad \Rightarrow\; u_2=xy+C_2'(t)\ ,\\
&\frac{\partial u_3}{\partial x}=3x^2z \; ; \quad \frac{\partial u_3}{\partial y}=0 \quad ; \quad \frac{\partial u_3}{\partial z}=x^3 \quad \Rightarrow\; u_3=x^3z+C_3'(t)\ .
\end{aligned}
$$

and imposing that $\mathbf{u}(\mathbf{x},t)|_{\mathbf{x}=(0,0,0)^T} \stackrel{not}{\equiv} [3t,\ 0,\ 0]^T$ yields

$$
C_1(t)=3t\ ;\ \ C_2(t)=C_3(t)=0 \quad \Longrightarrow \quad \mathbf{u}(\mathbf{x}) \stackrel{not}{\equiv} \left[4x^2-y^2+3t\ ,\ \ xy\ ,\ \ x^3z\right]^T .
$$

3.5 Compatibility Equations and Integration of the Strain Rate Field

Given the definitions of the infinitesimal strain tensor $\boldsymbol{\varepsilon}$, the infinitesimal rotation tensor $\boldsymbol{\Omega}$ and the infinitesimal rotation vector $\boldsymbol{\theta}$, there exists a clear correspondence between these magnitudes and *a)* the strain rate tensor $\mathbf{d}$, *b)* the rotation rate (or spin) tensor $\mathbf{w}$ and *c)* the spin vector $\boldsymbol{\omega}$ given in Chapter 2. These correspondences can be established in the following manner:

$$
\boxed{\begin{array}{c}
\mathbf{u}\\[4pt]
\boldsymbol{\varepsilon}(\mathbf{u})\\[4pt]
\varepsilon_{ij}=\dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j}+\dfrac{\partial u_j}{\partial x_i}\right)\\[8pt]
\Omega_{ij}=\dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j}-\dfrac{\partial u_j}{\partial x_i}\right)\\[8pt]
\boldsymbol{\theta}=\dfrac{1}{2}\nabla\times\mathbf{u}
\end{array}}
\quad \Longleftrightarrow \quad
\boxed{\begin{array}{c}
\mathbf{v}\\[4pt]
\mathbf{d}(\mathbf{v})\\[4pt]
d_{ij}=\dfrac{1}{2}\left(\dfrac{\partial \mathrm{v}_i}{\partial x_j}+\dfrac{\partial \mathrm{v}_j}{\partial x_i}\right)\\[8pt]
w_{ij}=\dfrac{1}{2}\left(\dfrac{\partial \mathrm{v}_i}{\partial x_j}-\dfrac{\partial \mathrm{v}_j}{\partial x_i}\right)\\[8pt]
\boldsymbol{\omega}=\dfrac{1}{2}\nabla\times\mathbf{v}
\end{array}}
\tag{3.31}
$$

Then, it is obvious that the concept of compatibility of a strain field $\boldsymbol{\varepsilon}$ introduced in Section 3.1 can be extended, by virtue of the correspondence with (3.31), to the compatibility of a strain rate field $\mathbf{d}(\mathbf{x},t)$.

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To integrate this field, the same procedure as that seen in Section 3.4.2 can be used, replacing $\boldsymbol{\varepsilon}$ by $\mathbf{d}$, $\mathbf{u}$ by $\mathbf{v}$, $\boldsymbol{\Omega}$ by $\mathbf{w}$ and $\boldsymbol{\theta}$ by $\boldsymbol{\omega}$. Certainly, this integration can only be performed if the compatibility equations in (3.16) are satisfied for the components of $\mathbf{d}(\mathbf{x},t)$.

Remark 3.9. The resulting compatibility equations and the integration process of the strain rate vector $\mathbf{d}(\mathbf{x},t)$ are not, in this case, restricted to the infinitesimal strain case.

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PROBLEMS

Problem 3.1 – *Determine the spatial description of the velocity field that corresponds to the strain rate tensor*

$$\mathbf{d}(\mathbf{x},t) \stackrel{not}{\equiv} \begin{bmatrix} t\mathrm{e}^{tx} & 0 & 0 \\ 0 & 0 & t\mathrm{e}^{y}+1 \\ 0 & t\mathrm{e}^{y}+1 & 0 \end{bmatrix} .$$

For $\mathbf{x}=\mathbf{0}$*,* $\boldsymbol{\omega}_0 \stackrel{not}{\equiv} [t-1,\ 0,\ 0]^T$ *and* $\mathbf{v}_0 \stackrel{not}{\equiv} [t,\ 0,\ t]^T$ *for* $\forall t$ *is satisfied.*

Solution

The problem is solved by integrating the corresponding differential equations, taking into account the existent parallelism between the variables:

$$\left.\begin{matrix} \mathbf{u} \\ \boldsymbol{\varepsilon} \\ \boldsymbol{\theta} \end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix} \mathbf{v} \\ \mathbf{d} \\ \boldsymbol{\omega} \end{matrix}\right.$$

Angular velocity of the rotation vector

$$\begin{array}{llll} \dfrac{\partial \omega_1}{\partial x} = 0 \ ; & \dfrac{\partial \omega_1}{\partial y} = t\mathrm{e}^y \ ; & \dfrac{\partial \omega_1}{\partial z} = 0 & \Rightarrow \quad \omega_1 = C_1(t) + t\mathrm{e}^y \ , \\ \dfrac{\partial \omega_2}{\partial x} = 0 \ ; & \dfrac{\partial \omega_2}{\partial y} = 0 \ ; & \dfrac{\partial \omega_2}{\partial z} = 0 & \Rightarrow \quad \omega_2 = C_2(t) \ , \\ \dfrac{\partial \omega_3}{\partial x} = 0 \ ; & \dfrac{\partial \omega_3}{\partial y} = 0 \ ; & \dfrac{\partial \omega_3}{\partial z} = 0 & \Rightarrow \quad \omega_3 = C_3(t) \ . \end{array}$$

The boundary conditions are imposed for $\mathbf{x}=\mathbf{0}$,

$$\boldsymbol{\omega}_0 \stackrel{not}{\equiv} \begin{bmatrix} t-1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t+C_1 \\ C_2 \\ C_3 \end{bmatrix} \Longrightarrow \begin{cases} C_1 = -1 \\ C_2 = 0 \\ C_3 = 0 \end{cases} ,$$

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and the final result is

$$\boldsymbol{\omega}(\mathbf{x},t) \stackrel{not}{\equiv} \begin{bmatrix} te^{y}-1 \\ 0 \\ 0 \end{bmatrix} .$$

Velocity vector

$$\begin{aligned}
&\frac{\partial \mathrm{v}_1}{\partial x} = te^{tx} \ ; \quad \frac{\partial \mathrm{v}_1}{\partial y} = 0 \ ; \quad \frac{\partial \mathrm{v}_1}{\partial z} = 0 \quad \Rightarrow \quad \mathrm{v}_1 = \mathrm{C}_1^{'}(\mathrm{t}) + \mathrm{e}^{\mathrm{tx}} \ , \\
&\frac{\partial \mathrm{v}_2}{\partial x} = 0 \ ; \quad \frac{\partial \mathrm{v}_2}{\partial y} = 0 \ ; \quad \frac{\partial \mathrm{v}_2}{\partial z} = 2 \quad \Rightarrow \quad \mathrm{v}_2 = \mathrm{C}_2^{'}(\mathrm{t}) + 2\mathrm{z} \ , \\
&\frac{\partial \mathrm{v}_3}{\partial x} = 0 \ ; \quad \frac{\partial \mathrm{v}_3}{\partial y} = 2te^{y} \ ; \quad \frac{\partial \mathrm{v}_3}{\partial z} = 0 \quad \Rightarrow \quad \mathrm{v}_3 = \mathrm{C}_3^{'}(\mathrm{t}) + 2\mathrm{te}^{\mathrm{y}} \ .
\end{aligned}$$

The boundary conditions are imposed for $\mathbf{x} = \mathbf{0}$,

$$\mathbf{v}_0 \stackrel{not}{\equiv} \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 + C_1^{'} \\ C_2^{'} \\ 2t + C_3^{'} \end{bmatrix} \Longrightarrow \begin{cases} C_1^{'} = t - 1 \\ C_2^{'} = 0 \\ C_3^{'} = -t \end{cases} ,$$

and the spatial description of the velocity field is

$$\boxed{\mathbf{v}(\mathbf{x}) \stackrel{not}{\equiv} \begin{bmatrix} e^{tx} + t - 1 \\ 2z \\ 2te^{y} - t \end{bmatrix}} .$$

EXERCISES

3.1 – *Deduce the displacement field that corresponds to the infinitesimal strain tensor*

$$\boldsymbol{\varepsilon}(\mathbf{x},t) \stackrel{not}{\equiv} \begin{bmatrix} 0 & t e^{ty} & 0 \\ t e^{ty} & 0 & 0 \\ 0 & 0 & t e^{tz} \end{bmatrix} .$$

At point $(1,1,1)$ *,* $\mathbf{u} \stackrel{not}{\equiv} [2e^t, \ e^t, \ e^t]^T$ *and* $\boldsymbol{\theta} \stackrel{not}{\equiv} [0, \ 0, \ -te^t]^T$ *is verified.*

3.2 – *Determine the spatial description of the velocity field that corresponds to the strain rate tensor*

$$\mathbf{d}(\mathbf{x},t) \stackrel{not}{\equiv} \begin{bmatrix} 0 & 0 & t e^{tz} \\ 0 & t e^{ty} & 0 \\ t e^{tz} & 0 & 0 \end{bmatrix} .$$

The following is known:

$$\begin{cases} \textit{for } z = 0 : & \mathrm{v_x} = \mathrm{v_z} = 0 \,, \quad \forall \mathrm{t} \,, \quad \mathrm{x,y} \\ \textit{for } y = 1 : & \mathrm{v_y} = 0 \,, \quad \forall \mathrm{t} \,, \quad \mathrm{x,z} \end{cases}$$

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Chapter 4
Stress

4.1 Forces Acting on a Continuum Body

Two types of forces that can act on a continuous medium will be considered: *body forces* and *surface forces*.

4.1.1 Body Forces

> **Definition 4.1.** The *body forces* are the forces that act at a distance on the internal particles of a continuous medium. Examples of this kind of forces are the gravitational, inertial or magnetic attraction forces.

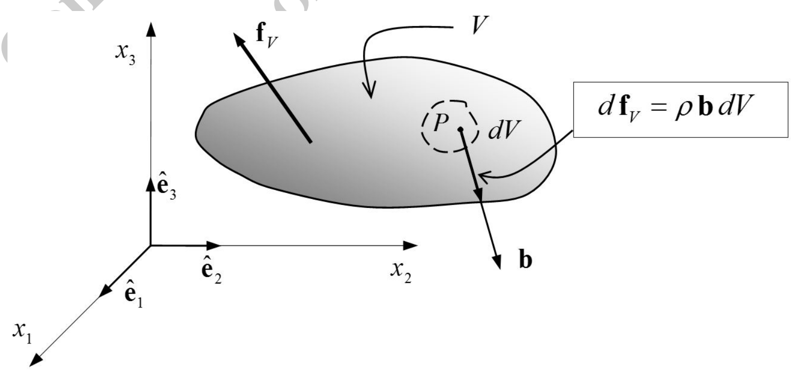


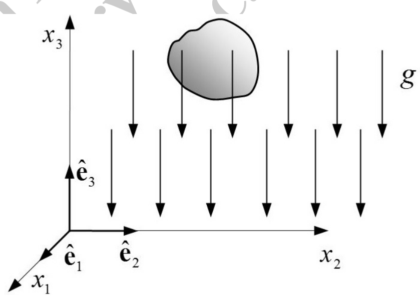
Figure 4.1: Body forces on a continuous medium.

Consider $\mathbf{b}(\mathbf{x},t)$ is the spatial description of the vector field of *body forces per unit of mass*. Multiplying the *vector of body forces* $\mathbf{b}(\mathbf{x},t)$ by the density ρ, the vector of body forces per unit of volume $\rho\mathbf{b}(\mathbf{x},t)$ (*density of body forces*) is obtained. The total resultant, $\mathbf{f}_V$, of the body forces on the material volume V in Figure 4.1 is

$$\mathbf{f}_V = \int\limits_V \rho\mathbf{b}(\mathbf{x},t)\,dV \ . \tag{4.1}$$

> *Remark 4.1.* In the definition of body forces given in (4.1), the existence of the vector *density of body forces* $\rho\mathbf{b}(\mathbf{x},t)$ is implicitly accepted. This means that, given an arbitrary sequence of volumes ΔV_i that contain the particle P, and the corresponding sequence of body forces $\mathbf{f}_{\Delta V_i}$, there exists the limit
>
> $$\rho\mathbf{b}(\mathbf{x},t) = \lim_{\Delta V_i \to 0} \frac{\mathbf{f}_{\Delta V_i}}{\Delta V_i}$$
>
> and, in addition, it is independent of the sequence of volumes considered.

Example 4.1 – *Given a continuous medium with volume V placed on the Earth's surface, obtain the value of the total resultant of the body forces in terms of the gravitational constant g.*



Solution

Assuming a system of Cartesian axes (see figure above) such that the x_3-axis is in the direction of the vertical from the center of the Earth, the vector field $\mathbf{b}(\mathbf{x},t)$ of gravitational force per unit of mass is

$$\mathbf{b}(\mathbf{x},t) \overset{not}{\equiv} [0\ ,\ 0\ ,\ -g]^T$$

and, finally, the vector of body forces is

$$\mathbf{f}_V = \int\limits_V \rho\mathbf{b}(\mathbf{x},t)\,dV \overset{not}{\equiv} \left[0\ ,\ 0\ ,\ -\int\limits_V \rho g\,dV\right]^T .$$

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4.1.2 Surface Forces

Definition 4.2. The *surface forces* are the forces that act on the *boundary* of the material volume considered. They can be regarded as produced by the *contact actions* of the particles located in the boundary of the medium with the exterior of this medium.

Consider the spatial description of the vector field of surface forces *per unit of surface* $\mathbf{t}(\mathbf{x},t)$ on the continuous medium shown in Figure 4.2. The resultant force on a differential surface element dS is $\mathbf{t}\,dS$ and the total resultant of the surface forces acting on the boundary ∂V of volume V can be written as

$$\mathbf{f}_S = \int_{\partial V} \mathbf{t}(\mathbf{x},t)\,dS \,. \tag{4.2}$$

Remark 4.2. In the definition of surface forces given in (4.2), the existence of the vector of surface forces per unit of surface $\mathbf{t}(\mathbf{x},t)$ (traction vector[1]) is implicitly accepted. In other words, if a sequence of surfaces ΔS_i, each containing point P, and the corresponding surface forces $\mathbf{f}_{\Delta S_i}$ are considered (see Figure 4.3), there exists the limit

$$\mathbf{t}(\mathbf{x},t) = \lim_{\Delta S_i \to 0} \frac{\mathbf{f}_{\Delta S_i}}{\Delta S_i}$$

and it is independent of the chosen sequence of surfaces.

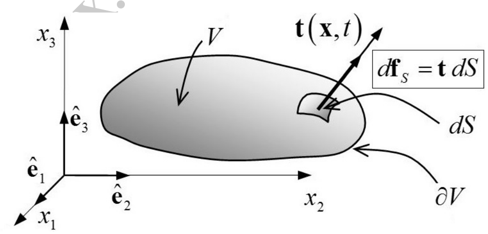

Figure 4.2: Surface forces on a continuous medium.

[1] In literature, the vector of surface forces per unit of surface $\mathbf{t}(\mathbf{x},t)$ is often termed *traction vector*, although this concept can be extended to points in the interior of the continuous medium.

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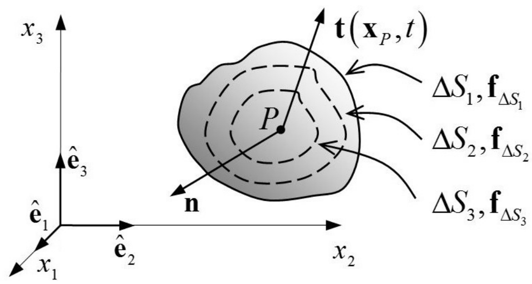

Figure 4.3: Traction vector.

4.2 Cauchy's Postulates

Consider a continuous medium on which body and surface forces are acting (see Figure 4.4). Consider also a particle *P* in the interior of the continuous medium and an arbitrary surface containing point *P* and with a unit normal vector **n** at this point, which divides the continuous medium into two parts (material volumes). The surface forces due to the contact between volumes will act on the imaginary separating surface, considered now a part of the boundary of each of these material volumes.

Consider the traction vector **t** that acts at the chosen point *P* as part of the boundary of the first material volume. In principle, this traction vector (defined now at a material point belonging to the interior of the original continuous medium) will depend on

1) the particle being considered,
2) the orientation of the surface (defined by means of the normal **n**) and
3) the separating surface itself.

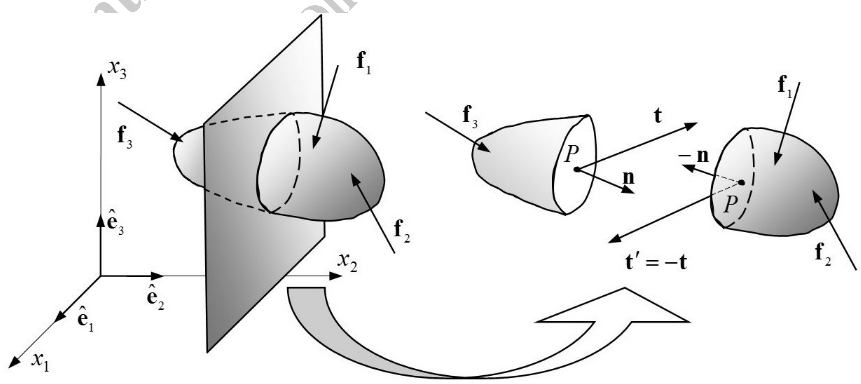

Figure 4.4: Cauchy's postulates.

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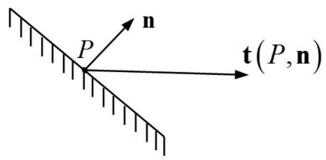
The following postulate[2] makes it independent of this last condition.

Definition 4.3. *Cauchy's* 1^{st} *postulate* establishes that the traction vector that acts at a material point P of a continuous medium according to a plane with unit normal vector $\mathbf{n}$ depends only on the point P and the normal $\mathbf{n}$.



$$\mathbf{t} = \mathbf{t}(P, \mathbf{n})$$

Remark 4.3. Consider a particle P of a continuous medium and different surfaces that contain this point P such that they all have the same unit normal vector $\mathbf{n}$ at said point. In accordance with Cauchy's postulate, the traction vectors at point P, according to each of these surfaces, coincide. On the contrary, if the normal to the surfaces at P is different, the corresponding traction vectors will not coincide (see Figure 4.5).

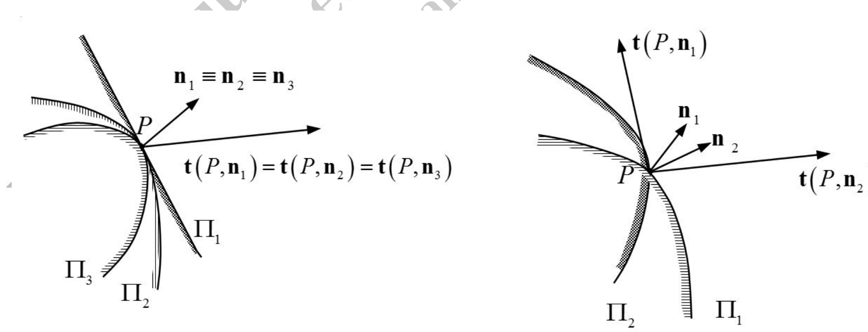


Figure 4.5: Traction vector at a point according to different surfaces.

[2] A *postulate* is a fundamental ingredient of a theory that is formulated as a principle of this theory and, as such, does not need proof.

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Definition 4.4. *Cauchy's 2^{nd} postulate - action and reaction law* establishes the traction vector at point P of a continuous medium, according to a plane with unit normal vector $\mathbf{n}$, has the same magnitude and opposite direction to the traction vector at the same point P according to a plane with unit normal vector $-\mathbf{n}$ at the same point (see Figure 4.4).

$$\mathbf{t}(P,\mathbf{n}) = -\mathbf{t}(P,-\mathbf{n})$$

4.3 Stress Tensor

4.3.1 Application of Newton's 2^{nd} Law to a Continuous Medium

Consider a discrete system of particles in motion such that a generic particle i of this system has mass m_i, velocity $\mathbf{v}_i$ and acceleration $\mathbf{a}_i = d\mathbf{v}_i/dt$. In addition, a force $\mathbf{f}_i$ acts on each particle i, which is related to the particle's acceleration through Newton's second law[3],

$$\mathbf{f}_i = m_i \mathbf{a}_i \ . \tag{4.3}$$

Then, the resultant $\mathbf{R}$ of the forces that act on all the particles of the system is

$$\mathbf{R} = \sum_i \mathbf{f}_i = \sum_i m_i \mathbf{a}_i \ . \tag{4.4}$$

The previous concepts can be generalized for the case of continuous mediums when these are understood as discrete systems constituted by an infinite number of particles. In this case, the application of Newton's second law to a continuous medium with total mass M, on which external forces characterized by the vector density of body forces $\rho\mathbf{b}(\mathbf{x},t)$ and the traction vector $\mathbf{t}(\mathbf{x},t)$ are acting, whose particles have an acceleration $\mathbf{a}(\mathbf{x},t)$, and that occupies at time t the space volume V_t results in

$$\mathbf{R} = \underbrace{\int_{V_t} \rho\mathbf{b}\, dV}_{\substack{\text{Resultant of}\\ \text{the body}\\ \text{forces}}} + \underbrace{\int_{\partial V_t} \mathbf{t}\, dS}_{\substack{\text{Resultant of}\\ \text{the surface}\\ \text{forces}}} = \int_M \mathbf{a}\, \underbrace{dm}_{\rho dV} = \int_{V_t} \rho\mathbf{a}\, dV \ . \tag{4.5}$$

[3] The Einstein notation introduced in (1.1) is not used here.

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4.3.2 Stress Tensor

Consider now the particular case of a material volume constituted by an elemental tetrahedron placed in the neighborhood of an arbitrary particle P of the interior of the continuous medium and oriented according to the scheme in Figure 4.6. Without loss of generality, the origin of coordinates can be placed at P.

The tetrahedron has a vertex at P and its faces are completely defined by means of a plane with normal $\mathbf{n} = [n_1, n_2, n_3]^T$ that intersects with the coordinate planes, defining a generic surface with area S (the base of the tetrahedron) at a distance h (the height of the tetrahedron) of point P. In turn, the coordinate planes define the other faces of the tetrahedron with areas S_1, S_2 and S_3, and (outward) normals $-\hat{\mathbf{e}}_1$, $-\hat{\mathbf{e}}_2$ and $-\hat{\mathbf{e}}_3$, respectively. Through geometric considerations, the relations

$$S_1 = n_1 S \qquad S_2 = n_2 S \qquad S_3 = n_3 S \tag{4.6}$$

can be established. The notation for the traction vectors on each of the faces of the tetrahedron is introduced in Figure 4.7 as well as the corresponding normals with which they are associated.

According to Cauchy's second postulate (see Definition 4.4), the traction vector on a generic point $\mathbf{x}$ belonging to one of the surfaces S_i (with outward normal $-\hat{\mathbf{e}}_i$) can be written as

$$\mathbf{t}(\mathbf{x}, -\hat{\mathbf{e}}_i) = -\mathbf{t}(\mathbf{x}, \hat{\mathbf{e}}_i) \overset{not}{=} -\mathbf{t}^{(i)}(\mathbf{x}) \qquad i \in \{1,2,3\} \ . \tag{4.7}$$

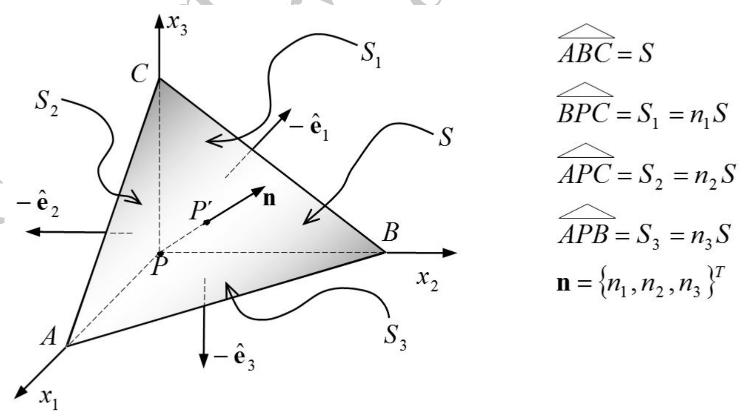

Figure 4.6: Elemental tetrahedron in the neighborhood of a material point P.

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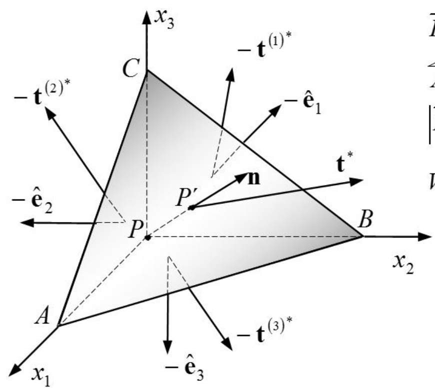

Figure 4.7: Traction vectors on an elemental tetrahedron.

Remark 4.4. The *mean value theorem* establishes that, given a (scalar, vectorial o tensorial) function that is *continuous* in the interior of a (compact) domain, the function reaches its mean value *in the interior* of said domain. In mathematical terms, given $f(\mathbf{x})$ continuous in Ω,

$$\exists\, \mathbf{x}^* \in \Omega \mid \int_{\Omega} f(\mathbf{x})\, d\Omega = \Omega \cdot f(\mathbf{x}^*)$$

where $f(\mathbf{x}^*)$ is the mean value of f in Ω. Figure 4.8 shows the graphical interpretation of the mean value theorem in one dimension.

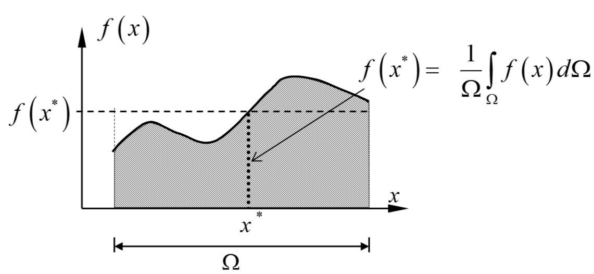

Figure 4.8: Mean value theorem.

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In virtue of the mean value theorem, the vector field $\mathbf{t}^{(i)}(\mathbf{x})$, assumed to be continuous in the domain S_i, attains its mean value in the interior of this domain. Let $\mathbf{x}^*_{s_I} \in S_i$ be the point where the mean value is reached and $\mathbf{t}^{(i)^*} = \mathbf{t}^{(i)}\left(\mathbf{x}^*_{s_I}\right)$ this mean value. Analogously, the vectors $\mathbf{t}^* = \mathbf{t}\left(\mathbf{x}^*_S\right)$, $\rho^*\,\mathbf{b}^* = \rho\left(\mathbf{x}^*_V\right)\,\mathbf{b}\left(\mathbf{x}^*_V\right)$ and $\rho^*\mathbf{a}^* = \rho\left(\mathbf{x}^*_V\right)\,\mathbf{a}\left(\mathbf{x}^*_V\right)$ are the mean values corresponding to the vector fields: traction vector $\mathbf{t}(\mathbf{x})$ in S, density of body forces $\rho\mathbf{b}(\mathbf{x})$ and inertial forces $\rho\mathbf{a}(\mathbf{x})$, respectively. These mean values are attained, again according to the mean value theorem, at points $\mathbf{x}^*_s \in S$ and $\mathbf{x}^*_V \in V$ *of the interior* of the corresponding domains. Therefore, one can write

$$\begin{aligned} &\int\limits_{S_i} \mathbf{t}^{(i)}(\mathbf{x})\,dS = \mathbf{t}^{(i)^*} S_i \qquad i \in \{1,2,3\}\,, \qquad \int\limits_{S} \mathbf{t}(\mathbf{x})\,dS = \mathbf{t}^* S\,, \\ &\int\limits_{V} \rho(\mathbf{x})\,\mathbf{b}(\mathbf{x})\,dV = \rho^*\mathbf{b}^* V \quad \text{and} \quad \int\limits_{V} \rho(\mathbf{x})\,\mathbf{a}(\mathbf{x})\,dV = \rho^*\mathbf{a}^* V\,. \end{aligned} \tag{4.8}$$

Applying now (4.5) on the tetrahedron considered, results in

$$\begin{aligned} &\int\limits_V \rho\mathbf{b}\,dV + \int\limits_S \mathbf{t}\,dS + \int\limits_{S_1} \mathbf{t}\,dS + \int\limits_{S_2} \mathbf{t}\,dS + \int\limits_{S_3} \mathbf{t}\,dS = \\ &= \int\limits_V \rho\mathbf{b}\,dV + \int\limits_S \mathbf{t}\,dS + \int\limits_{S_1} -\mathbf{t}^{(1)}dS + \int\limits_{S_2} -\mathbf{t}^{(2)}dS + \int\limits_{S_3} -\mathbf{t}^{(3)}dS = \int\limits_V \rho\mathbf{a}\,dV, \end{aligned} \tag{4.9}$$

where (4.7) has been taken into account. Replacing (4.8) in (4.9), the latter can be written in terms of the mean values as

$$\rho^*\,\mathbf{b}^*\,V + \mathbf{t}^* S - \mathbf{t}^{(1)^*} S_1 - \mathbf{t}^{(2)^*} S_2 - \mathbf{t}^{(3)^*} S_3 = \rho^*\,\mathbf{a}^*\,V\,. \tag{4.10}$$

Introducing now (4.6) and expressing the total volume of the tetrahedron as $V = Sh/3$, the equation above becomes

$$\begin{aligned} &\frac{1}{3}\rho^*\,\mathbf{b}^*\,hS + \mathbf{t}^* S - \mathbf{t}^{(1)^*} n_1 S - \mathbf{t}^{(2)^*} n_2 S - \mathbf{t}^{(3)^*} n_3 S = \frac{1}{3}\rho^*\,\mathbf{a}^*\,hS \quad \Longrightarrow \\ &\frac{1}{3}\rho^*\,\mathbf{b}^*\,h + \mathbf{t}^* - \mathbf{t}^{(1)^*} n_1 - \mathbf{t}^{(2)^*} n_2 - \mathbf{t}^{(3)^*} n_3 = \frac{1}{3}\rho^*\,\mathbf{a}^*\,h\,. \end{aligned} \tag{4.11}$$

Expression (4.11) is valid for *any* tetrahedron defined by a plane with unit normal vector $\mathbf{n}$ placed at a distance h of point P. Consider now an infinitesimal tetrahedron, also in the neighborhood of point P, by making the value of $\left|\overline{PP'}\right| = h$ tend to zero but maintaining the orientation of the plane constant ($\mathbf{n}$=constant). Then, the domains S_i, S and V in (4.11) collapse into point P (see Figure 4.7). Therefore, the points of the corresponding domains in which the

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mean values are obtained also tend to point P,

$$\begin{aligned}
&\mathbf{x}^*_{S_i} \to \mathbf{x}_P \quad \Longrightarrow \quad \lim_{h\to 0} \mathbf{t}^{(i)^*}\left(\mathbf{x}^*_{S_i}\right) = \mathbf{t}^{(i)}(P) \qquad i \in \{1,2,3\} \ , \\
&\mathbf{x}^*_{S} \to \mathbf{x}_P \quad \Longrightarrow \quad \lim_{h\to 0} \mathbf{t}^*\left(\mathbf{x}^*_{S}, \mathbf{n}\right) = \mathbf{t}(P,\mathbf{n}) \ ,
\end{aligned} \tag{4.12}$$

and, in addition,

$$\lim_{h\to 0}\left(\frac{1}{3}\rho^*\,\mathbf{b}^*\,h\right) = \lim_{h\to 0}\left(\frac{1}{3}\rho^*\,\mathbf{a}^*\,h\right) = \mathbf{0}\,. \tag{4.13}$$

Taking the limit of (4.11) and replacing expressions (4.12) and (4.13) in it leads to

$$\mathbf{t}(P,\mathbf{n}) - \mathbf{t}^{(1)}\mathbf{n}_1 - \mathbf{t}^{(2)}\mathbf{n}_2 - \mathbf{t}^{(3)}\mathbf{n}_3 = 0 \quad \Longrightarrow \quad \mathbf{t}(P,\mathbf{n}) - \mathbf{t}^{(i)}\mathbf{n}_i = \mathbf{0}\,. \tag{4.14}$$

The traction vector $\mathbf{t}^{(1)}$ can be written in terms of its corresponding Cartesian components (see Figure 4.9) as

$$\mathbf{t}^{(1)} = \sigma_{11}\hat{\mathbf{e}}_1 + \sigma_{12}\hat{\mathbf{e}}_2 + \sigma_{13}\hat{\mathbf{e}}_3 = \sigma_{1i}\hat{\mathbf{e}}_i\,. \tag{4.15}$$

Operating in an analogous manner on traction vectors $\mathbf{t}^{(2)}$ and $\mathbf{t}^{(3)}$ (see Figure 4.10) results in

$$\mathbf{t}^{(2)} = \sigma_{21}\hat{\mathbf{e}}_1 + \sigma_{22}\hat{\mathbf{e}}_2 + \sigma_{23}\hat{\mathbf{e}}_3 = \sigma_{2i}\hat{\mathbf{e}}_i \tag{4.16}$$

$$\mathbf{t}^{(3)} = \sigma_{31}\hat{\mathbf{e}}_1 + \sigma_{32}\hat{\mathbf{e}}_2 + \sigma_{33}\hat{\mathbf{e}}_3 = \sigma_{3i}\hat{\mathbf{e}}_i \tag{4.17}$$

and, for the general case,

$$\mathbf{t}^{(i)}(P) = \sigma_{ij}\,\hat{\mathbf{e}}_j \qquad i,j \in \{1,2,3\}\ . \tag{4.18}$$

$$\boxed{\sigma_{ij}(P) = t^{(i)}_j(P) \qquad i,j \in \{1,2,3\}} \tag{4.19}$$

Remark 4.5. Note that in expression (4.19) the functions σ_{ij} are functions of (the components of) the traction vectors $t^{(i)}_j(P)$ *on the surfaces specifically oriented* at point P. Thus, it is emphasized that these functions depend on point P but not on the unit normal vector $\mathbf{n}$.

$$\sigma_{ij} = \sigma_{ij}(P)$$

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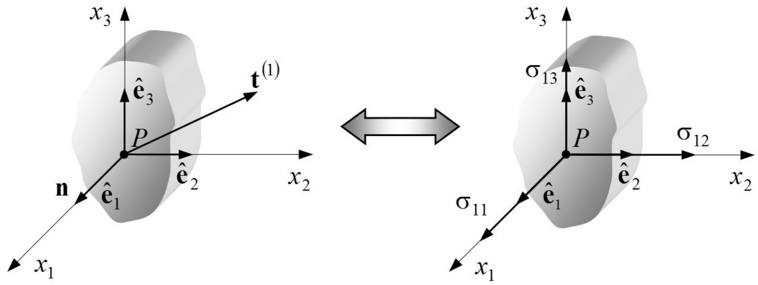

Figure 4.9: Decomposition of the traction vector $\mathbf{t}^{(1)}$ into its components.

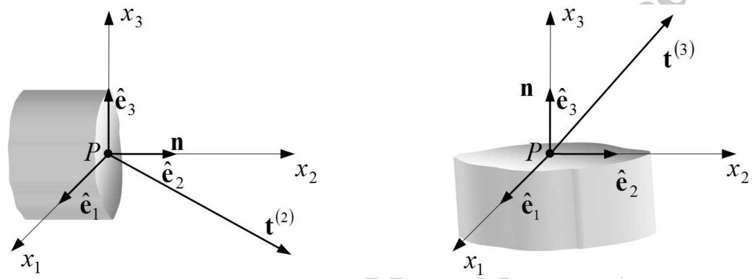

Figure 4.10: Traction vectors $\mathbf{t}^{(2)}$ and $\mathbf{t}^{(3)}$.

Replacing (4.19) in (4.14) yields

$$\mathbf{t}(P,\mathbf{n}) = n_i \mathbf{t}^{(i)} \implies t_j(P,\mathbf{n}) = n_i t_j^{(i)}(P) = n_i \sigma_{ij}(P) \quad i,j \in \{1,2,3\} \implies$$

$$\boxed{\mathbf{t}(P,\mathbf{n}) = \mathbf{n} \cdot \boldsymbol{\sigma}(P)} \tag{4.20}$$

where the *Cauchy stress tensor* $\boldsymbol{\sigma}$ is defined as

$$\boldsymbol{\sigma} = \sigma_{ij}\, \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \ . \tag{4.21}$$

> *Remark 4.6.* Note that expression (4.20) is consistent with Cauchy's first postulate (see Definition 4.3) and that the second postulate (see Definition 4.4) is satisfied from
>
> $$\left.\begin{aligned} \mathbf{t}(P,\mathbf{n}) &= \mathbf{n} \cdot \boldsymbol{\sigma} \\ \mathbf{t}(P,-\mathbf{n}) &= -\mathbf{n} \cdot \boldsymbol{\sigma} \end{aligned}\right\} \implies \mathbf{t}(P,\mathbf{n}) = -\mathbf{t}(P,-\mathbf{n}) \ .$$

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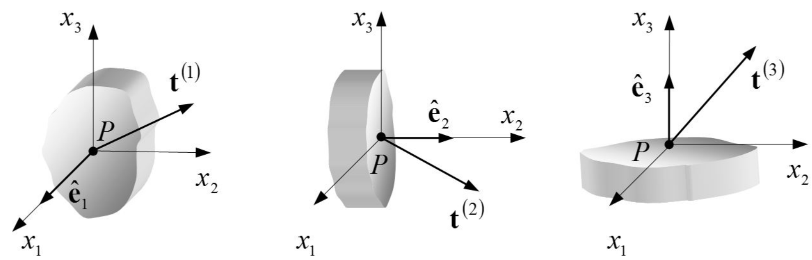

Figure 4.11: Traction vectors for the construction of the Cauchy stress tensor.

> *Remark 4.7.* In accordance with (4.18) and (4.21), the Cauchy stress tensor is constructed from the traction vectors according to three coordinate planes that include point P (see Figure 4.11). However, by means of (4.20), the stress tensor $\boldsymbol{\sigma}(P)$ is seen to contain information on the traction vectors corresponding to *any plane* (identified by its normal $\mathbf{n}$) *that contains this point*.

4.3.3 Graphical Representation of the Stress State in a Point

It is common to resort to graphical representations of the stress tensor based on elemental parallelepipeds in the neighborhood of the particle considered, with faces oriented in accordance to the Cartesian planes and in which the corresponding traction vectors are decomposed into their normal and tangent components following expressions (4.15) through (4.20) (see Figure 4.12).

4.3.3.1 Scientific Notation

The representation in Figure 4.12 corresponds to what is known as *scientific notation*. In this notation, the matrix of components of the stress tensor is written as

$$\boldsymbol{\sigma} \stackrel{not}{\equiv} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{4.22}$$

and each component σ_{ij} can be characterized in terms of its indices:

- Index i indicates the plane on which the stress acts (plane perpendicular to the x_i-axis).
- Index j indicates the direction of the stress (direction of the x_j-axis).

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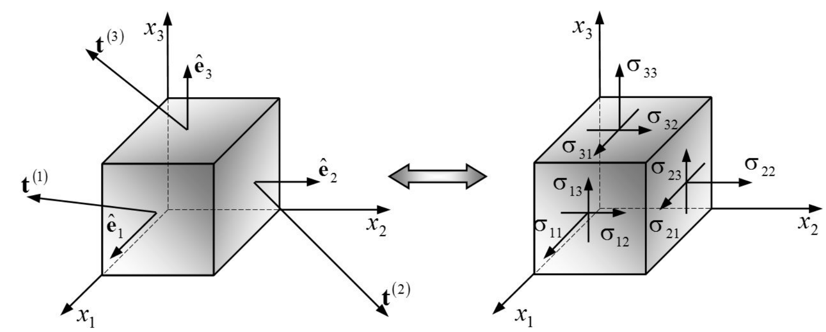

Figure 4.12: Graphical representation of the stress tensor (scientific notation).

4.3.3.2 Engineering Notation

In engineering notation, the components of the Cauchy stress tensor (see Figure 4.13) are written as

$$\boldsymbol{\sigma} \stackrel{not}{\equiv} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{bmatrix} \tag{4.23}$$

and each component can be characterized as follows:

- The component σ_a is the *normal* stress acting on the plane perpendicular to the a-axis.
- The component τ_{ab} is the *tangential* (shear) stress acting on the plane perpendicular to the a-axis in the direction of the b-axis.

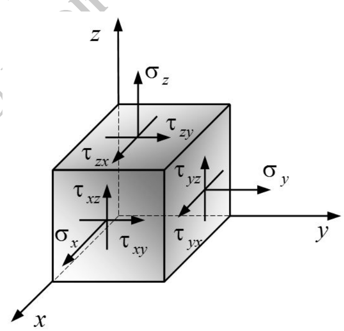

Figure 4.13: Graphical representation of the stress tensor (engineering notation).

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4.3.3.3 Sign Criterion

Consider a particle P of the continuous medium and a plane with unit normal vector $\mathbf{n}$ that contains this particle (see Figure 4.14). The corresponding traction vector $\mathbf{t}$ can be decomposed into its normal component $\boldsymbol{\sigma}_n$ and its tangential component $\boldsymbol{\tau}_n$. The sign of the projection of $\mathbf{t}$ on $\mathbf{n}$ ($\sigma = \mathbf{t} \cdot \mathbf{n}$) defines the *tensile* ($\boldsymbol{\sigma}_n$ tends to *pull* on the plane) or *compressive* ($\boldsymbol{\sigma}_n$ tends to *compress* the plane) character of the normal component.

This concept can be used to define the sign of the components of the stress tensor. For this purpose, in the elemental parallelepiped of Figure 4.12, the distinction is made between the positive or *visible faces* (its outward normal has the same direction as the positive base vector and the faces *can be seen* in the figure) and the negative or *hidden faces*.

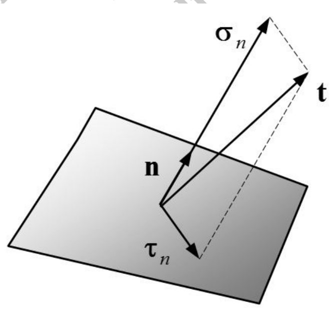
The sign criterion for the visible faces is

$$\text{Normal stresses} \quad \boxed{\sigma_{ij} \text{ or } \sigma_a \begin{cases} \textit{positive } (+) \Rightarrow \text{ tension} \\ \textit{negative } (-) \Rightarrow \text{ compression} \end{cases}} \qquad \text{and}$$

$$\text{Tangential stresses} \quad \boxed{\tau_{ab} \begin{cases} \textit{positive } (+) \Rightarrow \text{ direction of } b\text{-axis} \\ \textit{negative } (-) \Rightarrow \text{ opposite direction to } b\text{-axis} \end{cases}}$$

In accordance with this criterion, the directions of the stresses represented in Figure 4.13 (on the visible faces of the parallelepiped) correspond to positive values of the respective components of the stress tensor[4].

In virtue of the action and reaction law (see Definition 4.4) and for the hidden faces of the parallelepiped, the aforementioned positive values of the components of the stress tensor correspond to opposite directions in their graphical representation (see Figure 4.15).



$$\boldsymbol{\sigma}_n = \sigma \mathbf{n}$$

$$\boxed{\sigma = \mathbf{t} \cdot \mathbf{n} \begin{cases} > 0 \text{ tension} \\ < 0 \text{ compression} \end{cases}}$$

Figure 4.14: Decomposition of the traction vector.

[4] It is obvious that the negative values of the components of the stress tensor will result in graphical representations of opposite direction to the positive values indicated in the figures.

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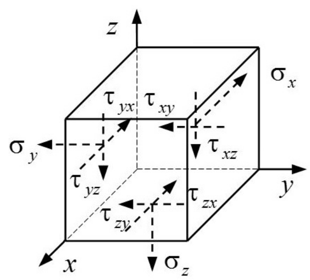

Figure 4.15: Positive stresses in the hidden faces.

4.4 Properties of the Stress Tensor

Consider an arbitrary material volume V in a continuous medium and its boundary ∂V. The body forces $\mathbf{b}(\mathbf{x},t)$ act on V and the prescribed traction vector $\mathbf{t}^*(\mathbf{x},t)$ acts on ∂V. The acceleration vector field of the particles is $\mathbf{a}(\mathbf{x},t)$ and the Cauchy stress tensor field is $\boldsymbol{\sigma}(\mathbf{x},t)$ (see Figure 4.16).

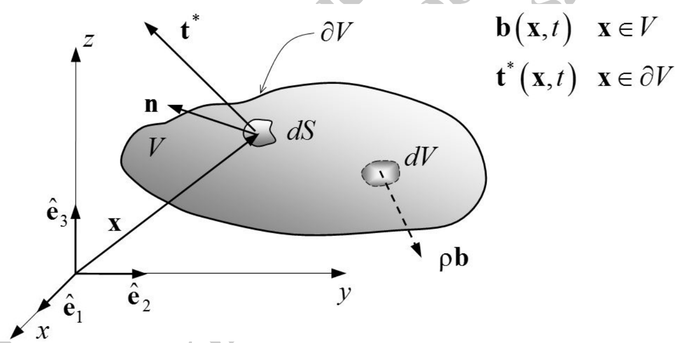

Figure 4.16: Forces acting on a continuous medium.

4.4.1 Cauchy Equation. Internal Equilibrium Equation

The stress tensor, the body forces and the accelerations are related through *Cauchy's equation*,

$$\boxed{\text{Cauchy's equation} \quad \begin{cases} \nabla\cdot\boldsymbol{\sigma}+\rho\mathbf{b}=\rho\mathbf{a} & \forall\mathbf{x}\in V \\ \dfrac{\partial\sigma_{ij}}{\partial x_i}+\rho b_j=\rho a_j & j\in\{1,2,3\} \end{cases}} \tag{4.24}$$

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whose explicit expression in engineering notation is

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} + \rho b_x = \rho a_x \ , \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} + \rho b_y = \rho a_y \ , \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + \rho b_z = \rho a_z \ . \end{cases} \tag{4.25}$$

If the system is in equilibrium, the acceleration is null ($\mathbf{a} = \mathbf{0}$), and (4.24) is reduced to

$$\boxed{\begin{array}{c}\text{Internal} \\ \text{equilibrium} \\ \text{equation}\end{array} \begin{cases} \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \mathbf{0} & \forall \mathbf{x} \in V \\ \dfrac{\partial \sigma_{ij}}{\partial x_i} + \rho b_j = 0 & j \in \{1,2,3\} \end{cases}} \tag{4.26}$$

which is known as the *internal equilibrium equation* of the continuous medium.

Cauchy's equation of motion is derived from the *principle of balance of linear momentum*, which will be studied in Chapter 5.

4.4.2 Equilibrium Equation at the Boundary

Equation (4.20) is applied on the boundary points taking into account that the traction vector is now known in said points ($\mathbf{t} = \mathbf{t}^*$). The result is denoted as *equilibrium equation at the boundary*.

$$\boxed{\begin{array}{c}\text{Equilibrium} \\ \text{equation at} \\ \text{the boundary}\end{array} \begin{cases} \mathbf{n}(\mathbf{x},t) \cdot \boldsymbol{\sigma}(\mathbf{x},t) = \mathbf{t}^*(\mathbf{x},t) & \forall \mathbf{x} \in \partial V \\ n_i \ \sigma_{ij} = t_j^* & j \in \{1,2,3\} \end{cases}} \tag{4.27}$$

4.4.3 Symmetry of the Cauchy Stress Tensor

The Cauchy stress tensor is proven to be *symmetric* by applying the principle of balance of angular momentum (see Chapter 5).

$$\begin{cases} \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \\ \sigma_{ij} = \sigma_{ji} \qquad i,j \in \{1,2,3\} \end{cases} \tag{4.28}$$

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Remark 4.8. The symmetry of the stress tensor allows the Cauchy's equation (4.24) and the equilibrium equation at the boundary (4.27) to be written, respectively, as

$$\begin{cases} \nabla\cdot\boldsymbol{\sigma}+\rho\mathbf{b}=\boldsymbol{\sigma}\cdot\nabla+\rho\mathbf{b}=\rho\mathbf{a} & \forall\mathbf{x}\in V \\ \dfrac{\partial\sigma_{ij}}{\partial x_i}+\rho b_j=\dfrac{\partial\sigma_{ji}}{\partial x_i}+\rho b_j=\rho a_j & j\in\{1,2,3\} \end{cases}$$

$$\begin{cases} \mathbf{n}\cdot\boldsymbol{\sigma}=\boldsymbol{\sigma}\cdot\mathbf{n}=\mathbf{t}^*(\mathbf{x},t) & \forall\mathbf{x}\in\partial V \\ n_i\,\sigma_{ij}=\sigma_{ji}\,n_i=t_j^* & j\in\{1,2,3\} \end{cases}$$

Example 4.2 – *A continuous medium moves with a velocity field whose spatial description is* $\mathbf{v}(\mathbf{x},t)\overset{not}{\equiv}[z,\ x,\ y]^T$. *The Cauchy stress tensor is*

$$\boldsymbol{\sigma}\overset{not}{\equiv}\begin{bmatrix} y & g(x,z,t) & 0 \\ h(y) & z(1+t) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Determine the functions g, h *and the spatial form of the body forces* $\mathbf{b}(\mathbf{x},t)$ *that generate the motion.*

Solution

The stress tensor is symmetric, therefore

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}^T \quad\Longrightarrow\quad h(y)=g(x,z,t) \quad\Longrightarrow\quad \begin{cases} h(y)=C\,, \\ g(x,z,t)=C\,, \end{cases}$$

where C is a constant. In addition, the divergence of the tensor is null,

$$\nabla\cdot\boldsymbol{\sigma}\overset{not}{\equiv}\left[\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right]\begin{bmatrix} y & C & 0 \\ C & z(1+t) & 0 \\ 0 & 0 & 0 \end{bmatrix}=[0,\ 0,\ 0]\ .$$

Thus, Cauchy's equation is reduced to

$$\left.\begin{array}{l} \nabla\cdot\boldsymbol{\sigma}+\rho\mathbf{b}=\rho\mathbf{a} \\ \nabla\cdot\boldsymbol{\sigma}=\mathbf{0} \end{array}\right\}\quad\Longrightarrow\quad \mathbf{b}=\mathbf{a}\ .$$

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Applying the expression for the material derivative of velocity,

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v} \qquad \text{with}$$

$$\frac{\partial \mathbf{v}}{\partial t}=\mathbf{0} \qquad \text{and} \qquad \nabla\mathbf{v}=\nabla\otimes\mathbf{v}\overset{not}{\equiv}\begin{bmatrix}\dfrac{\partial}{\partial x}\\ \dfrac{\partial}{\partial y}\\ \dfrac{\partial}{\partial z}\end{bmatrix}[z,\ x,\ y]=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}.$$

the acceleration

$$\mathbf{a}=\mathbf{v}\cdot\nabla\mathbf{v}\overset{not}{\equiv}[z,\ x,\ y]\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}=[y,\ z,\ x]$$

is obtained. Finally, the body forces are

$$\boxed{\mathbf{b}(\mathbf{x},t)=\mathbf{a}(\mathbf{x},t)\overset{not}{\equiv}[y,\ z,\ x]^T}.$$

4.4.4 Diagonalization. Principal Stresses and Directions

Consider the stress tensor $\boldsymbol{\sigma}$. Since it is a symmetric second-order tensor, it diagonalizes[5] in an orthonormal basis and its eigenvalues are real. Consider, then, its matrix of components in the Cartesian basis $\{x,\ y,\ z\}$ (see Figure 4.17),

$$\boldsymbol{\sigma}\overset{not}{\equiv}\begin{bmatrix}\sigma_x&\tau_{xy}&\tau_{xz}\\ \tau_{yx}&\sigma_y&\tau_{yz}\\ \tau_{zx}&\tau_{yz}&\sigma_z\end{bmatrix}_{\{x,\,y,\,z\}}. \tag{4.29}$$

In the Cartesian system $\{x',\ y',\ z'\}$ in which $\boldsymbol{\sigma}$ diagonalizes, its matrix of components will be

$$\boldsymbol{\sigma}\overset{not}{\equiv}\begin{bmatrix}\sigma_1&0&0\\0&\sigma_2&0\\0&0&\sigma_3\end{bmatrix}_{\{x',\,y',\,z'\}}. \tag{4.30}$$

[5] A theorem of tensor algebra guarantees that all symmetric second-order tensor diagonalizes in an orthonormal basis and its eigenvalues are real.

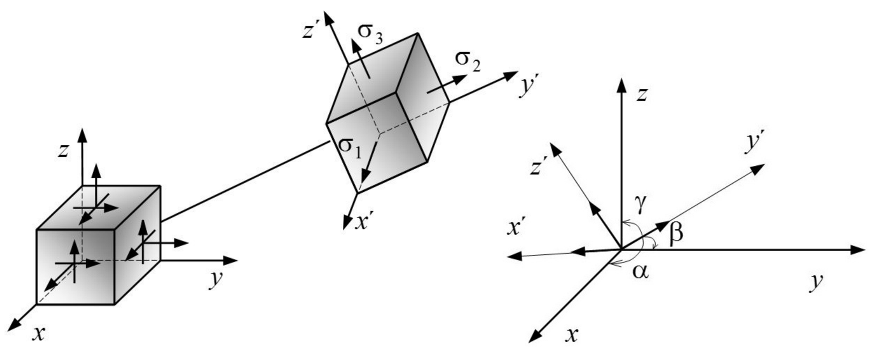

Figure 4.17: Diagonalization of the stress tensor.

Definition 4.5. The *principal stress directions* are the directions, associated with the axes $\{x', y', z'\}$, in which the stress tensor diagonalizes.
The *principal stresses* are the eigenvalues of the stress tensor $(\sigma_1,\ \sigma_2,\ \sigma_3)$. In general, they will be assumed to be arranged in the form $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

To obtain the principal stress directions and the principal stresses, the eigenvalue problem associated with tensor $\boldsymbol{\sigma}$ must be posed. That is, if λ and $\mathbf{v}$ are an eigenvalue and its corresponding eigenvector, respectively, then

$$\boldsymbol{\sigma}\cdot\mathbf{v} = \lambda\,\mathbf{v} \quad \Longrightarrow \quad (\boldsymbol{\sigma} - \lambda\mathbf{1})\cdot\mathbf{v} = \mathbf{0}\ . \tag{4.31}$$

The solution to this system will not be trivial (will be different to $\mathbf{v} = \mathbf{0}$) when the determinant of (4.31) is equal to zero, that is

$$\det(\boldsymbol{\sigma} - \lambda\mathbf{1}) \stackrel{not}{=} |\boldsymbol{\sigma} - \lambda\mathbf{1}| = 0\ . \tag{4.32}$$

Equation (4.32) is a third-grade polynomial equation in λ. Since tensor $\boldsymbol{\sigma}$ is symmetric, its three solutions $(\lambda_1 \equiv \sigma_1,\ \lambda_2 \equiv \sigma_2,\ \lambda_3 \equiv \sigma_3)$ are real. Once the eigenvalues have been found and ordered according to the criterion $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the eigenvector $\mathbf{v}^{(i)}$can be obtained for each stress σ_i by resolving the system in (4.31),

$$(\boldsymbol{\sigma} - \sigma_i\mathbf{1})\cdot\mathbf{v}^{(i)} = \mathbf{0} \qquad i \in \{1,2,3\}\ . \tag{4.33}$$

This equation provides a non-trivial solution of the eigenvectors $\mathbf{v}^{(i)}$, orthogonal between themselves, which, once it has been normalized, defines the three elements of the base corresponding to the three principal directions.

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Remark 4.9. In accordance with the graphical interpretation of the components of the stress tensor in Section 4.3.3, only normal stresses act on the faces of the elemental parallelepiped associated with the principal stress directions, which are, precisely, the principal stresses (see Figure 4.17).

4.4.5 Mean Stress and Mean Pressure

Definition 4.6. The *mean stress* is the mean value of the principal stresses.

$$\sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

Considering the matrix of components of the stress tensor in the principal stress directions (4.30), results in

$$\sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}\,\mathrm{Tr}(\boldsymbol{\sigma}) \ . \tag{4.34}$$

Definition 4.7. The *mean pressure* is the mean stress with its sign changed.

$$\text{mean pressure} \overset{not}{=} \bar{p} = -\sigma_m = -\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

Definition 4.8. A *spherical or hydrostatic stress state* is a state in which all three principal stress directions have the same value.

$$\sigma_1 = \sigma_2 = \sigma_3 \quad \Longrightarrow \quad \boldsymbol{\sigma} \overset{not}{\equiv} \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} \overset{not}{\equiv} \sigma \mathbf{1}$$

> *Remark 4.10.* In a hydrostatic stress state, the stress tensor is isotropic[6] and, thus, its components are the same in every Cartesian coordinate system.
> As a consequence, any direction is a principal stress direction and the stress state (traction vector) is the same in any plane.

4.4.6 *Decomposition of the Stress Tensor into its Spherical and Deviatoric Parts*

The stress tensor $\boldsymbol{\sigma}$ can be split[7] into a spherical part (or component) $\boldsymbol{\sigma}_{sph}$ and a deviatoric part $\boldsymbol{\sigma}'$,

$$\boxed{\boldsymbol{\sigma} = \underbrace{\boldsymbol{\sigma}_{sph}}_{\substack{\text{spherical}\\ \text{part}}} + \underbrace{\boldsymbol{\sigma}'}_{\substack{\text{deviatoric}\\ \text{part}}}} \,. \tag{4.35}$$

The spherical part is defined as

$$\boldsymbol{\sigma}_{sph} :\overset{def}{=} \frac{1}{3}\,\mathrm{Tr}\,(\boldsymbol{\sigma})\,\mathbf{1} = \sigma_m \mathbf{1} \overset{not}{\equiv} \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} , \tag{4.36}$$

where σ_m is the mean stress defined in (4.34). According to definition (4.35), the deviatoric part of the stress tensor is

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \boldsymbol{\sigma}_{sph} \overset{not}{\equiv} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} - \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} \tag{4.37}$$

resulting in

$$\boldsymbol{\sigma}' \overset{not}{\equiv} \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_m \end{bmatrix} = \begin{bmatrix} \sigma_x{}' & \tau_{xy}{}' & \tau_{xz}{}' \\ \tau_{xy}{}' & \sigma_y{}' & \tau_{yz}{}' \\ \tau_{xz}{}' & \tau_{yz}{}' & \sigma_z{}' \end{bmatrix} . \tag{4.38}$$

---

[6] A tensor is defined as *isotropic* when it remains invariant under any change of orthogonal basis. The general expression of an isotropic second-order tensor is $\mathbf{T} = \alpha \mathbf{1}$ where α can be any scalar.

[7] This type of decomposition can be applied to any second-order tensor.

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Remark 4.11. The spherical part of the stress tensor $\boldsymbol{\sigma}_{sph}$ is an isotropic tensor (and defines a hydrostatic stress state), therefore, it remains invariant under any change of orthogonal basis.

Remark 4.12. The deviatoric component of the tensor is an *indicator* of how far from a hydrostatic stress state the present state is (see (4.37) and Remark 4.11).

Remark 4.13. The principal directions of the stress tensor and of its deviatoric tensor coincide. Proof is trivial considering that, from Remark 4.11, the spherical part $\boldsymbol{\sigma}_{sph}$ is diagonal in any coordinate system. Consequently, if $\boldsymbol{\sigma}$ diagonalizes for a certain basis in (4.37), $\boldsymbol{\sigma}'$ will also diagonalize for that basis.

Remark 4.14. The trace of the deviatoric (component) tensor is null. Taking into account (4.34) and (4.37),

$$\mathrm{Tr}\left(\boldsymbol{\sigma}'\right) = \mathrm{Tr}\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{sph}\right) = \mathrm{Tr}(\boldsymbol{\sigma}) - \mathrm{Tr}\left(\boldsymbol{\sigma}_{sph}\right) = 3\sigma_m - 3\sigma_m = 0 \, .$$

4.4.7 Tensor Invariants

The three fundamental invariants of the stress tensor[8] (*or I invariants*) are

$$I_1 = \mathrm{Tr}(\boldsymbol{\sigma}) = \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3 \, , \tag{4.39}$$

$$I_2 = \frac{1}{2}\left(\boldsymbol{\sigma} : \boldsymbol{\sigma} - I_1^2\right) = -\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right) \, , \tag{4.40}$$

$$I_3 = \det(\boldsymbol{\sigma}) \, . \tag{4.41}$$

[8] The tensor invariants are scalar algebraic combinations of the components of a tensor that do not vary when the basis changes.

Any combination of the I invariants is, in turn, another invariant. In this manner, the J *invariants*

$$J_1 = I_1 = \sigma_{ii} \ , \tag{4.42}$$

$$J_2 = \frac{1}{2}\left(I_1^2 + 2I_2\right) = \frac{1}{2}\sigma_{ij}\sigma_{ji} = \frac{1}{2}\left(\boldsymbol{\sigma} : \boldsymbol{\sigma}\right) \ , \tag{4.43}$$

$$J_3 = \frac{1}{3}\left(I_1^3 + 3I_1 I_2 + 3I_3\right) = \frac{1}{3}\mathrm{Tr}\left(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\right) = \frac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki} \ , \tag{4.44}$$

are defined.

Remark 4.15. For a purely deviatoric tensor $\boldsymbol{\sigma}'$, the corresponding J *invariants* are (see Remark 4.14 and equations (4.39) to (4.44))

$$\left.\begin{array}{l} J_1 = I_1 = 0 \\ J_2 = I_2 \\ J_3 = I_3 \end{array}\right\} \Longrightarrow \boldsymbol{\sigma}' \Longrightarrow \begin{cases} J_1{}' = I_1{}' = 0 \\ J_2{}' = I_2{}' = \dfrac{1}{2}\left(\boldsymbol{\sigma}' : \boldsymbol{\sigma}'\right) = \dfrac{1}{2}\sigma'_{ij}\sigma'_{ji} \\ J_3{}' = I_3{}' = \dfrac{1}{3}\left(\sigma'_{ij}\sigma'_{jk}\sigma'_{ki}\right) \end{cases}$$

4.5 Stress Tensor in Curvilinear Orthogonal Coordinates

4.5.1 Cylindrical Coordinates

Consider a point in space defined by the cylindrical coordinates $\{r,\theta,z\}$ (see Figure 4.18). A physical (orthonormal) basis $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z\}$ and a Cartesian system of local axes $\{x', y', z'\}$ defined as dextrorotatory are considered at this point.

The components of the stress tensor in this basis are

$$\boldsymbol{\sigma} \overset{not}{\equiv} \begin{bmatrix} \sigma_{x'} & \tau_{x'y'} & \tau_{x'z'} \\ \tau_{x'y'} & \sigma_{y'} & \tau_{y'z'} \\ \tau_{x'z'} & \tau_{y'z'} & \sigma_{z'} \end{bmatrix} = \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_\theta & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_z \end{bmatrix} . \tag{4.45}$$

The graphical representation on an elemental parallelepiped is shown in Figure 4.19, where the components of the stress tensor have been drawn on the visible faces. Note that, here, the visible faces of the figure do not coincide with the positive faces, defined (in the same direction as in Section 4.3.3.3) as those whose unit normal vector has the same direction as a vector of the physical basis.

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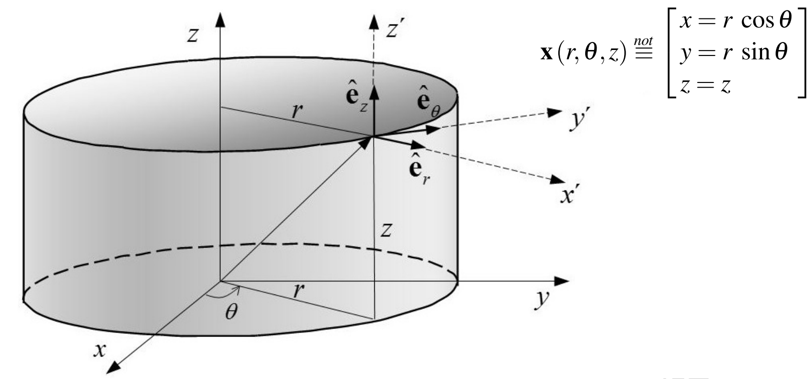

Figure 4.18: Cylindrical coordinates.

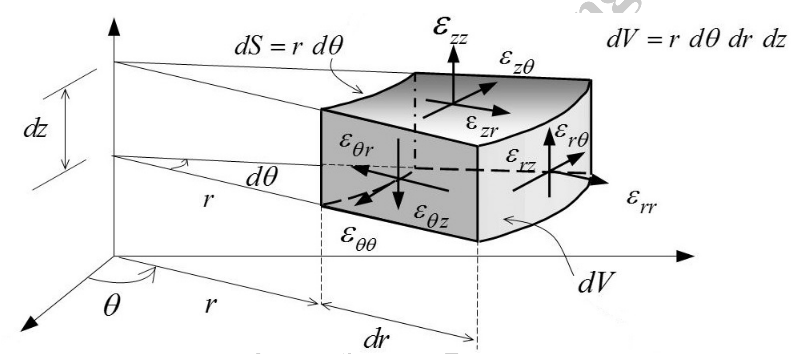

Figure 4.19: Differential element in cylindrical coordinates.

4.5.2 Spherical Coordinates

A point in space is defined by the spherical coordinates $\{r,\theta,\phi\}$ (see Figure 4.20). A physical (orthonormal) basis $\{\hat{\mathbf{e}}_r,\hat{\mathbf{e}}_\theta,\hat{\mathbf{e}}_\phi\}$ and a Cartesian system of local axes $\{x',y',z'\}$ defined as dextrorotatory are considered at this point.

The components of the stress tensor in this basis are

$$\boldsymbol{\sigma} \stackrel{not}{\equiv} \begin{bmatrix} \sigma_{x'} & \tau_{x'y'} & \tau_{x'z'} \\ \tau_{x'y'} & \sigma_{y'} & \tau_{y'z'} \\ \tau_{x'z'} & \tau_{y'z'} & \sigma_{z'} \end{bmatrix} = \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{r\phi} \\ \tau_{r\theta} & \sigma_\theta & \tau_{\theta\phi} \\ \tau_{r\phi} & \tau_{\theta\phi} & \sigma_\phi \end{bmatrix} . \qquad (4.46)$$

The graphical representation on an elemental parallelepiped is shown in Figure 4.21, where the components of the stress tensor have been drawn on the visible faces.

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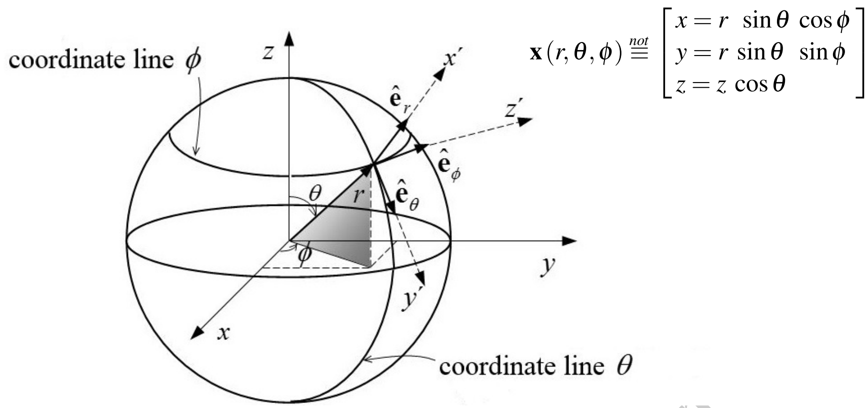

Figure 4.20: Spherical coordinates.

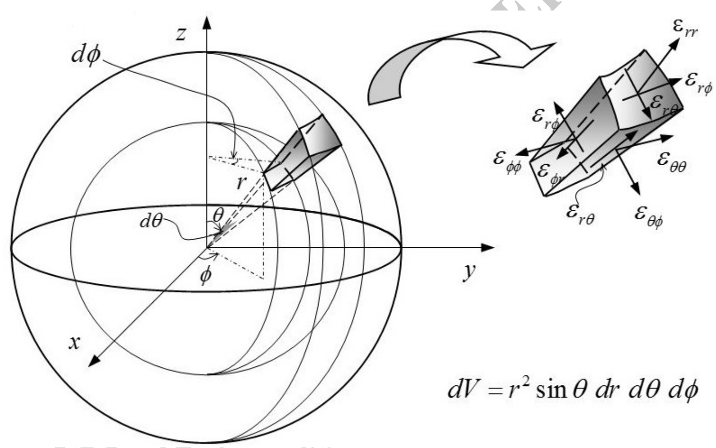

Figure 4.21: Differential element in spherical coordinates.

4.6 Mohr's Circle in 3 Dimensions

4.6.1 Graphical Interpretation of the Stress States

The stress tensor plays such a crucial role in engineering that, traditionally, several procedures have been developed, essentially graphical ones, to visualize and interpret it. The most common are the so-called *Mohr's circles*.

Consider an arbitrary point in the continuous medium P and the stress tensor $\boldsymbol{\sigma}(P)$ at this point. Consider also an arbitrary plane, with unit normal vector $\mathbf{n}$, that contains P (see Figure 4.22). The traction vector acting on point P corresponding to this plane is $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$. This vector can now be decomposed into its components $\boldsymbol{\sigma}_n$, normal to the plane considered, and $\boldsymbol{\tau}_n$, tangent to said plane.

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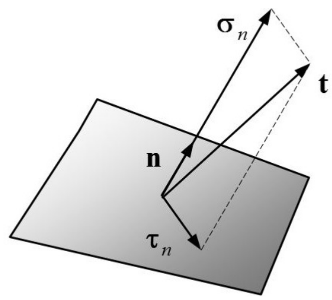

Figure 4.22: Decomposition of the traction vector.

Consider now the normal component $\boldsymbol{\sigma}_n = \sigma\,\mathbf{n}$, where σ is the normal component of the stress on the plane, defined in accordance with the sign criterion detailed in Section 4.3.3.3,

$$\boldsymbol{\sigma}_n = \sigma \cdot \mathbf{n} \qquad \begin{cases} \sigma > 0 \text{ tension }, \\ \sigma < 0 \text{ compression }. \end{cases} \tag{4.47}$$

Consider now the tangential component $\boldsymbol{\tau}_n$, of which only its module is of interest,

$$\boldsymbol{\tau}_n = \mathbf{t} - \boldsymbol{\sigma}_n \qquad |\boldsymbol{\tau}_n| = \tau \geq 0 \,. \tag{4.48}$$

The stress state on the plane with unit normal vector $\mathbf{n}$ at the point considered can be characterized by means of the pair

$$(\sigma, \tau) \;\rightarrow\; \begin{cases} \sigma \in \mathbb{R} \\ \tau \in \mathbb{R}^+ \end{cases} \tag{4.49}$$

which, in turn, determine a point of the half-plane $(x \equiv \sigma, y \equiv \tau) \in \mathbb{R} \times \mathbb{R}^+$ in Figure 4.23. If the infinite number of planes that contain point P are now considered (characterized by all the possible unit normal vectors $\mathbf{n}_{(i)}$) and the corresponding values of the normal stress σ_i and tangential stress τ_i are obtained and, finally, are represented in the half-space mentioned above, a point cloud is obtained. One can then wonder whether the point cloud occupies all the half-space or is *limited to a specific locus*. The answer to this question is provided by the following analysis.

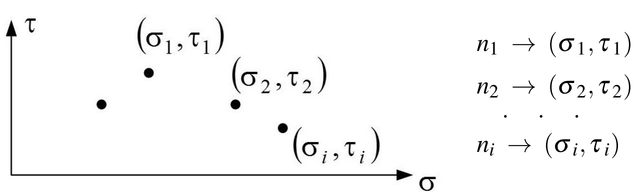

Figure 4.23: Locus of points (σ, τ).

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4.6.2 Determination of the Mohr's Circles

Consider the system of Cartesian axes associated with the principal directions of the stress tensor. In this basis, the components of the stress tensor are

$$\boldsymbol{\sigma} \stackrel{not}{\equiv} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \quad \text{with } \sigma_1 \geq \sigma_2 \geq \sigma_3 \tag{4.50}$$

and the components of the traction vector are

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} \stackrel{not}{\equiv} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 \, n_1 \\ \sigma_2 \, n_2 \\ \sigma_3 \, n_3 \end{bmatrix}, \tag{4.51}$$

where n_1, n_2, n_3 are the components of the unit normal vector $\mathbf{n}$ in the basis associated with the principal stress directions. In view of (4.51), the normal component of the stress (σ), defined in (4.47), is

$$\mathbf{t} \cdot \mathbf{n} \stackrel{not}{\equiv} [\sigma_1 \, n_1, \;\; \sigma_2 \, n_2, \;\; \sigma_3 \, n_3] \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \boxed{\sigma_1 \, n_1^2 + \sigma_2 \, n_2^2 + \sigma_3 \, n_3^2 = \sigma} \tag{4.52}$$

and the module of the traction vector is

$$|\mathbf{t}|^2 = \mathbf{t} \cdot \mathbf{t} = \sigma_1^2 \, n_1^2 + \sigma_2^2 \, n_2^2 + \sigma_3^2 \, n_3^2 \,. \tag{4.53}$$

The modules of the traction vector and of its normal and tangential components can also be related through

$$|\mathbf{t}|^2 = \boxed{\sigma_1^2 \, n_1^2 + \sigma_2^2 \, n_2^2 + \sigma_3^2 \, n_3^2 = \sigma^2 + \tau^2}\,, \tag{4.54}$$

where (4.53) has been taken into account. Finally, the condition that $\mathbf{n}$ is a unit normal vector can be expressed in terms of its components as

$$|\mathbf{n}| = 1 \quad \Longrightarrow \quad \boxed{n_1^2 + n_2^2 + n_3^2 = 1}\,. \tag{4.55}$$

Equations (4.54), (4.52) and (4.55) can be summarized in the following matrix equation.

$$\underbrace{\begin{bmatrix} \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} n_1^2 \\ n_2^2 \\ n_3^2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} \sigma^2 + \tau^2 \\ \sigma \\ 1 \end{bmatrix}}_{\mathbf{b}} \quad \Longrightarrow \quad \mathbf{A} \cdot \mathbf{x} = \mathbf{b} \tag{4.56}$$

System (4.56) can be interpreted as a linear system with:

a) A matrix of coefficients, $\mathbf{A}(\boldsymbol{\sigma})$, defined by the stress tensor at point P (by means of the principal stresses).

b) An independent term, $\mathbf{b}$, defined by the coordinates of a certain point in the half-space $\sigma - \tau$ (representative, in turn, of the stress state on a certain plane).

c) A vector of unknowns $\mathbf{x}$ that determines (by means of the components of the unit normal vector $\mathbf{n}$) in which plane the values of the selected σ and τ correspond.

> *Remark 4.16.* Only the solutions of system (4.56) whose components $\mathbf{x} \stackrel{not}{\equiv} \left[n_1^2,\, n_2^2,\, n_3^2\right]^T$ are positive and smaller than 1 will be feasible (see (4.55)), i.e.,
>
> $$0 \le n_1^2 \le 1 \ , \quad 0 \le n_2^2 \le 1 \quad \text{and} \quad 0 \le n_3^2 \le 1 \ .$$
>
> Every pair (σ, τ) that leads to a solution $\mathbf{x}$ that satisfies this requirement will be considered a *feasible point* of the half-space $\sigma - \tau$, which is *representative of the stress state on a plane that contains P*. The locus of feasible points (σ, τ) is named *feasible zone* of the half-space $\sigma - \tau$.

Consider now the goal of finding the feasible region. Through some algebraic operations, system (4.56) can be rewritten as

$$\begin{cases} \sigma^2 + \tau^2 - (\sigma_1 + \sigma_3)\,\sigma + \sigma_1\,\sigma_3 - \dfrac{A}{(\sigma_1 - \sigma_3)} n_1^2 = 0 & \text{(I)} \\ \sigma^2 + \tau^2 - (\sigma_2 + \sigma_3)\,\sigma + \sigma_2\,\sigma_3 - \dfrac{A}{(\sigma_2 - \sigma_3)} n_2^2 = 0 & \text{(II)} \\ \sigma^2 + \tau^2 - (\sigma_1 + \sigma_2)\,\sigma + \sigma_1\,\sigma_2 - \dfrac{A}{(\sigma_1 - \sigma_2)} n_3^2 = 0 & \text{(III)} \end{cases} \tag{4.57}$$

$$\text{with} \quad A = (\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)(\sigma_1 - \sigma_3) \ .$$

Given, for example, equation (III) of the system in (4.57), it is easily verifiable that it can be written as

$$\begin{aligned} &(\sigma - a)^2 + \tau^2 = R^2 \qquad \text{with} \quad a = \frac{1}{2}(\sigma_1 + \sigma_2) \\ &\text{and} \quad R = \sqrt{\frac{1}{4}(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)(\sigma_1 - \sigma_3)\, n_3^2}\ , \end{aligned} \tag{4.58}$$

which corresponds to the equation of a semicircle in the half-space $\sigma - \tau$ of center C_3 and a radius R_3, given by

$$
\begin{aligned}
C_3 &= \left(\frac{1}{2} (\sigma_1 + \sigma_2) \, , \, 0 \right) \quad \text{and} \\
R_3 &= \sqrt{\frac{1}{4} (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)(\sigma_1 - \sigma_3) \, n_3^2} \, .
\end{aligned}
\tag{4.59}
$$

The different values of $n_3^2 \in [0,1]$ determine a set of concentric semicircles of center C_3 and radii $R_3(n_3)$ belonging to the half-space $\sigma - \tau$ and whose points occupy a certain region of this half-space. This region is delimited by the maximum and minimum values of $R_3(n_3)$. Observing that the radical in the expression of R_3 in (4.59) is positive, these values are obtained for $n_3^2 = 0$ (the minimum radius) and $n_3^2 = 1$ (the maximum radius).

$$
\begin{aligned}
n_3^2 = 0 \quad &\Longrightarrow \quad R_3^{\min} = \frac{1}{2} (\sigma_1 - \sigma_2) \\
n_3^2 = 1 \quad &\Longrightarrow \quad R_3^{\max} = \frac{1}{2} (\sigma_1 + \sigma_2) - \sigma_3
\end{aligned}
\tag{4.60}
$$

The domain delimited by both semicircles defines an initial limitation of the feasible domain, shown in Figure 4.24.

This process is repeated for the other two equations, (I) and (II), in (4.57), resulting in:

– Equation (I) : $C_1 = \Big(\underbrace{\frac{1}{2} (\sigma_2 + \sigma_3)}_{a_1}, \, 0 \Big) \quad \Longrightarrow \quad \begin{cases} R_1^{\min} = \dfrac{1}{2} (\sigma_2 - \sigma_3) \\ R_1^{\max} = |\sigma_1 - a_1| \end{cases}$

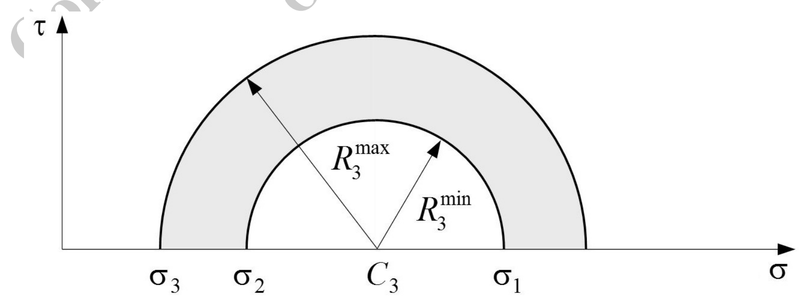

Figure 4.24: Initial limitation of the feasible domain.

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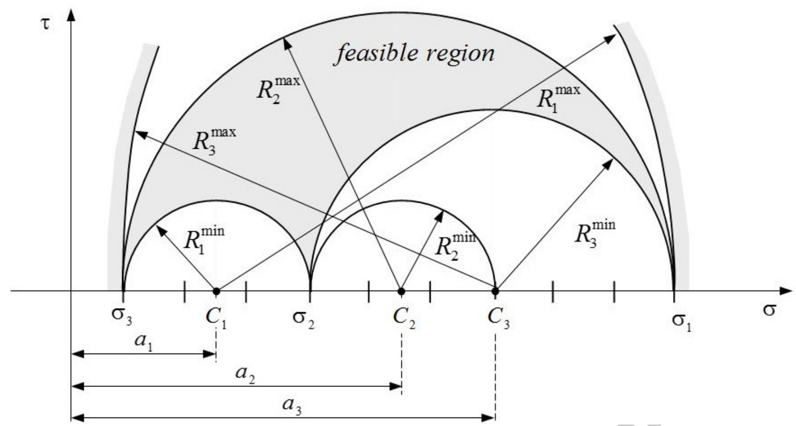

Figure 4.25: Feasible region.

– Equation (II) : $C_2 = \Big(\underbrace{\frac{1}{2}(\sigma_1 + \sigma_3)}_{a_2}, 0\Big) \quad \Longrightarrow \quad \begin{cases} R_2^{\min} = \frac{1}{2}(\sigma_1 - \sigma_3) \\ R_2^{\max} = |\sigma_2 - a_2| \end{cases}$

– Equation (III) : $C_3 = \Big(\underbrace{\frac{1}{2}(\sigma_1 + \sigma_2)}_{a_3}, 0\Big) \quad \Longrightarrow \quad \begin{cases} R_3^{\min} = \frac{1}{2}(\sigma_1 - \sigma_2) \\ R_3^{\max} = |\sigma_3 - a_3| \end{cases}$

For each case, a feasible region that consists in a semi-annulus defined by the minimum and maximum radii is obtained. Obviously, the final feasible region must be in the intersection of these semi-annuli, as depicted in Figure 4.25.

Figure 4.26 shows the final construction that results of the three Mohr's semicircles that contain points σ_1, σ_2 and σ_3. It can also be shown that every point within the domain enclosed by the Mohr's circles is feasible (in the sense that the corresponding values of σ and τ correspond to stress states on a certain plane that contains point P).

The construction of Mohr's circle is trivial (once the three principal stresses are known) and is useful for discriminating possible stress states on planes, determining maximum values of shear stresses, etc.

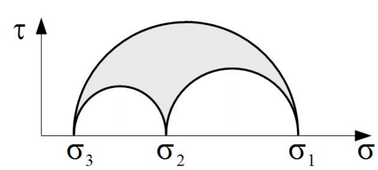

Figure 4.26: Mohr's circle in three dimensions.

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Example 4.3 – *The principal stresses at a certain point in a continuous medium are*

$$\sigma_1 = 10\,, \quad \sigma_2 = 5 \quad \text{and} \quad \sigma_3 = 2\,.$$

The normal and tangential stresses on a plane that contains this point are σ *and* τ*, respectively. Justify if the following values of* σ *and* τ *are possible or not.*

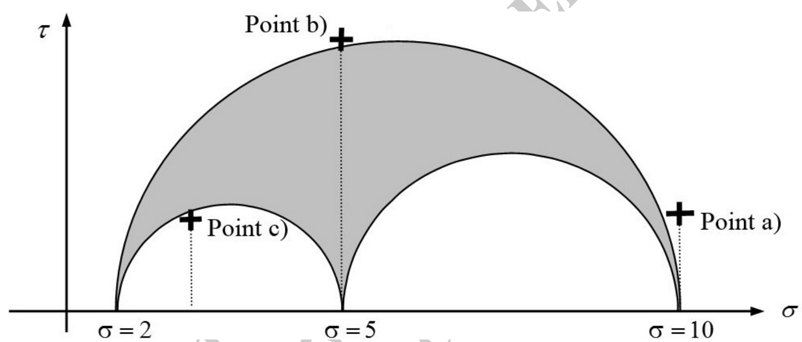
a) $\sigma = 10$ and $\tau = 1$.

b) $\sigma = 5$ and $\tau = 4$.

c) $\sigma = 3$ and $\tau = 1$.

Solution

The *Mohr's circle* for the defined stress state is drawn and the given points are marked in the half-space $\sigma - \tau$.



Only the points belonging to the gray zone represent stress states (feasible points). It is verified that *none of the given points are feasible*.

4.7 Mohr's Circle in 2 Dimensions

Many real-life problems in engineering are assimilated to an ideal bi-dimensional stress state[9] in which one of the principal stress directions is known (or assumed) *a priori*. In these cases, the Cartesian axis x_3 (or z-axis) is made to coincide with said principal direction (see Figure 4.25) and, thus, the components of the stress tensor can be written as

$$\boldsymbol{\sigma} \stackrel{not}{\equiv} \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} . \tag{4.61}$$

[9] This type of problems will be analyzed in depth in Chapter 7, dedicated to bi-dimensional elasticity.

Consider now *only the family of planes parallel to the x_3-axis* (therefore, the component n_3 of its unit normal vector is null). The corresponding traction vector is

$$\mathbf{t}(P,\mathbf{n}) = \boldsymbol{\sigma}\cdot\mathbf{n} \quad \Longrightarrow \quad \begin{bmatrix} t_1 \\ t_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ 0 \end{bmatrix} \tag{4.62}$$

and its component t_3 vanishes. In (4.61) and (4.62) the components of the stress tensor, $\boldsymbol{\sigma}$, of the unit normal vector defining the plane, $\mathbf{n}$, and of the traction vector, $\mathbf{t}$, associated with direction x_3 are either well known (this is the case for $\sigma_{13}, \sigma_{23}, n_3$ or t_3), or do not intervene in the problem (as is the case for σ_{33}). This circumstance suggests ignoring the third dimension and reducing the analysis to the two dimensions associated with the x_1- and x_2-axes (or x- and y-axes), as indicated in Figure 4.27. Then, the problem can be defined *in the plane* through the components of the stress tensor

$$\boldsymbol{\sigma} \overset{not}{\equiv} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \tag{4.63}$$

and the components of the traction vector

$$\mathbf{t}(P,\mathbf{n}) = \boldsymbol{\sigma}\cdot\mathbf{n} \overset{not}{\equiv} \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} . \tag{4.64}$$

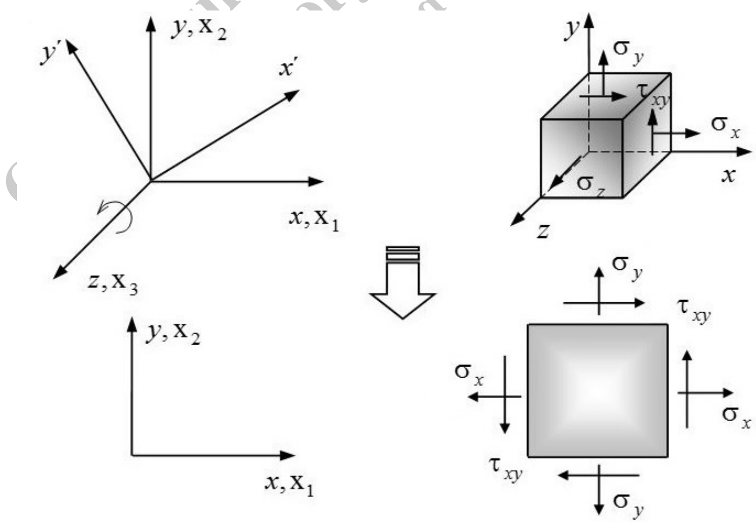


Figure 4.27: Reduction of the problem from three to two dimensions.

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4.7.1 Stress State on a Given Plane

Consider a plane (always parallel to the z-axis) whose unit normal vector $\mathbf{n}$ forms an angle θ with the x-axis. A unit vector $\mathbf{m}$ is defined in the *tangential* direction to the trace of the plane as indicated in Figure 4.28.

Remark 4.17. The unit normal vector $\mathbf{n}$, the unit tangent vector $\mathbf{m}$, and the angle θ in Figure 4.28 have the following *positive directions* associated with them.

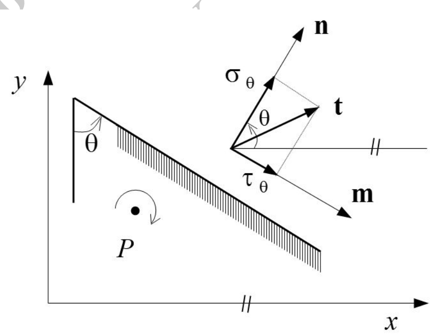
- Unit *normal* vector $\mathbf{n}$: towards the *exterior* of the plane (with respect to the position of point P).
- Unit *tangent* vector $\mathbf{m}$: generates a *clockwise rotation* with respect to point P.
- Angle θ: defined as *counterclockwise*.

Consider $\boldsymbol{\sigma}$, the stress tensor at a given point, whose components are defined in a Cartesian base,

$$\boldsymbol{\sigma} \overset{not}{\equiv} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} . \tag{4.65}$$

Using (4.64), the traction vector on the given point, which belongs to the plane considered, is

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} \overset{not}{\equiv} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \sigma_x \cos\theta + \tau_{xy} \sin\theta \\ \tau_{xy} \cos\theta + \sigma_y \sin\theta \end{bmatrix} . \tag{4.66}$$



$$\begin{cases} \mathbf{n} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \\ \mathbf{m} = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix} \end{cases}$$

Figure 4.28: Stress state on a given plane.

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Taking into consideration the expression $\mathbf{t} = \sigma_\theta \mathbf{n} + \tau_\theta \mathbf{m}$, the *normal stress* σ_θ and the *tangent stress* τ_θ on the plane with inclination θ (see Figure 4.28) are defined, respectively, as

$$\begin{aligned}\sigma_\theta &= \mathbf{t}\cdot\mathbf{n} \stackrel{not}{\equiv} [\sigma_x \cos\theta + \tau_{xy}\sin\theta,\ \ \tau_{xy}\cos\theta + \sigma_y \sin\theta] \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} = \\ &= \sigma_x \cos^2\theta + \tau_{xy} 2\sin\theta\cos\theta + \sigma_y \sin^2\theta\end{aligned} \tag{4.67}$$

and

$$\begin{aligned}\tau_\theta &= \mathbf{t}\cdot\mathbf{m} \stackrel{not}{\equiv} [\sigma_x \cos\theta + \tau_{xy}\sin\theta,\ \ \tau_{xy}\cos\theta + \sigma_y \sin\theta] \begin{bmatrix}\sin\theta \\ -\cos\theta\end{bmatrix} = \\ &= \sigma_x \sin\theta\cos\theta - \sigma_y \sin\theta\cos\theta + \tau_{xy}\left(\sin^2\theta - \cos^2\theta\right),\end{aligned} \tag{4.68}$$

which can be rewritten as[10]

$$\boxed{\begin{aligned}\sigma_\theta &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\theta) + \tau_{xy}\sin(2\theta) \\ \tau_\theta &= \frac{\sigma_x - \sigma_y}{2}\sin(2\theta) - \tau_{xy}\cos(2\theta)\end{aligned}} \tag{4.69}$$

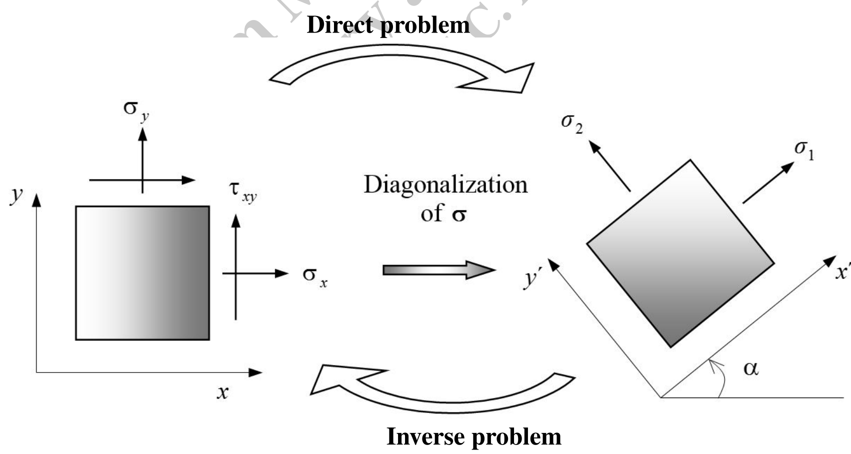

Figure 4.29: Direct and inverse problems.

[10] The following trigonometric relations are used here: $\sin(2\theta) = 2\sin\theta\cos\theta$, $\cos^2\theta = (1+\cos(2\theta))/2$ and $\sin^2\theta = (1-\cos(2\theta))/2$.

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4.7.2 Direct Problem: Diagonalization of the Stress Tensor

The direct problem consists in obtaining the principal stresses and the principal stress directions given the components of the stress tensor (4.65) in a certain system of axes $x-y$ (see Figure 4.29).

The principal stress directions associated with the x'- and y'-axes defined by the angles α and $\pi/2+\alpha$ (see Figure 4.29) determine the inclinations of the two planes on which the stresses only have a normal component σ_α, *being the tangent component* τ_α *null*. Imposing this condition on (4.69) yields

$$
\begin{aligned}
&\tau_\alpha = \frac{\sigma_x-\sigma_y}{2}\sin(2\alpha) - \tau_{xy}\cos(2\alpha) = 0 \implies \tan(2\alpha) = \frac{\tau_{xy}}{\dfrac{\sigma_x-\sigma_y}{2}} , \\
&\sin(2\alpha) = \pm\frac{1}{\sqrt{1+\dfrac{1}{\tan^2(2\alpha)}}} = \pm\frac{\tau_{xy}}{\left|\sqrt{\left(\dfrac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\right|} , \\
&\cos(2\alpha) = \pm\frac{1}{\sqrt{1+\tan^2(2\alpha)}} = \pm\frac{\dfrac{\sigma_x-\sigma_y}{2}}{\left|\sqrt{\left(\dfrac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}\right|} .
\end{aligned}
\tag{4.70}
$$

Equation (4.70) provides two solutions (associated with the $+$ and $-$ signs) α_1 and $\alpha_2=\alpha_1+\pi/2$, which define the two principal stress directions (orthogonal) to the plane being analyzed[11]. The corresponding principal stress directions are obtained replacing the angle $\theta=\alpha$ in (4.70) in (4.69), resulting in

$$
\sigma_\alpha = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos(2\alpha) + \tau_{xy}\sin(2\alpha) \ . \tag{4.71}
$$

$$
\boxed{\sigma_\alpha \ \rightarrow \ \begin{cases} \sigma_1 = \dfrac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\dfrac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2} \\[2ex] \sigma_2 = \dfrac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\dfrac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2} \end{cases}} \tag{4.72}
$$

[11] The third principal stress direction is the direction perpendicular to the plane being analyzed (z- or x_3-axis), see (4.61) and Figure 4.27.

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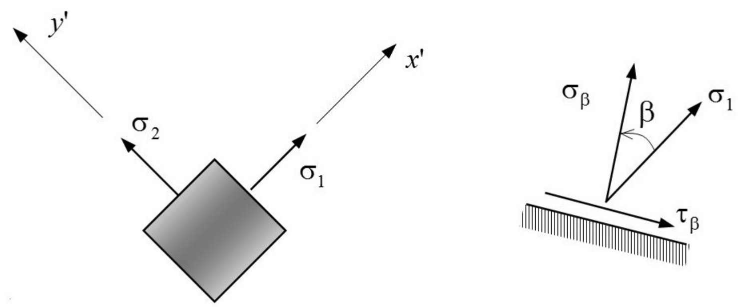

Figure 4.30: Inverse problem.

4.7.3 Inverse Problem

The problem consists in obtaining the stress state on any plane given the principal stresses and the principal stress directions σ_1 and σ_2 in the plane being analyzed. The stress state on any plane is characterized by *the angle* β *that forms the unit normal vector of the plane with the principal stress direction corresponding to* σ_1. As a particular case, the components of the stress tensor on an elemental rectangle associated with the system of axes $x-y$ can be obtained (see Figure 4.29).

Consider now the Cartesian system $x'-y'$, associated with the principal stress directions (see Figure 4.30). Applying (4.69) with $\sigma_{x'}=\sigma_1$, $\sigma_{y''}=\sigma_2$, $\tau_{x'y'}=0$ and $\theta\equiv\beta$ results in

$$\boxed{\begin{aligned}&\sigma_\beta=\frac{\sigma_1+\sigma_2}{2}+\frac{\sigma_1-\sigma_2}{2}\cos(2\beta)\\&\tau_\beta=\frac{\sigma_1-\sigma_2}{2}\sin(2\beta)\end{aligned}} \tag{4.73}$$

4.7.4 Mohr's Circle for Plane States (in 2 Dimensions)

Consider all the possible planes that contain point P and the values of the normal and tangent stresses, σ_θ and τ_θ, defined in (4.69) for all the possible values of $\theta\in[0,2\pi]$. The stress state in the point for an inclined plane θ can now be characterized by means of the pair

$$(\sigma=\sigma_\theta,\ \tau=\tau_\theta)\quad\text{where}\quad\sigma\in\mathbb{R}\quad\text{and}\quad\tau\in\mathbb{R}\,, \tag{4.74}$$

which, in turn, determines a point $(x\equiv\sigma,\ y\equiv\tau)\in\mathbb{R}\times\mathbb{R}$ of the plane $\sigma-\tau$ in Figure 4.31. To determine the locus of points of said plane that characterizes

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all the possible stress states for planes that contain the point being analyzed, the ensuing procedure is followed.

Considering a reference system that coincides with the principal stress directions (as in Figure 4.30) and characterizing the inclination of the planes by means of the angle β with the principal stress direction σ_1, one obtains from (4.73)

$$\begin{cases} \sigma - \dfrac{\sigma_1 + \sigma_2}{2} = \dfrac{\sigma_1 - \sigma_2}{2} \cos(2\beta) \\ \tau = \dfrac{\sigma_1 - \sigma_2}{2} \sin(2\beta) \end{cases} \tag{4.75}$$

and, squaring both equations and adding them up results in

$$\left(\sigma - \frac{\sigma_1 + \sigma_2}{2} \right)^2 + \tau^2 = \left(\frac{\sigma_1 - \sigma_2}{2} \right)^2 . \tag{4.76}$$

Note that this equation, which will be valid for any value of the angle β, or, in other words, for any arbitrarily oriented plane that contains the point, corresponds to a circle with center C and radius R in the plane $\sigma - \tau$ given by (see Figure 4.31)

$$C = \left(\frac{\sigma_1 + \sigma_2}{2} , 0 \right) \quad \text{and} \quad R = \frac{\sigma_1 - \sigma_2}{2} . \tag{4.77}$$

Consequently, the locus of points representative of a stress state on the planes that contain P is a circle (named *Mohr's circle*), whose construction is defined in Figure 4.31.

The inverse proposition is also true: given a point of Mohr's circle with coordinates $(\sigma,\ \tau)$, there exists a plane that contains P whose normal and tangent stresses are σ and τ, respectively. In effect, using (4.75) the following trigonometric expressions are obtained.

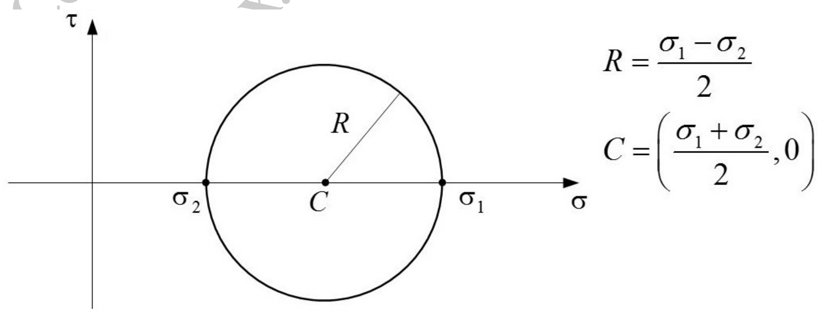

Figure 4.31: Mohr's circle for plane stress states.

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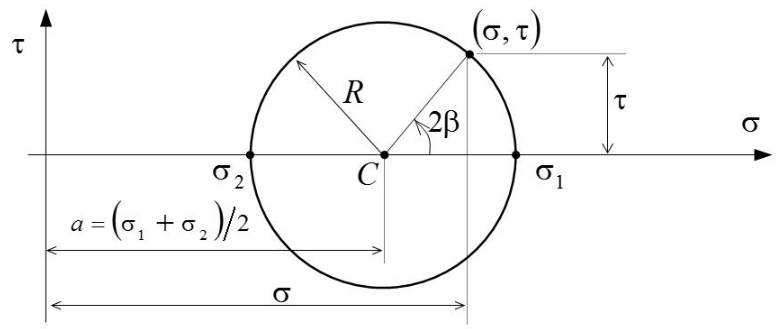

Figure 4.32: Interpretation of the angle β.

$$\begin{aligned} \cos(2\beta) &= \frac{\left(\sigma - \dfrac{\sigma_1 + \sigma_2}{2}\right)}{\left(\dfrac{\sigma_1 - \sigma_2}{2}\right)} = \frac{\sigma - a}{R} \\ \sin(2\beta) &= \frac{\tau}{\left(\dfrac{\sigma_1 - \sigma_2}{2}\right)} = \frac{\tau}{R} \end{aligned} \tag{4.78}$$

These expressions uniquely define the angle β between the normal direction to the plane and the principal stress direction σ_1. The plane obtained corresponds to the aforementioned stresses σ and τ. Figure 4.32 provides an interpretation of the angle 2β in the Mohr's circle itself.

4.7.5 *Properties of the Mohr's Circle*

a) *Obtaining the point in Mohr's circle that is representative of the stress state on a plane whose normal direction forms an angle β with the principal stress direction σ_1.*

   Take a representative point of the plane on which the principal stress direction σ_1 acts (point $(\sigma_1, 0)$) and rotate an angle 2β in the direction going from σ_1 to σ_β (see Figure 4.32 and Figure 4.33).

b) *The representative points in Mohr's circle of two orthogonal planes are aligned with the center of the circle* (as a consequence of property a)) for $\beta_2 = \beta_1 + \pi/2$ (see Figure 4.34).

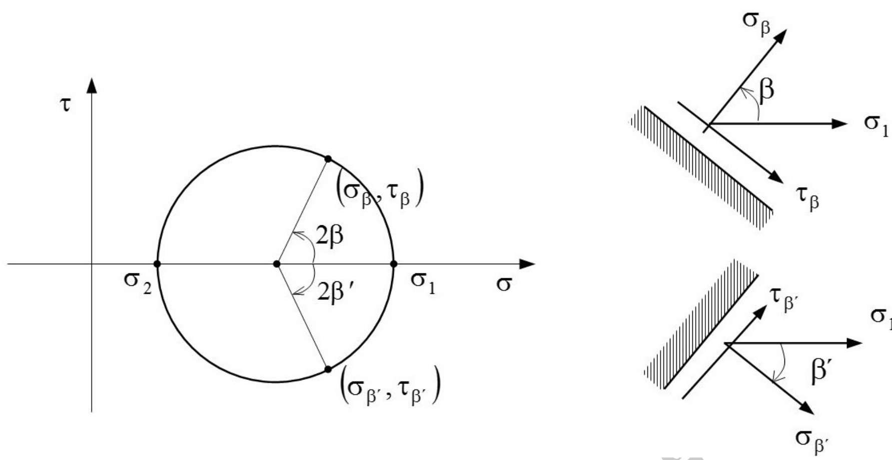

Figure 4.33: Representative point associated with angle β in Mohr's circle.

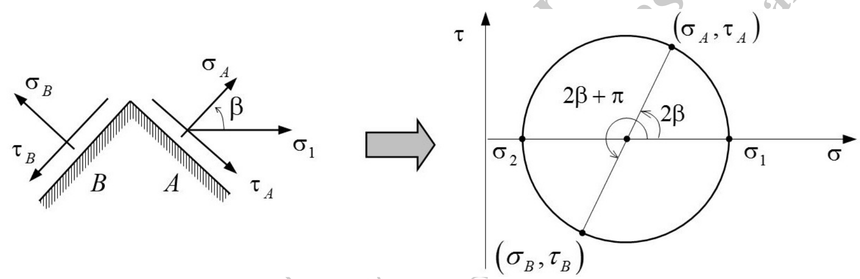

Figure 4.34: Representative points for two orthogonal planes in Mohr's circle.

c) *Mohr's circle can be drawn if the stress state on two orthogonal planes is known.*

In effect, by means of property b) the points representative of these two orthogonal planes in plane $\sigma - \tau$ are aligned with the center of Mohr's circle. Therefore, joining both points provides the intersection with the σ-axis that corresponds to the center of the circle. Since two additional points of the circle are known, the circle can be drawn.

d) *Mohr's circle can be drawn if the components of the stress tensor in a certain orthonormal base are known.*

This is a particular case of property c) in which the points representative of a stress state on Cartesian planes are known (see Figure 4.35). Note, in this figure, how the radius and the diametrical points of the circle can be obtained. In addition, note that the application of property a) on the point representative of the plane perpendicular to the x-axis implies moving in the

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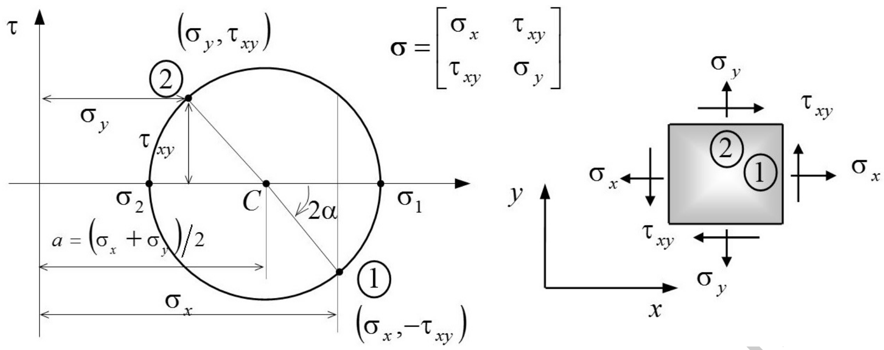

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_1 = a + R = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = a - R = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

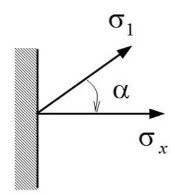

Figure 4.35: Calculation of the radius and diametrical points of Mohr's circle for a stress state on Cartesian planes.

opposite direction to that of angle α (angle of σ_x with σ_1= - angle of σ_1 with $\sigma_x = -\alpha$).

4.7.6 The Pole of Mohr's Circle

Theorem 4.1. *There exists a point in Mohr's circle denoted* pole *or* origin of planes *that has the following properties:*

- *Any straight line drawn from the pole P will intersect Mohr's circle at a point A that represents the stress state on a plane parallel in space to that line (see Figure 4.36).*
- *The inverse is also verified, that is, if a straight line, parallel to a given plane, is drawn from the pole P, the intersection point B represents the stress state on this particular plane (see Figure 4.37).*

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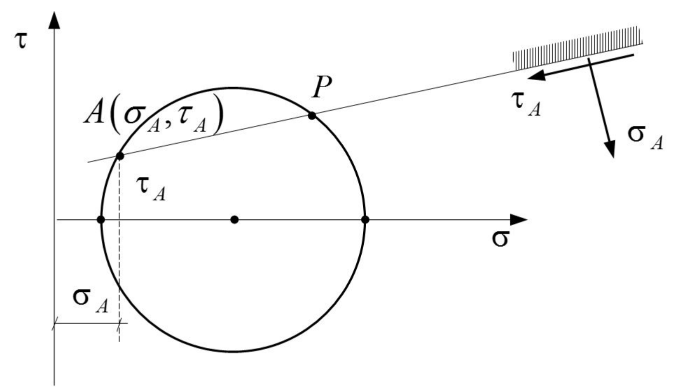

Figure 4.36: First property of the pole of Mohr's circle.

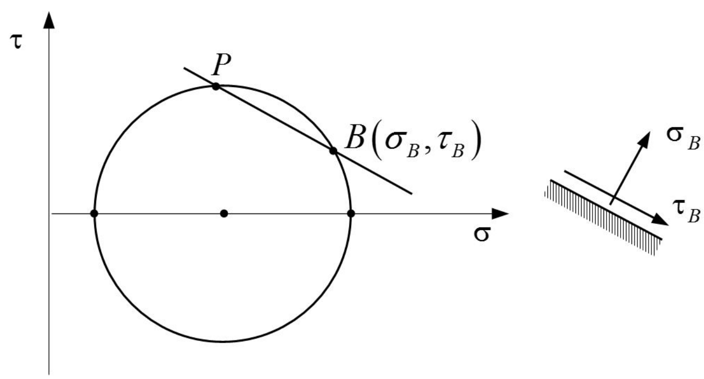

Figure 4.37: Second property of the pole of Mohr's circle.

Proof

Consider the stress tensor at the point being analyzed and its graphical representation on the Cartesian planes of Figure 4.38 (left)[12] denoted as plane A (vertical plane) and plane B (horizontal plane). A and B are the corresponding points in the Mohr's circle drawn in Figure 4.38 (right).

1) Assuming property a) is verified, the pole of Mohr's circle can be obtained by drawing a vertical line from point A (parallel to plane A). Then, the pole P is located at the intersection of this line with the Mohr's circle. Also, drawing a horizontal line from point B (parallel to plane B) determines the location of

[12] Note that, following the sign criterion of Mohr's circle, the tangent stress on plane A is $\tau = -\tau_{xy}$.

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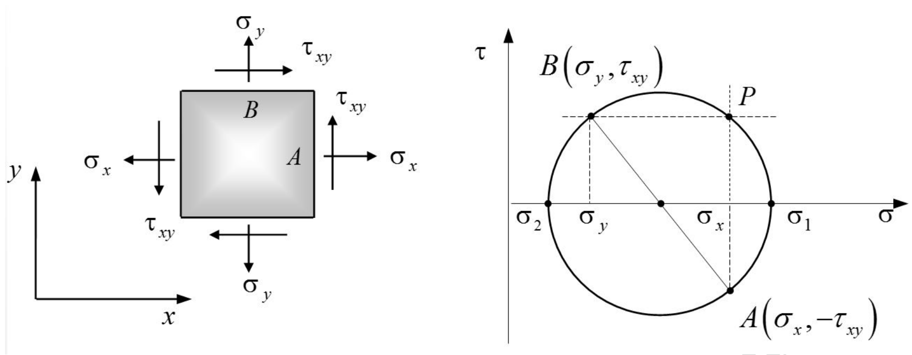

Figure 4.38: Proof of the properties of the pole of Mohr's circle (1).

the pole at the intersection of this line with the Mohr's circle. The same point P is obtained in both cases, as is verified in the Figure 4.38.

2) Consider now an arbitrary plane whose normal direction forms an angle θ with the horizontal direction (see Figure 4.39, left) and consider also the normal and tangent stresses, σ_θ and τ_θ, respectively, according to this plane. Assuming that the major principal stress direction σ_1 forms an angle α with the direction of stress σ_x, then, the direction of stress σ_θ forms an angle $(\theta-\alpha)$ with the major principal stress direction σ_1.

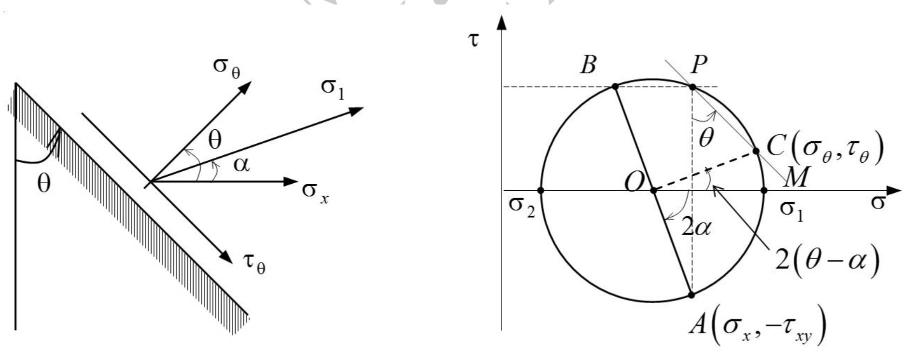

Figure 4.39: Proof of the properties of the pole of Mohr's circle (2).

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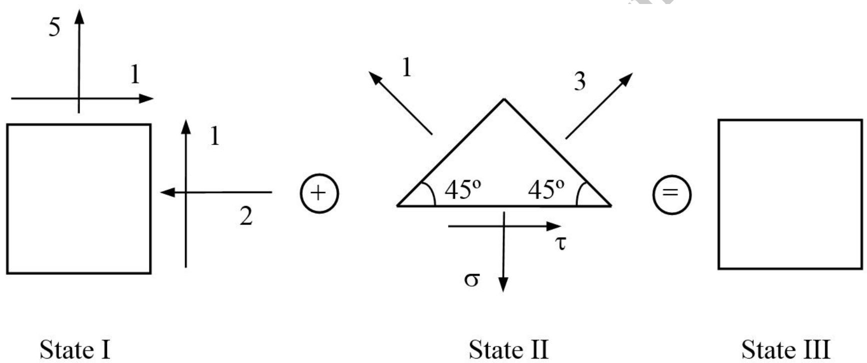
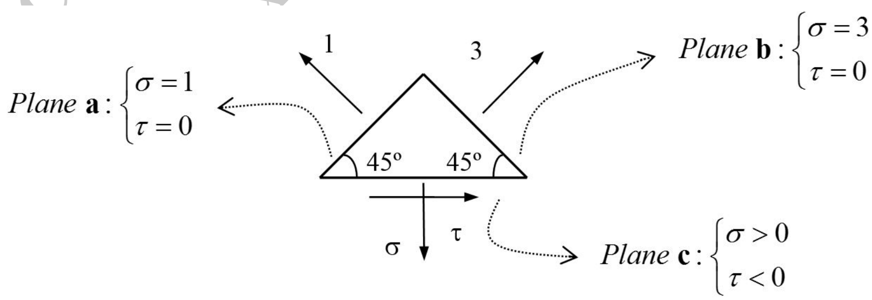
3) Consider the Mohr's circle and the pole P obtained in step 1) (see Figure 4.39, right)[13]. Using property a) of Section 4.7.5, point C can be obtained. This point is representative of the Mohr's circle that corresponds to the plane considered, obtained by rotating from point M a double angle equal to $2(\theta-\alpha)$ such that the angle MOC is $2(\theta-\alpha)$. By construction, angle AOM is 2α and angle AOC, the sum of both, is $2(\theta-\alpha)+2\alpha=2\theta$. The arc included by this angle is $AMC=2\theta$. Then, the angle semi-inscribed in APC, which includes arc AMC, will be θ, which proves that the straight line *PC is parallel to the trace of the plane considered*. Since this plane could be any plane, the validity of the property is proven.

Example 4.4 – *Calculate the stresses acting on state* III = I + II:



Solution

To be able to add states I and II, the stresses must act on the same planes. Since the two states present planes with different orientations, the stresses acting in state II must be found for the planes given in state I. To this aim, the *Mohr's circle* for state II must be drawn.



[13] The following geometric properties are used here: a) the value of a central angle of a circle is the same as the arc it includes; and b) the value of an angle semi-inscribed in a circle is equal to half the arc it includes.

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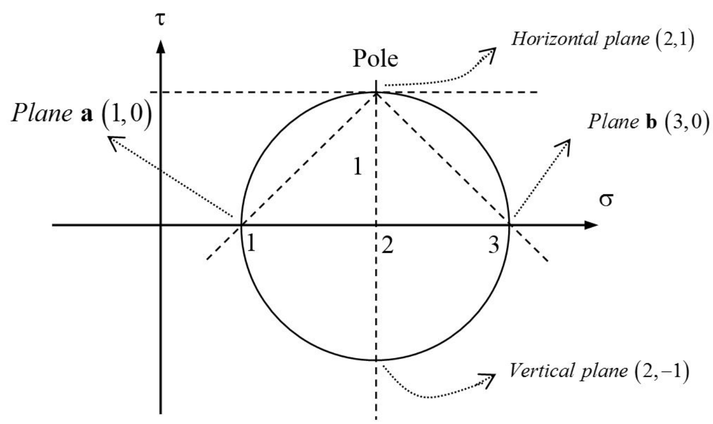
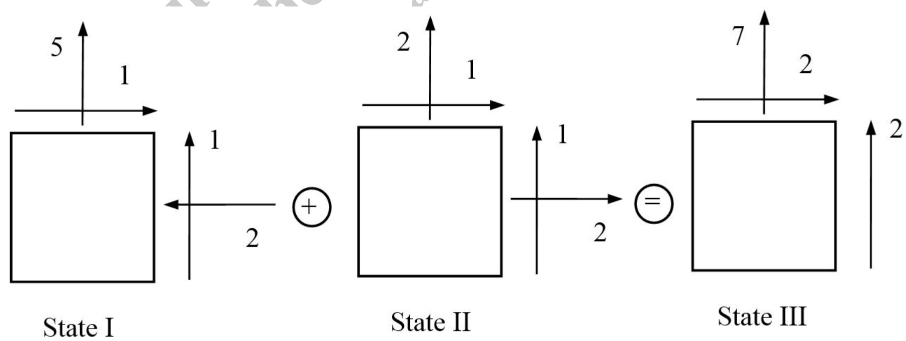


To draw the circle, planes **a** and **b** are represented since their stress states are known. The corresponding points in the Mohr's circle belong to the abscissa and determine, thus, the diameter of the circle.
The pole is obtained as the intersection of the lines that are parallel to the two planes inclined at 45° and that contain the points that they represent. Once the pole is determined, a horizontal line is drawn from it, whose intersection with the Mohr's circle (because it is tangent to the point, the intersection in this case is the same pole) determines the point representative of the horizontal plane $(2,1)$. The same procedure is repeated for a vertical plane to obtain point $(2,-1)$. With this information, state II can be reconstructed on the horizontal and vertical planes. Then, the stresses obtained are added to those of state I to finally obtain state III.



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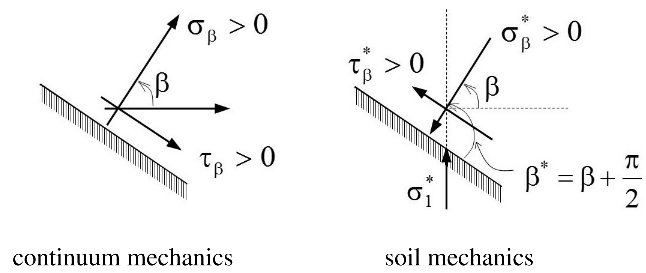

Figure 4.40: Differences in the sign criterion for continuum mechanics and soil mechanics.

4.7.7 *Mohr's Circle with the Soil Mechanics Sign Criterion*

The sign criterion, with respect to the normal and tangent stresses, used in soil mechanics is the inverse of the one used in continuum mechanics (see Figure 4.40). The differences are:

- The positive stresses in soil mechanics are in the opposite direction (normal stresses are positive when they are *compressive*, and the direction of the positive tangent stresses is defined by a *counterclockwise* rotation with respect to the plane).
- The sign criterion for angles is the same (counterclockwise angles are positive).

Consequently, if the order of the principal stresses is respected $(\sigma_1 \geq \sigma_2)$, the order of the principal stresses will be inverted in soil mechanics with respect to continuum mechanics for a same stress state (see Figure 4.41).

Consider the fundamental expressions in (4.73), which are the starting point in the construction and determination of the properties of the Mohr's circle. Using the two sign criteria for a same stress state results in:

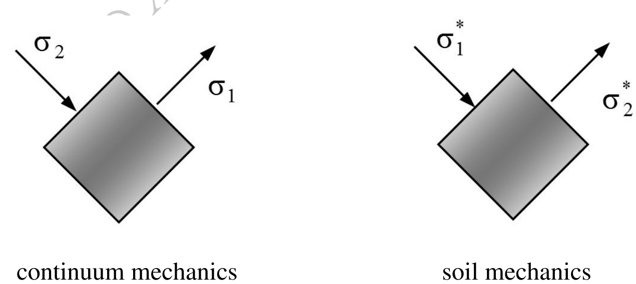

Figure 4.41: Direction of the principal stresses for continuum mechanics and soil mechanics.

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$$
\begin{array}{ll}
\text{Continuum mechanics:} & \sigma_\beta,\ \tau_\beta,\ \sigma_1,\ \sigma_2,\ \beta \\
\text{Soil mechanics:} & \begin{cases} \sigma_\beta^* = -\sigma_\beta \\ \tau_\beta^* = -\tau_\beta \\ \sigma_1^* = -\sigma_2 \\ \sigma_2^* = -\sigma_1 \\ \beta^* = \beta + \pi/2 \end{cases}
\end{array}
\tag{4.79}
$$

Replacing (4.79) in (4.73) yields

$$
\begin{cases}
-\sigma_\beta^* = \dfrac{-\sigma_2^* - \sigma_1^*}{2} + \dfrac{-\sigma_2^* + \sigma_1^*}{2} \underbrace{\cos\left(2\beta^* - \pi\right)}_{-\cos\left(2\beta^*\right)} , \\
-\tau_\beta^* = \dfrac{-\sigma_2^* + \sigma_1^*}{2} \underbrace{\sin\left(2\beta^* - \pi\right)}_{-\sin\left(2\beta^*\right)} ,
\end{cases}
\tag{4.80}
$$

and, operating on these expressions finally results in

$$
\begin{cases}
\sigma_\beta^* = \dfrac{\sigma_1^* + \sigma_2^*}{2} + \dfrac{\sigma_1^* - \sigma_2^*}{2} \cos\left(2\beta^*\right) , \\
\tau_\beta^* = \dfrac{\sigma_1^* - \sigma_2^*}{2} \sin\left(2\beta^*\right) .
\end{cases}
\tag{4.81}
$$

Note that the fundamental expressions in (4.81), obtained on the basis of the sign criterion in soil mechanics, are the same as those in (4.73), obtained on the basis of the sign criterion in continuum mechanics. Therefore, *the construction of the Mohr's circle and the determination of its properties is the same in both cases*.

4.8 Mohr's Circle for Particular Cases

4.8.1 Hydrostatic Stress State

In an hydrostatic stress state, characterized by $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$, the Mohr's circles in three dimensions collapses into a point (see Figure 4.42).

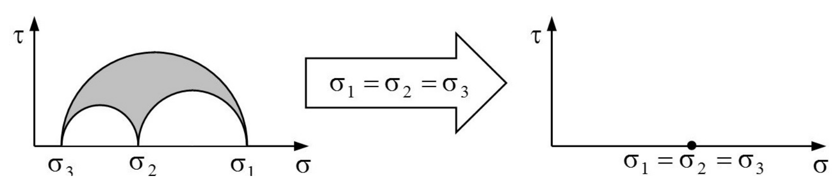

Figure 4.42: Mohr's circle for a hydrostatic stress state.

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4.8.2 Mohr's Circles for a Tensor and its Deviator

The Mohr's circles in three dimensions associated with a stress state and its deviator differ in a translation equal to the mean stress (see Figure 4.43).

$$\boldsymbol{\sigma} = \underbrace{\boldsymbol{\sigma}_{sph}}_{\substack{\text{spherical}\\ \text{part}}} + \underbrace{\boldsymbol{\sigma}'}_{\substack{\text{deviator}\\ \text{part}}} \quad ; \qquad \boldsymbol{\sigma}_{sph} \overset{not}{\equiv} \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} \Longrightarrow \begin{cases} \sigma_1 = \sigma_m + \sigma_1{}' \\ \sigma_2 = \sigma_m + \sigma_2{}' \\ \sigma_3 = \sigma_m + \sigma_3{}' \end{cases}$$

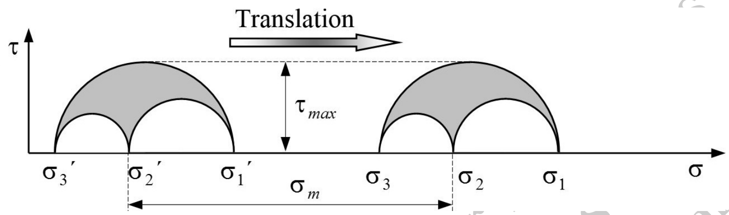

Figure 4.43: Mohr's circle for a stress state and its deviator.

4.8.3 Mohr's Circles for a Plane Pure Shear Stress State

> **Definition 4.9.** A *plane pure shear stress state* occurs at a point when there are two orthogonal planes on which there is only tangent (shear) stress (see Figure 4.44).

The Mohr's circle corresponding to a *pure shear stress state* characterized by a tangent stress τ^* *has as center the origin of axes and as radius* $R = |\tau^*|$. The proof is immediate from the construction criteria of the Mohr's circle (see Figure 4.44, left).

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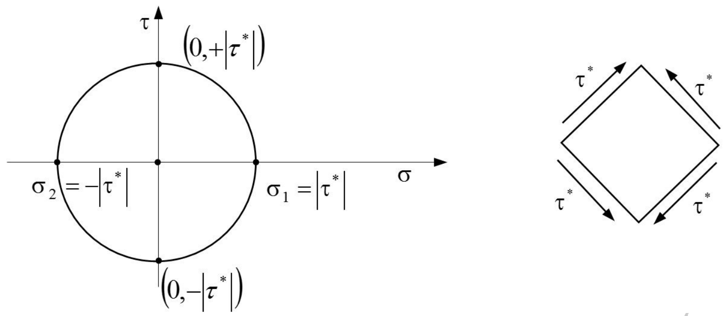
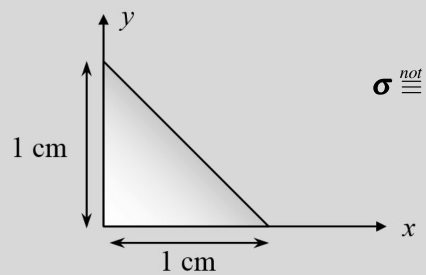


Figure 4.44: Mohr's circle for a plane pure shear stress state.

PROBLEMS

Problem 4.1 – *The solid below is subjected to the following stress state in equilibrium.*



$$\boldsymbol{\sigma} \stackrel{not}{\equiv} \begin{bmatrix} xy & 5y \\ 5y & 4x \end{bmatrix} \qquad (in\ MPa)$$

Determine:

1) *The expression of the forces per unit of mass acting on the solid.*
2) *The expression of the normal and tangent components of the forces acting on the boundary, indicating their sign according to the Mohr's circle criterion.*

Solution

1) The expression of the body forces is obtained directly from the internal equilibrium equation (4.26),

$$\mathbf{b} = -\frac{1}{\rho}\nabla\cdot\boldsymbol{\sigma} \quad \Longrightarrow \quad \mathbf{b} \stackrel{not}{\equiv} -\frac{1}{\rho}\begin{bmatrix} \dfrac{\partial}{\partial x}, & \dfrac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} xy & 5y \\ 5y & 4x \end{bmatrix} = -\frac{1}{\rho}\begin{bmatrix} y+5 \\ 0 \end{bmatrix}.$$

2) The normal (σ) and tangent (τ) components of the body forces acting on the boundary are given by

$$\sigma = \mathbf{t}\cdot\mathbf{n} \quad \text{and} \quad \tau = \mathbf{t}\cdot\mathbf{m} \quad \text{with} \quad \mathbf{t} = \mathbf{n}\cdot\boldsymbol{\sigma},$$

where $\mathbf{n}$ and $\mathbf{m}$ are the unit normal vector and the unit tangent vector of the boundary, respectively. The boundary of the solid can be divided into three parts, according to their $\mathbf{n}$ and $\mathbf{m}$ vectors:

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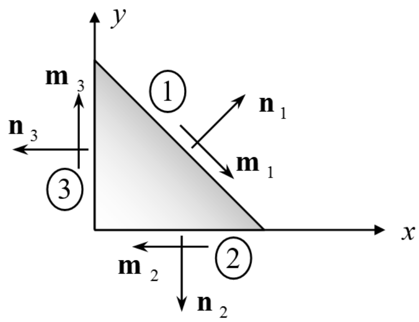

Boundary 1
The traction vector for this surface is

$$\mathbf{t}_1 = \mathbf{n}_1 \cdot \boldsymbol{\sigma} \overset{not}{\equiv} \frac{1}{\sqrt{2}} [1,\ 1] \begin{bmatrix} xy & 5y \\ 5y & 4x \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} xy + 5y \\ 5y + 4x \end{bmatrix} .$$

Then, the corresponding normal and tangent components of the body forces are

$$\sigma_1 = \mathbf{t}_1 \cdot \mathbf{n}_1 \overset{not}{\equiv} \frac{1}{\sqrt{2}} [xy + 5y,\ \ 5y + 4x] \ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} (4x + 10y + xy) \ ,$$

$$\tau_1 = \mathbf{t}_1 \cdot \mathbf{m}_1 \overset{not}{\equiv} \frac{1}{\sqrt{2}} [xy + 5y,\ \ 5y + 4x] \ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2} (-4x + xy) \ .$$

This is now particularized for the x and y values corresponding to the boundary, that is, for $y = 1 - x$ and $x \in [0, 1]$,

$$\begin{cases} \sigma_1 = \dfrac{1}{2} \left(10 - 5x - x^2 \right) & \text{with } \ x \in [0, 1] \ , \\ \tau_1 = \dfrac{1}{2} \left(-3x - x^2 \right) & \text{with } \ x \in [0, 1] \ . \end{cases}$$

Boundary 2
The traction vector for this surface is

$$\mathbf{t}_2 = \mathbf{n}_2 \cdot \boldsymbol{\sigma} \overset{not}{\equiv} [0,\ -1] \begin{bmatrix} xy & 5y \\ 5y & 4x \end{bmatrix} = \begin{bmatrix} -5y \\ -4x \end{bmatrix} .$$

Then, the corresponding normal and tangent components of the body forces are

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$$\sigma_2 = \mathbf{t}_2 \cdot \mathbf{n}_2 \overset{not}{\equiv} [-5y,\ \ -4x] \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 4x\,,$$

$$\tau_2 = \mathbf{t}_2 \cdot \mathbf{m}_2 \overset{not}{\equiv} [-5y,\ \ -4x] \begin{bmatrix} -1 \\ 0 \end{bmatrix} = 5y\,.$$

This is now particularized for the x and y values corresponding to the boundary, that is, for $y = 0$ and $x \in [0,1]$,

$$\begin{cases} \sigma_2 = 4x \quad \text{with } \ x \in [0,1]\,, \\ \tau_2 = 0\,. \end{cases}$$

Boundary 3
The traction vector for this surface is

$$\mathbf{t}_3 = \mathbf{n}_3 \cdot \boldsymbol{\sigma} \overset{not}{\equiv} [-1,\ 0] \begin{bmatrix} xy & 5y \\ 5y & 4x \end{bmatrix} = \begin{bmatrix} -xy \\ -5y \end{bmatrix}.$$

Then, the corresponding normal and tangent components of the body forces are

$$\sigma_3 = \mathbf{t}_3 \cdot \mathbf{n}_3 \overset{not}{\equiv} [-xy,\ \ -5y] \begin{bmatrix} -1 \\ 0 \end{bmatrix} = xy\,,$$

$$\tau_3 = \mathbf{t}_3 \cdot \mathbf{m}_3 \overset{not}{\equiv} [-xy,\ \ -5y] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -5y\,.$$

This is now particularized for the x and y values corresponding to the boundary, that is, for $x = 0$ and $y \in [0,1]$,

$$\begin{cases} \sigma_3 = 0\,, \\ \tau_3 = -5y \quad \text{with } \ y \in [0,1]\,. \end{cases}$$

Note that the results for boundaries 2 and 3 could have been obtained by direct comparison since they are a horizontal and a vertical surface, respectively:

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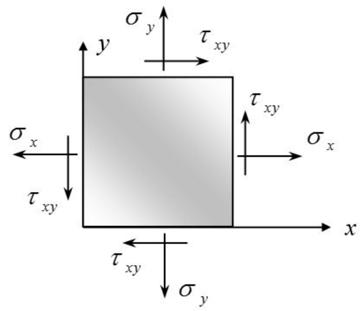
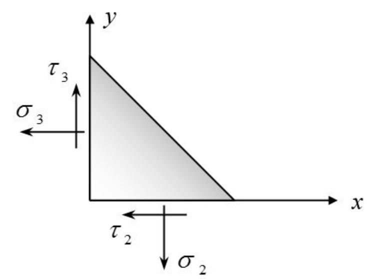
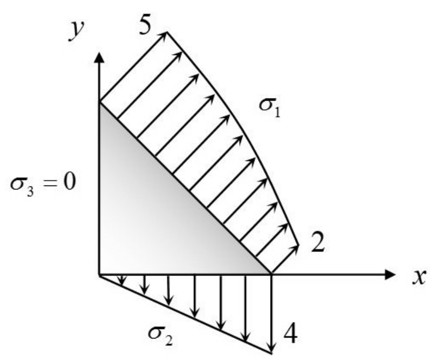
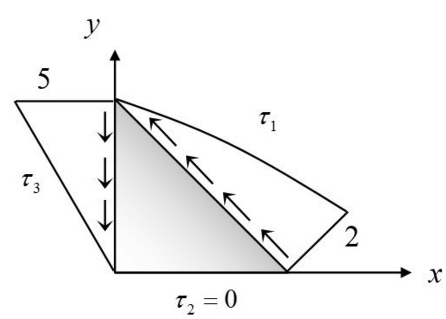




$$\begin{cases} \sigma_2 = \sigma_y & \text{with } x \in [0,1] \\ \tau_2 = \tau_{xy} & \text{with } y = 0 \end{cases} \qquad \begin{cases} \sigma_3 = \sigma_x & \text{with } x = 0 \\ \tau_3 = -\tau_{xy} = -5y & \text{with } y \in [0,1] \end{cases}$$

Finally, the expression of the normal and tangent components of the forces acting on the boundary of the solid are drawn, indicating the most significant values.





Problem 4.2 – *The following is known of a stress state.*

1) *The z-direction is a principal stress direction and* $\sigma_{zz} = a$.
2) *The mean stress is* $\sigma_m = a > 0$.
3) *The maximum shear stress in the planes that are parallel to the z-axis is* $\tau_{max} = b > 0$.

Draw, indicating the most significant values, the Mohr's circle in three dimensions of the stress tensor and its deviatoric tensor.

Solution

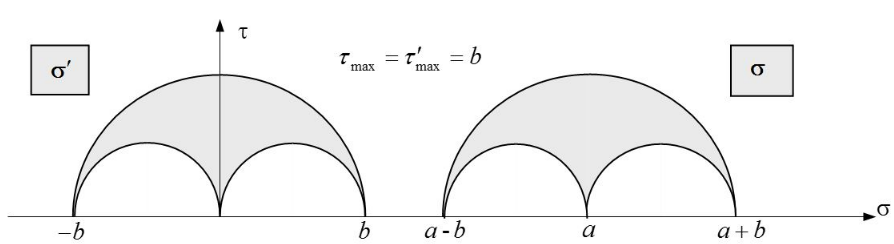
Note that the only difference there will be between the two circles is that one will be translated a distance σ_m with respect to the other.
By means of the definition of the deviatoric stress tensor,

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \sigma_m \mathbf{1} \quad \Longrightarrow \quad \sigma'_{zz} = \sigma_{zz} - \sigma_m = a - a = 0 \quad \Longrightarrow \quad \sigma'_{zz} = 0$$

is deduced. The fact that the trace is an invariant and that the trace of the deviatoric stress tensor is zero, $\mathrm{Tr}(\boldsymbol{\sigma}') = 0$, results in

$$\sigma'_{xx} + \sigma'_{yy} = 0 \quad \Longrightarrow \quad \begin{cases} \sigma'_{zz} = \sigma'_2 = 0 \, , \\ \sigma'_1 + \sigma'_3 = 0 \, . \end{cases}$$

Finally, the radius of the major circumference (between σ'_1 and σ'_3) is determined through the application of condition *3)*. The two Mohr's circles are shown below.



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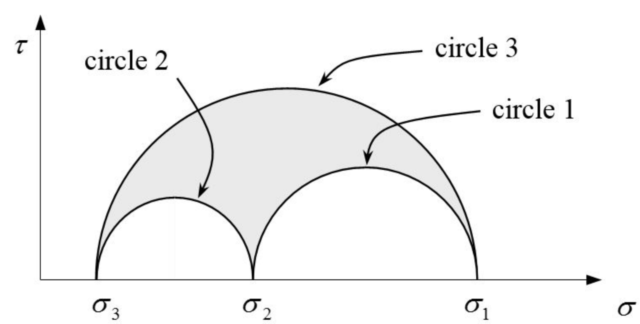
Problem 4.3 – *Given the following information of a stress state in a certain point,*

1) $\sigma_x = 1$ *(where the x-axis is a principal stress direction).*
2) *The maximum shear stress in the planes that are parallel to the x-axis is 3.*
3) *The maximum shear stress in the planes that are parallel to the minor principal stress direction is 2.*

obtain all the possible Mohr's circles corresponding to this state, indicating the values of the principal stresses.

Solution

The following property of the Mohr's circle in 3D must be taken into account to solve this problem.



Circle number:

1 – corresponds to planes parallel to the principal stress direction of σ_3.
2 – corresponds to planes parallel to the principal stress direction of σ_1.
3 – corresponds to planes parallel to the principal stress direction of σ_2.

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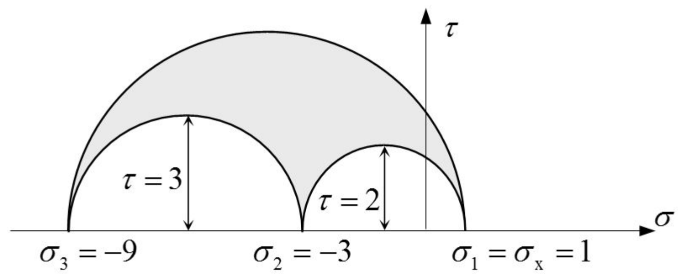
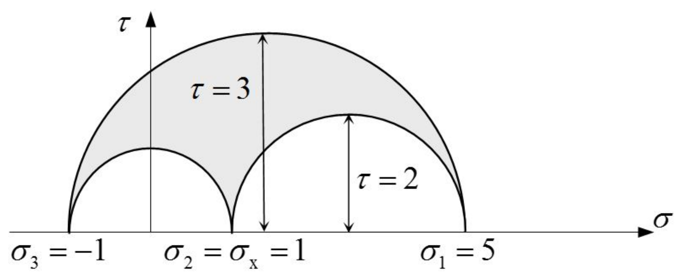
Then, the following possibilities are considered.

1. σ_x is the major principal stress, which results in the following Mohr's circle.



2. σ_x is the intermediate principal stress, which results in the following Mohr's circle.



3. σ_x is the minor principal stress, which is an impossible situation because conditions *2)* and *3)* cannot be satisfied at the same time since they refer to the maximum shear stress on the same plane.

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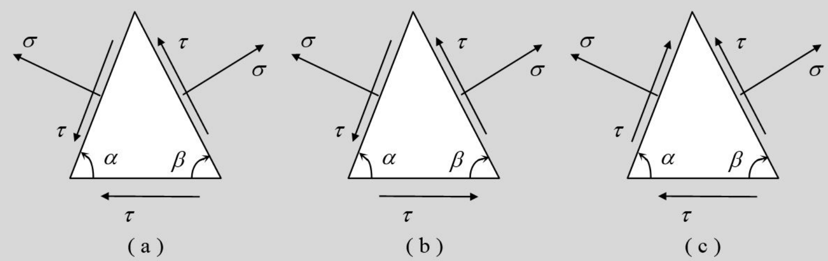
Problem 4.4 – *Determine the values of* α *and* β *for which the following stress states are possible, considering that* $\sigma > 0$ *and* $\tau = 0.5\sigma$.



Solution

The problem is solved following the same steps in all three cases, which are:

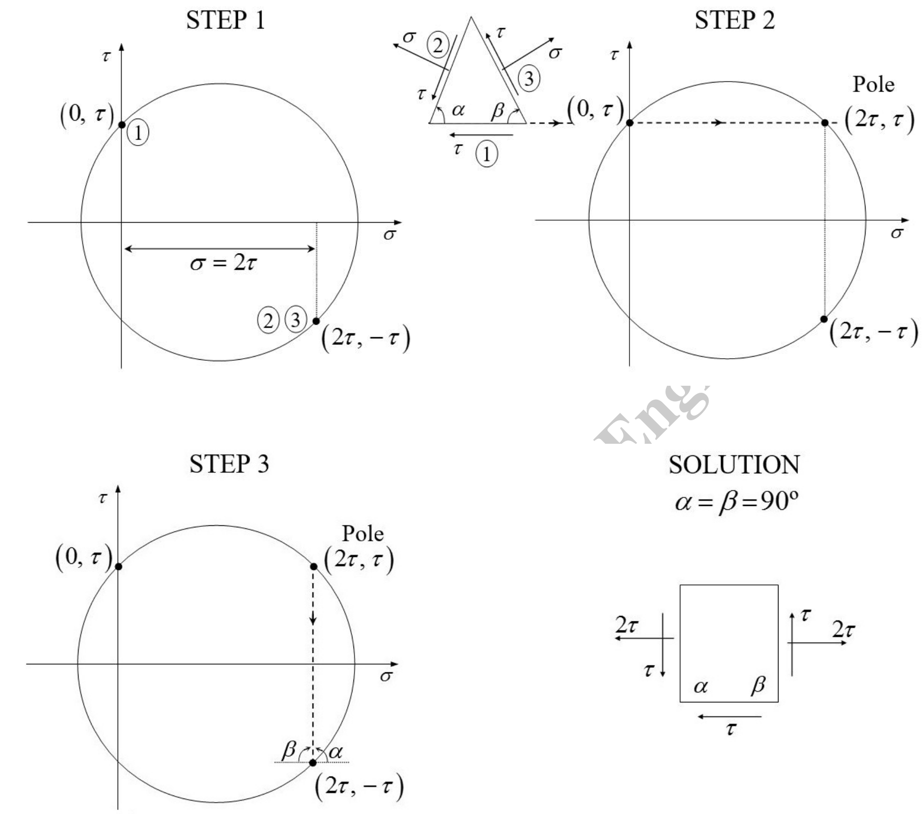
Step 1: Draw the Mohr's circle corresponding to the stress state. Even for the stress states in which only two different pairs of points (σ, τ) belonging to the Mohr's circle are given, the circle can be drawn taking into account that it must be symmetric with respect to the longitudinal axis.

Step 2: Identify the pole. In all cases, a straight horizontal line is drawn, which must contain the point of the Mohr's circle corresponding to the horizontal plane. Then, the pole is identified as the point where the line crosses the circle again. The horizontal plane is used to identify the pole because, of the three planes shown for each stress state, it is the only one with a known orientation.

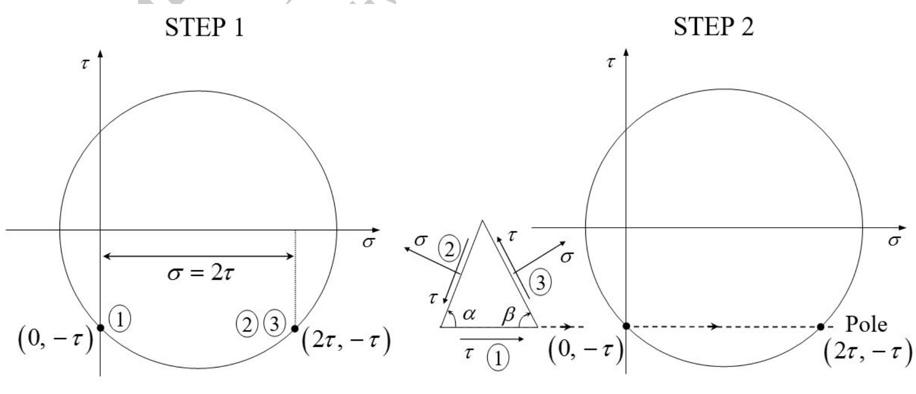
Step 3: Draw a straight line joining the pole and the two (σ, τ) points corresponding to the planes whose inclination must be obtained. The inclination of these planes and, thus, the angles α and β are given directly by the orientation of the lines drawn.

Step 4: The schematic description of the stress states on the three planes can be redrawn with the appropriate inclination.

(a)



(b)



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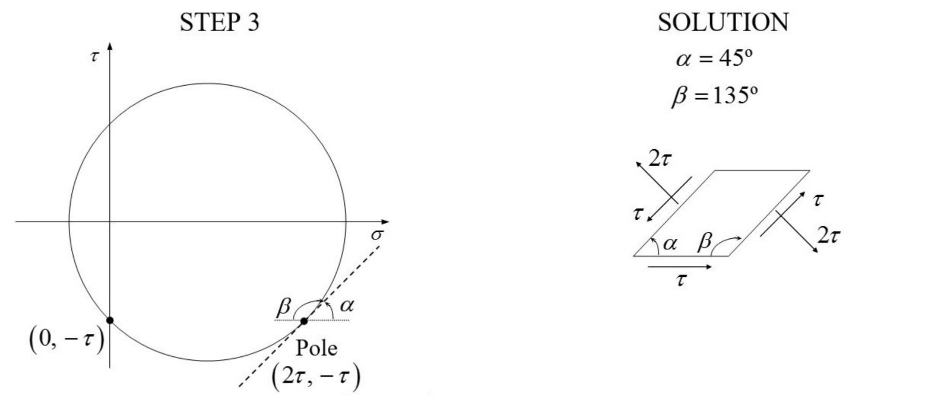
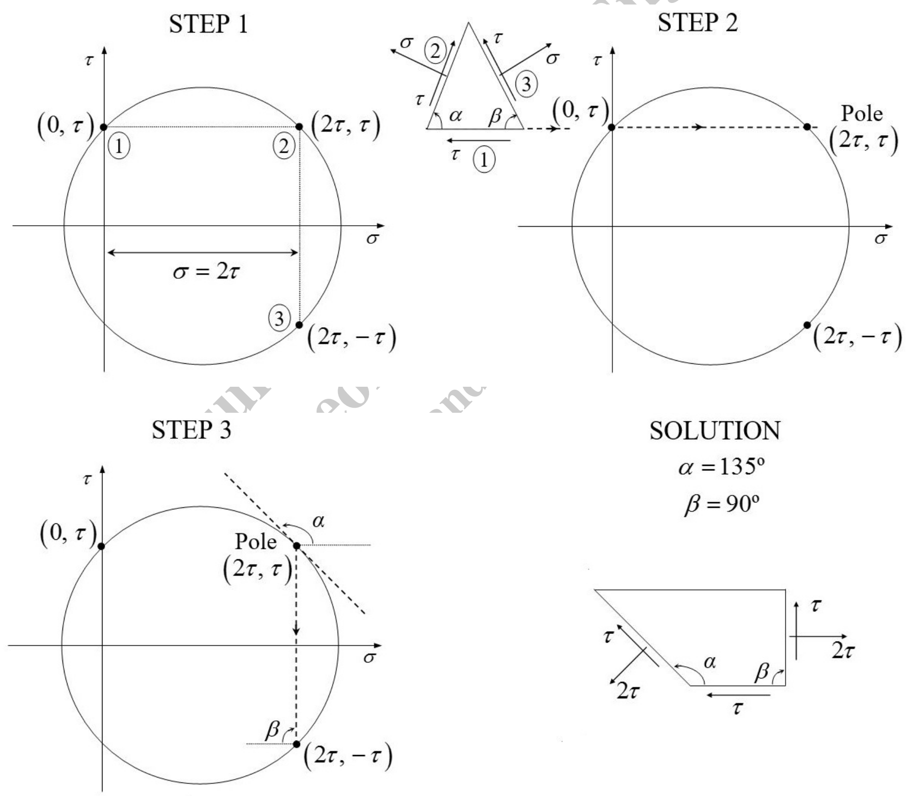


(c)



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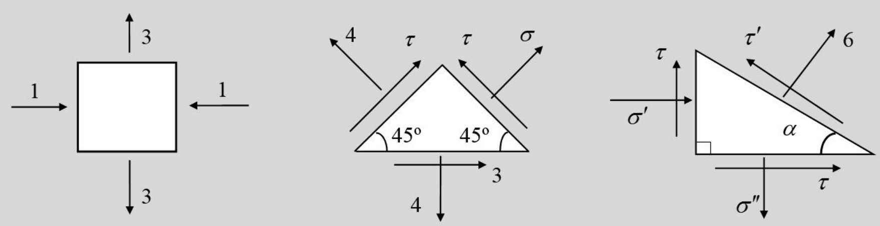
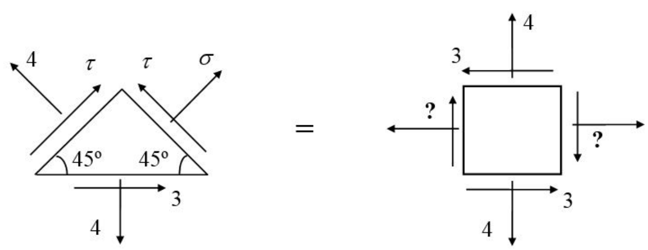
Problem 4.5 – *Calculate the possible values of σ, σ', σ'', τ, τ' and α for which state III is the sum of states I and II, considering that $\tau \geq 0$.*



STATE I STATE II STATE III

Solution

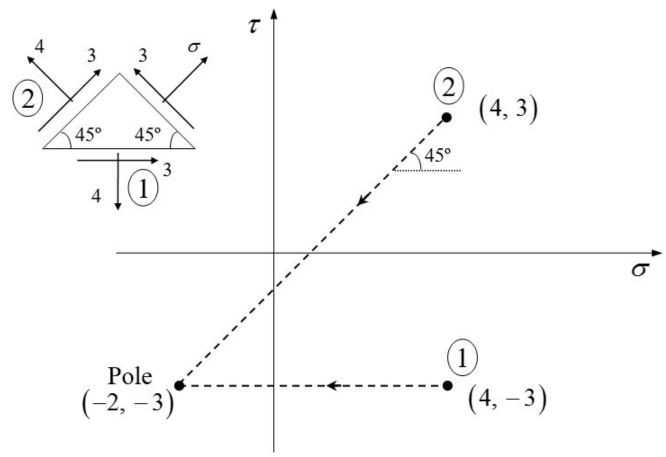
Stress state II on the vertical plane must be found to be able to add states I and II together.



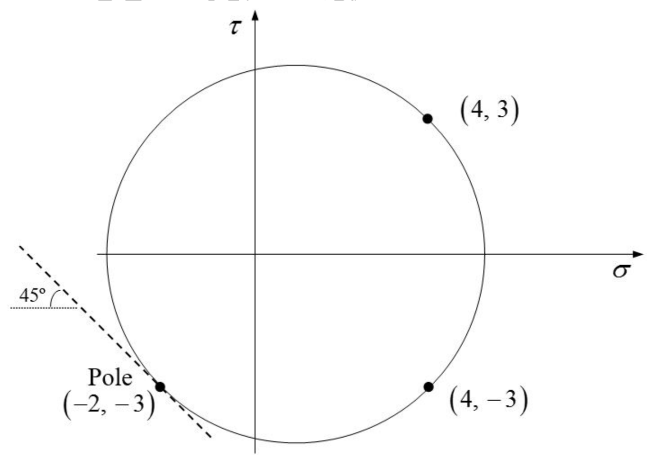
The Mohr's circle of state II will allow determining the normal and shear stress on the vertical plane. The known stress state on the horizontal plane $(4,-3)$ belongs to the Mohr's circle. Since it is known to be symmetric with respect to the longitudinal axis, the stress state $(4,3)$ must also belong to the Mohr's circle. Observing the figure representing state II, and considering that $\tau \geq 0$, it is concluded that this point, $(4,3)$, corresponds to the stress state on the plane inclined at 45° in the counterclockwise direction. Thus,

$$\boxed{\tau = 3}\,.$$

Now, a third point belonging to the Mohr's circle must be obtained in order to be able to draw the complete circle. Because there exists only one pole and it must belong to the Mohr's circle, finding this point will allow completing the circle. A straight horizontal line (parallel to the horizontal plane) is draw at point $(4,-3)$, which corresponds to the stress state on a horizontal plane. Another straight line, parallel to the other plane with a known stress state, the plane inclined at 45° in the counterclockwise direction, is drawn passing through the corresponding stress state, $(4,3)$. The point where these to lines meet provide the pole of the Mohr's circle, which is found to be at $(-2,-3)$:



Once these three points are known, the Mohr's circle can be drawn. Before calculating the stress state on the vertical plane, the value of σ is sought. To obtain the stress state on the plane inclined at 45° in the clockwise direction, a straight line must be drawn, parallel to this plane, that crosses the pole.



This results in a line tangent to the pole, therefore, the stress state corresponding to the pole is also the stress state on this plane and

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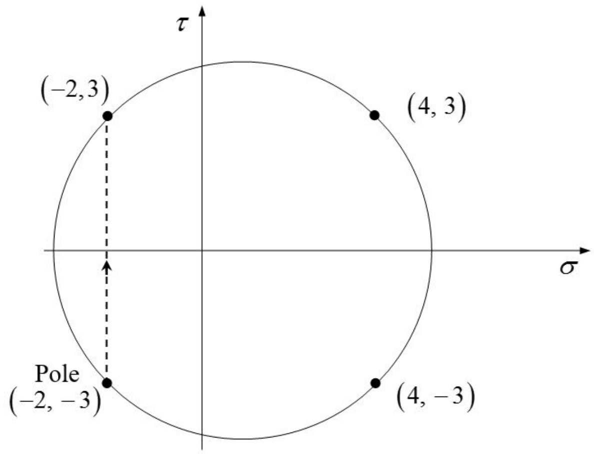
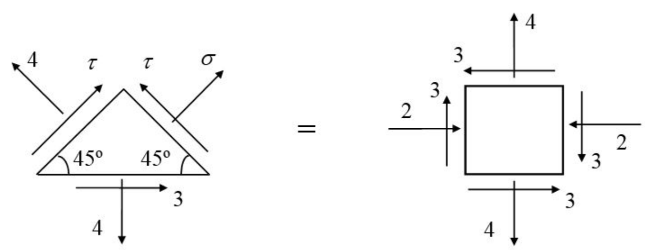
$$\boxed{\sigma = -2}\,.$$

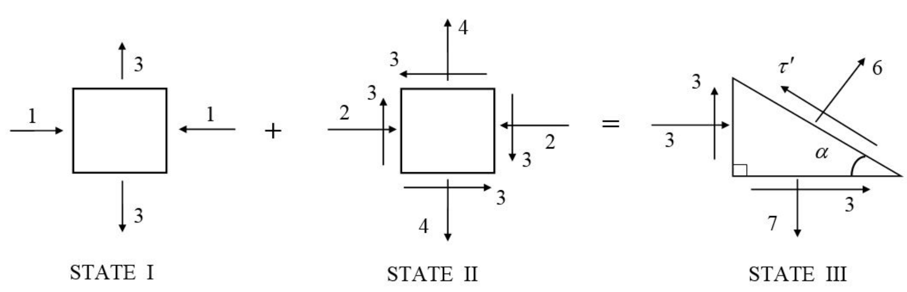
Finally, a vertical line is drawn from the pole and the intersection of this line with the Mohr's circle provides the stress state on the vertical plane, which results in $(-2,3)$.



Then, stress state II is defined on a vertical and horizontal plane as follows.



This allows adding stress states I and II to obtain state III,



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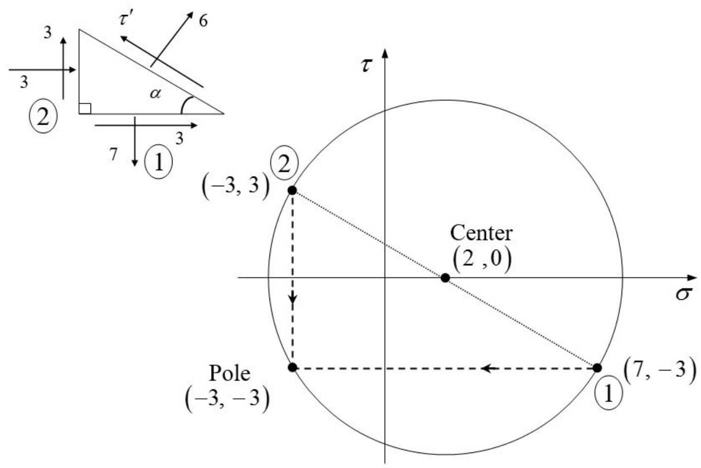
revealing the values of σ' and σ''.

$$\boxed{\begin{array}{l} \sigma' = 3 \\ \sigma'' = 7 \end{array}}$$

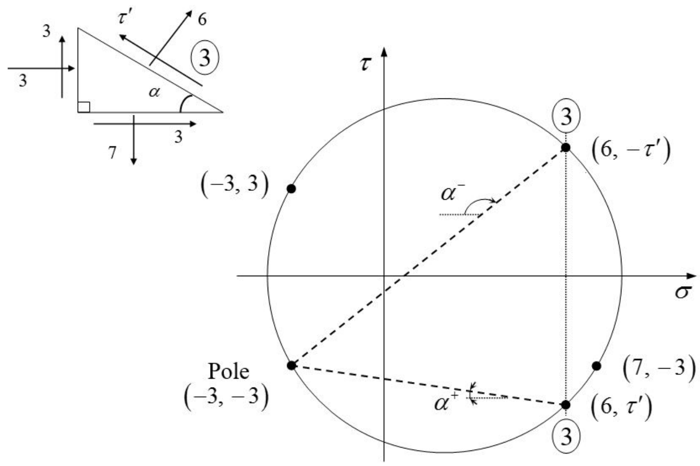
The values of τ' and α remain to be found. To this aim, the Mohr's circle of stress state III must be drawn. The points corresponding to the known stress states on the vertical and horizontal planes are marked on the $\sigma - \tau$ space and, in a procedure analogous to the one used for the Mohr's circle of state II, the pole is obtained. The circle can now be drawn and simple trigonometry allows calculating its center at $(2,0)$, which will be useful in the calculation of τ' and α.



Drawing a vertical line at $\sigma = 6$ provides the values of τ' at the intersection of this line with the circle. Two options are possible, one corresponding to a positive value of τ' and another corresponding to the same value but with a negative sign. Following the sign criterion for the Mohr's circle, and to be consistent with the directions drawn in the figure representing state III, the value of τ' must be

$$\boxed{\tau' = -3\sqrt{2}}\ .$$

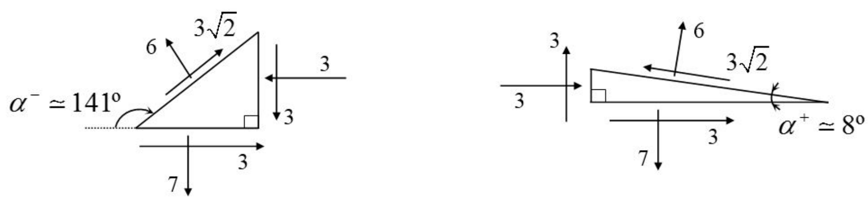
Since there are two possible values of τ', two values of α will exist, each corresponding to one of the τ' values. To obtain the values of α, a straight line is drawn from the pole to each of the points representing the possible stress states of the plane inclined at α in a clockwise direction.



Determining the inclination of these two lines will result directly in the possible values of α.

$$\boxed{\begin{array}{ll} \tau' = -3\sqrt{2} & \Rightarrow\ \alpha^{+} = 180^{\circ} - \arctan\left(\frac{1+\sqrt{2}}{3}\right) \simeq 141^{\circ} \\ -\tau' = 3\sqrt{2} & \Rightarrow\ \alpha^{-} = \arctan\left(\frac{\sqrt{2}-1}{3}\right) \simeq 8^{\circ} \end{array}}$$

The two possible configurations of stress state III are pictured below.



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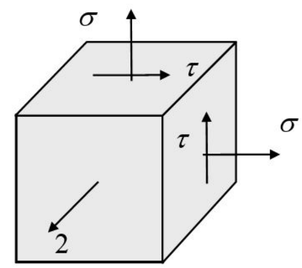
EXERCISES

4.1 – *Determine all the possible values of* $\sigma\,(\sigma > 0)$ *and* $\tau\,(\tau > 0)$ *in the figure knowing that the maximum shear stress on any plane at the point is* $\tau_{max} = 1$.



4.2 – *The following is known of the stress state in a point of a continuous medium. The maximum shear stress in planes parallel to the principal stress direction of* σ_1 *is* $\tau_{max} = 2$. *Obtain all the values of* σ_1, σ_2 *and* σ_3 *that make possible the stress state* $\sigma = 2$ *and* $\tau = 2$ *on a certain plane for the following cases (separately).*

a) The maximum shear stress in planes parallel to the principal stress direction of σ_2 *is* $\tau_2^{max} = 2$.

b) The maximum shear stress in planes parallel to the principal stress direction of σ_3 *is* $\tau_3^{max} = 0$.

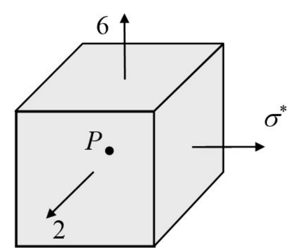
c) The maximum shear stress in planes parallel to the principal stress direction of σ_2 *is* $\tau_2^{max} = 4$.

4.3 – *Determine for which values of* σ^* *the following stress states are possible in the planes belonging to P.*

a) $\sigma = 4$ *and* $\tau = 2$.

b) $\sigma = 4$ *and* $\tau = 1$.

c) $\sigma = 7$ *and* $\tau = 0$.



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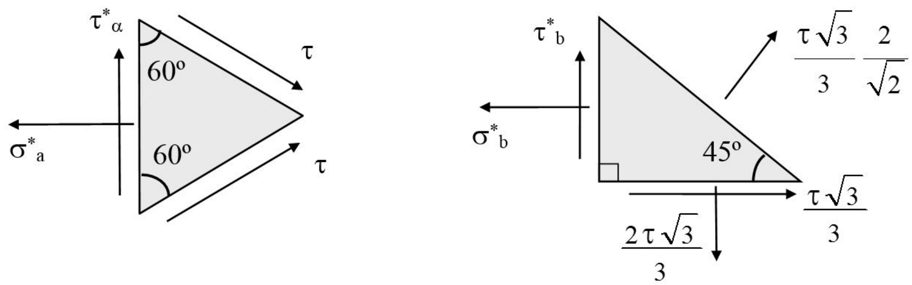
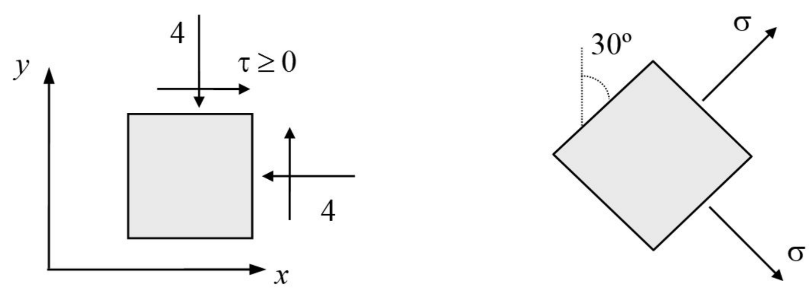
4.4 – *Obtain, in terms of* τ*, the principal stresses and the value of the maximum shear stress of the state that results from the sum of states I and II.*



STATE I STATE II

4.5 – *Given states I and II, determine the possible values of* σ *and* τ *for which state III = I + II verifies that the principal stress* σ_2 *is positive and its direction forms a* 30° *angle with the y-axis.*



STATE I STATE II

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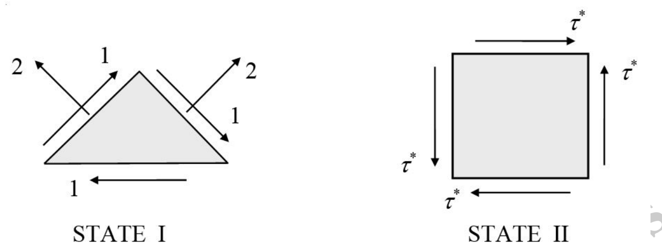
4.6 – *Determine all the possible values of τ^* for which the stress state that is the sum of states I and II verifies the following conditions (separately).*



a) There do not exist tensile stresses on any plane.
b) There do not exist compressive stresses on any plane.
c) The maximum shear stress (τ_{max}) is less than 2.
d) It is a pure shear stress state.
e) It is a hydrostatic stress state.

Chapter 5
Balance Principles

5.1 Introduction

Continuum Mechanics is based on a series of general postulates or principles that are assumed to always be valid, regardless of the type of material and the range of displacements or deformations. Among these are the so-called *balance principles*:

- Conservation of mass
- Balance of linear momentum
- Balance of angular momentum
- Balance of energy (or first law of thermodynamics)

A restriction that cannot be rigorously understood as a balance principle must be added to these laws, which is introduced by the

- Second law of thermodynamics

5.2 Mass Transport or Convective Flux

In continuum mechanics, the term *convection* is associated with *mass transport* in the medium, which derives from the motion of its particles. The continuous medium is composed of particles, some of whose properties are associated with the amount of mass: specific weight, angular momentum, kinetic energy, etc. Then, when particles move and transport their mass, a transport of the these properties occurs, named *convective transport* (see Figure 5.1).

Consider $\mathcal{A}$, an arbitrary (scalar, vector or tensor) property of the continuous medium, and $\Psi(\mathbf{x},t)$, the description of the amount of said property per *unit of mass* of the continuous medium. Consider also S, a control surface, i.e., a surface fixed in space (see Figure 5.2). Due to the motion of the particles in the medium, these cross the surface along time and, in consequence, there exists a certain amount of the property $\mathcal{A}$ that, associated with the mass transport, crosses the control surface S per unit of time.

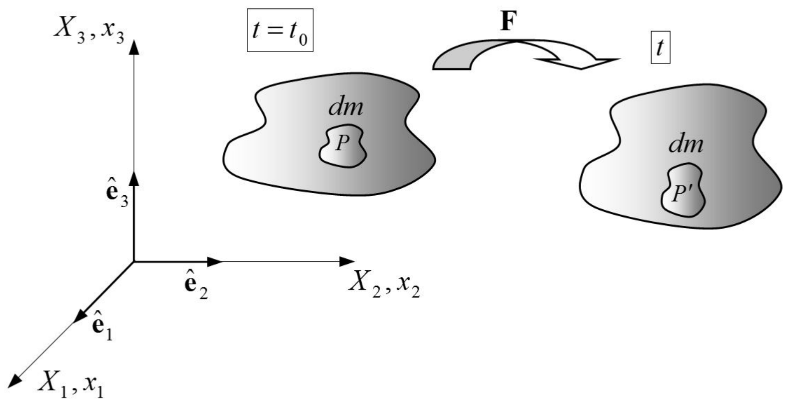


Figure 5.1: Convective transport in the continuous medium.

Definition 5.1. The *convective flux* (or mass transport flux) of a generic property $\mathcal{A}$ through a control surface S is the amount of $\mathcal{A}$ that, due to mass transport, crosses the surface S per unit of time.

$$\begin{array}{l}\text{convective flux} \\ \text{of } \mathcal{A} \text{ through } S\end{array} \stackrel{not}{=} \Phi_S = \frac{\text{amount of } \mathcal{A} \text{ crossing } S}{\text{unit of time}}$$

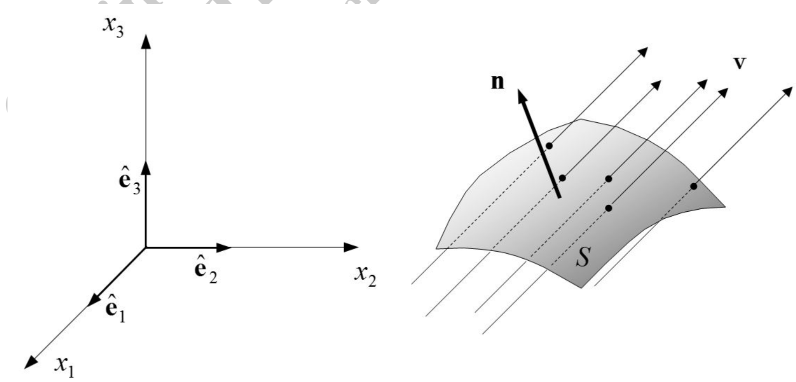


Figure 5.2: Convective flux through a control surface.

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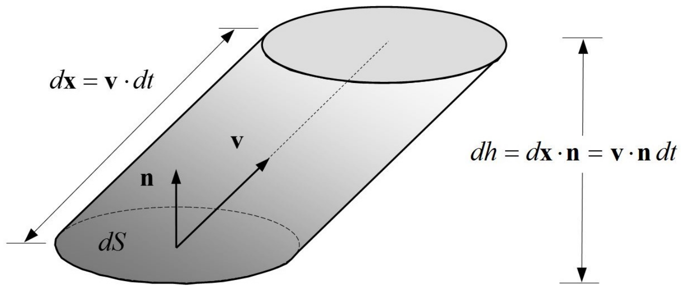

Figure 5.3: Cylinder occupied by the particles that have crossed dS in the time interval $[t,\, t+dt]$.

To obtain the mathematical expression of the convective flux of $\mathcal{A}$ through the surface S, consider a differential surface element dS and the velocity vector $\mathbf{v}$ of the particles that at time t are on dS (see Figure 5.3). In a time differential dt, these particles will have followed a pathline $d\mathbf{x} = \mathbf{v}\, dt$, such that at the instant of time $t+dt$ they will occupy a new position in space. Taking now into account all the particles that have crossed dS in the time interval $[t,\, t+dt]$, these will occupy a cylinder generated by translating the base dS along the directrix $d\mathbf{x} = \mathbf{v}\, dt$, and whose volume is given by

$$dV = dS\, dh = \mathbf{v} \cdot \mathbf{n}\, dt\, dS\ . \tag{5.1}$$

Since the volume (dV) of the particles crossing dS in the time interval $[t,\, t+dt]$ is known, the mass crossing dS in this same time interval can be obtained by multiplying (5.1) by the density,

$$dm = \rho\, dV = \rho \mathbf{v} \cdot \mathbf{n}\, dt\, dS\ . \tag{5.2}$$

Finally, the amount of $\mathcal{A}$ crossing dS in the time interval $[t,\, t+dt]$ is calculated by multiplying (5.2) by the function Ψ (amount of $\mathcal{A}$ per unit of mass),

$$\Psi\, dm = \rho\, \Psi\, \mathbf{v} \cdot \mathbf{n}\, dt\, dS\ . \tag{5.3}$$

Dividing (5.3) by dt yields the amount of the property that crosses the differential control surface dS per *unit of time*,

$$d\,\Phi_S = \frac{\Psi\, dm}{dt} = \rho\, \Psi\, \mathbf{v} \cdot \mathbf{n}\, dS\ . \tag{5.4}$$

Integrating (5.4) over the control surface S results in the amount of the property $\mathcal{A}$ crossing the whole surface S per unit of time, that is, the *convective flux of the property $\mathcal{A}$ through S*.

$$\boxed{\left.\begin{array}{l}\text{convective flux}\\ \text{of } \mathcal{A} \text{ through } S\end{array}\right\}\ \Phi_S = \int\limits_S \rho\, \Psi \mathbf{v} \cdot \mathbf{n}\, dS} \tag{5.5}$$

Example 5.1 – *Compute the magnitude Ψ and the convective flux Φ_S corresponding to the following properties: a) volume, b) mass, c) linear momentum, d) kinetic energy.*

Solution

a) If the property $\mathcal{A}$ is the volume occupied by the particles, then Ψ is the volume per unit of mass, that is, the inverse of the density. Therefore,

$$\mathcal{A} \equiv V \quad \text{and} \quad \Psi = \frac{1}{\rho} \quad \text{lead to} \quad \Phi_S = \int_S \mathbf{v} \cdot \mathbf{n} \, dS = \text{volume flow rate} \, .$$

b) If the property $\mathcal{A}$ is the mass, then Ψ is the mass per unit of mass, that is, the unit. Therefore,

$$\mathcal{A} \equiv M \quad \text{and} \quad \Psi = 1 \quad \text{lead to} \quad \Phi_S = \int_S \rho \, \mathbf{v} \cdot \mathbf{n} \, dS \, .$$

c) If the property $\mathcal{A}$ is the linear momentum (= mass $\times$ velocity), then Ψ is the linear momentum per unit of mass, that is, the velocity. Therefore,

$$\mathcal{A} \equiv m \mathbf{v} \quad \text{and} \quad \Psi = \mathbf{v} \quad \text{lead to} \quad \Phi_S = \int_S \rho \, \mathbf{v} \, (\mathbf{v} \cdot \mathbf{n}) \, dS \, .$$

(Note that in this case Ψ and the convective flux Φ_S are vectors).

d) If the property $\mathcal{A}$ is the kinetic energy then Ψ is the kinetic energy per unit of mass. Therefore,

$$\mathcal{A} \equiv \frac{1}{2} m \, |\mathbf{v}|^2 \quad \text{and} \quad \Psi = \frac{1}{2} \, |\mathbf{v}|^2 \quad \text{lead to} \quad \Phi_S = \int_S \frac{1}{2} \rho \, |\mathbf{v}|^2 \, (\mathbf{v} \cdot \mathbf{n}) \, dS \, .$$

Remark 5.1. In a closed control surface[1], $S = \partial V$, the expression of the convective flux corresponds to the *net outflow*, defined as the *outflow minus the inflow* (see Figure 5.4), that is,

$$\text{net convective flux of } \mathcal{A} \overset{not}{=} \Phi_{\partial V} = \int_{\partial V} \rho \, \Psi \, \mathbf{v} \cdot \mathbf{n} \, dS \, .$$

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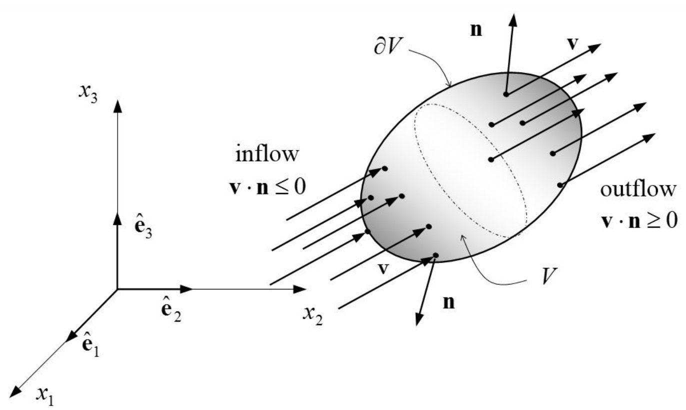

Figure 5.4: Net outflow through a closed control surface.

Remark 5.2. The convective flux of any property through a material surface is always null. Indeed, the convective flux of any property is associated, by definition, with the mass transport (of particles) and, on the other hand, a material surface is always formed by the same particles and cannot be crossed by them. Consequently, there is no mass transport through a material surface and, therefore, there is no convective flux through it.

Remark 5.3. Some properties can be transported within a continuous medium in a manner not necessarily associated with mass transport. This form of *non-convective* transport receives several names (conduction, diffusion, etc.) depending on the physical problem being studied. A typical example is heat flux by conduction.
The non-convective transport of a property is characterized by the *non-convective flux vector (or tensor)* $\mathbf{q}(\mathbf{x},t)$, which allows defining the (non-convective) flux through a surface S with unit normal vector $\mathbf{n}$ as

$$\text{non-convective flux} = \int_S \mathbf{q}\cdot\mathbf{n}\, dS \, .$$

[1] Unless stated otherwise, when dealing with closed surfaces, the positive direction of the unit normal vector **n** is taken in the *outward direction* of the surface.

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5.3 Local and Material Derivatives of a Volume Integral

Consider $\mathcal{A}$, an arbitrary (scalar, vector or tensor) property of the continuous medium, and μ, the description of the amount of said property per *unit of volume*[2],

$$\mu(\mathbf{x},t) = \frac{\text{amount of } \mathcal{A}}{\text{unit of volume}} \ . \tag{5.6}$$

Consider an arbitrary volume V in space. At time t, the total amount $Q(t)$ of the property contained in this volume is

$$Q(t) = \int_V \mu(\mathbf{x},t) \ dV \ . \tag{5.7}$$

To compute the content of property $\mathcal{A}$ at a different time $t+\Delta t$, the following two situations arise:

1) A *control volume* V is considered and, therefore, it is fixed in space and crossed by the particles along time.
2) A *material volume* that at time t occupies the spatial volume $V_t \equiv V$ is considered and, thus, the volume occupies different positions in space along time.

Different values of the amount $Q(t+\Delta t)$ are obtained for each case, and computing the difference between the amounts $Q(t+\Delta t)$ and $Q(t)$ when $\Delta t \to 0$ yields

$$Q'(t) = \lim_{\Delta t \to 0} \frac{Q(t+\Delta t) - Q(t)}{\Delta t} \ , \tag{5.8}$$

resulting in two *different* definitions of the time derivative, which lead to the concepts of *local derivative* and *material derivative* of a volume integral.

5.3.1 Local Derivative

> **Definition 5.2.** The *local derivative of the volume integral*,
>
> $$Q(t) = \int_V \mu(\mathbf{x},t) \ dV \ ,$$
>
> is the time derivative of $Q(t)$ when the volume V is a volume fixed in space (control volume), see Figure 5.5. The notation
>
> $$\text{local derivative} \stackrel{not}{=} \frac{\partial}{\partial t} \int_V \mu(\mathbf{x},t) \ dV$$
>
> will be used.

[2] μ is related to $\Psi = (\text{amount of } \mathcal{A})/(\text{unit of mass})$ through $\mu = \rho \Psi$ and has the same tensor order as the property $\mathcal{A}$.

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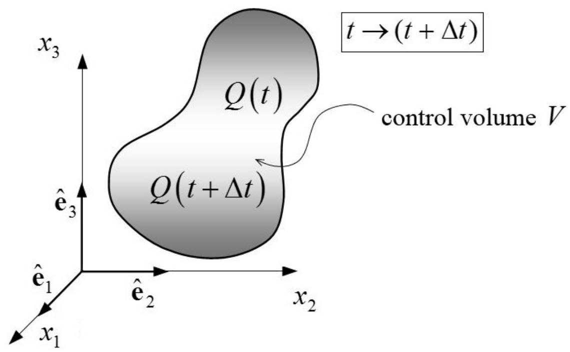

Figure 5.5: Local derivative of a volume integral.

The amount Q of the generic property $\mathcal{A}$ in the *control volume* V at times t and $t+\Delta t$ is, respectively,

$$Q(t) = \int_V \mu(\mathbf{x},t)\, dV \quad \text{and} \quad Q(t+\Delta t) = \int_V \mu(\mathbf{x},t+\Delta t)\, dV \,. \tag{5.9}$$

Using (5.9) in addition to the concept of time derivative of $Q(t)$ results in[3]

$$\begin{aligned} Q'(t) &= \frac{\partial}{\partial t}\int_V \mu(\mathbf{x},t)\, dV = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big(Q(t+\Delta t) - Q(t)\Big) = \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\int_V \mu(\mathbf{x},t+\Delta t)\, dV - \int_V \mu(\mathbf{x},t)\, dV\right) = \\ &= \int_V \underbrace{\lim_{\Delta t \to 0} \frac{\mu(\mathbf{x},t+\Delta t) - \mu(\mathbf{x},t)}{\Delta t}}_{\displaystyle \frac{\partial \mu(\mathbf{x},t)}{\partial t}} dV = \int_V \underbrace{\frac{\partial \mu(\mathbf{x},t)}{\partial t}}_{\substack{\text{local} \\ \text{derivative} \\ \text{of } \mu}} dV \,, \end{aligned} \tag{5.10}$$

which yields the mathematical expression of the local derivative of a volume integral.

> *Local derivative of a volume integral*
>
> $$\frac{\partial}{\partial t}\int_V \mu(\mathbf{x},t)\, dV = \int_V \frac{\partial \mu(\mathbf{x},t)}{\partial t}\, dV \tag{5.11}$$

[3] Note that the integration domain does not vary when the volume V is considered as a control volume and, therefore, is fixed in space.

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5.3.2 Material Derivative

Definition 5.3. The *material derivative of the volume integral*,

$$Q(t) = \int_V \mu(\mathbf{x},t)\, dV \ ,$$

is the time derivative of $Q(t)$ when the volume V_t is a material volume (mobile in space), see Figure 5.6. The notation

$$\text{material derivative} \overset{not}{=} \frac{d}{dt} \int_{V_t} \mu(\mathbf{x},t)\, dV$$

will be used.

The content Q of the generic property $\mathcal{A}$ in the *material volume* V_t at times t and $t+\Delta t$ is, respectively,

$$Q(t) = \int_{V_t} \mu(\mathbf{x},t)\, dV \quad \text{and} \quad Q(t+\Delta t) = \int_{V_{t+\Delta t}} \mu(\mathbf{x},t+\Delta t)\, dV \ . \tag{5.12}$$

Then, the material derivative is mathematically expressed as[4]

$$\begin{aligned} Q'(t) &= \left. \frac{d}{dt} \int_{V_t} \mu(\mathbf{x},t)\, dV \right|_{V_t \equiv V} = \lim_{\Delta t \to 0} \frac{Q(t+\Delta t) - Q(t)}{\Delta t} = \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(\int_{V_{t+\Delta t}} \mu(\mathbf{x},t+\Delta t)\, dV - \int_{V_t} \mu(\mathbf{x},t)\, dV \right) \ . \end{aligned} \tag{5.13}$$

The following step consists in introducing two variable substitutions, each suitable for one of the two integrals in (5.13), which lead to the same integration domain in both expressions. These variable substitutions are given by the equation of motion $\mathbf{x} = \varphi(\mathbf{X},t)$, particularized for times t and $t+\Delta t$,

$$\begin{cases} \mathbf{x}_t = \varphi(\mathbf{X},t) \ \rightarrow \ \underbrace{(dx_1\, dx_2\, dx_3)_t}_{dV_t} = |\mathbf{F}(\mathbf{X},t)| \underbrace{(dX_1\, dX_2\, dX_3)}_{dV_0} , \\ \mathbf{x}_{t+\Delta t} = \varphi(\mathbf{X},t+\Delta t) \ \rightarrow \ \underbrace{(dx_1\, dx_2\, dx_3)_{t+\Delta t}}_{dV_{t+\Delta t}} = |\mathbf{F}(\mathbf{X},t+\Delta t)| \underbrace{(dX_1\, dX_2\, dX_3)}_{dV_0} , \end{cases} \tag{5.14}$$

[4] Note that the integration domains are now different at times t and $t+\Delta t$.

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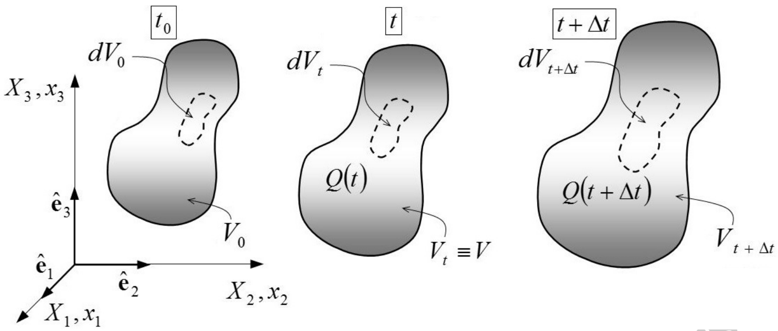

Figure 5.6: Material derivative of a volume integral.

where the identity $dV_t = |\mathbf{F}(\mathbf{X},t)|\, dV_0$ has been taken into account. The variable substitutions in (5.14) are introduced in (5.13), resulting in

$$
\begin{aligned}
\frac{d}{dt}\int_{V_t} \mu(\mathbf{x},t)\; dV &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_{V_0} \overbrace{\mu\left(\mathbf{x}(\mathbf{X},t+\Delta t),t+\Delta t\right)}^{\bar{\mu}(\mathbf{X},t+\Delta t)}\, |\mathbf{F}(\mathbf{X},t+\Delta t)|\; dV_0 \right. \\
&\qquad \left. -\int_{V_0} \underbrace{\mu\left(\mathbf{x}(\mathbf{X},t),t\right)}_{\bar{\mu}(\mathbf{X},t)}\, |\mathbf{F}(\mathbf{X},t)|\; dV_0 \right) = \\
&= \int_{V_0} \underbrace{\lim_{\Delta t\to 0}\frac{\bar{\mu}(\mathbf{X},t+\Delta t)\,|\mathbf{F}(\mathbf{X},t+\Delta t)| - \bar{\mu}(\mathbf{X},t)\;|\mathbf{F}(\mathbf{X},t)|}{\Delta t}}_{\frac{\partial}{\partial t}\left(\bar{\mu}(\mathbf{X},t)\;|\mathbf{F}(\mathbf{X},t)|\right) = \frac{d}{dt}\left(\mu(\mathbf{x},t)\;|\mathbf{F}(\mathbf{x},t)|\right)} dV_0 = \\
&= \int_{V_0} \frac{d}{dt}\Big(\mu\;|\mathbf{F}|\Big)\, dV_0\,.
\end{aligned}
\tag{5.15}
$$

Finally, expanding the last integral in (5.15)[5] and considering the equality $d|\mathbf{F}|/dt = |\mathbf{F}|\;\nabla\cdot\mathbf{v}$ yields

$$
\begin{aligned}
\frac{d}{dt}\int_{V_t} \mu(\mathbf{x},t)\; dV &= \int_{V_0}\frac{d}{dt}\Big(\mu\;|\mathbf{F}|\Big)\, dV_0 = \int_{V_0}\left(\frac{d\mu}{dt}\,|\mathbf{F}| + \underbrace{\frac{d|\mathbf{F}|}{dt}}_{|\mathbf{F}|\;\nabla\cdot\mathbf{v}}\;\mu\right) dV_0 = \\
&= \int_{V_0}\left(\frac{d\mu}{dt} + \mu\nabla\cdot\mathbf{v}\right)\underbrace{|\mathbf{F}|\; dV_0}_{dV_t} = \int_{V_t}\left(\frac{d\mu}{dt} + \mu\nabla\cdot\mathbf{v}\right) dV\;,
\end{aligned}
\tag{5.16}
$$

[5] The change of variable $\mathbf{x}_t = \varphi(\mathbf{X},t)$ is undone here.

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that is[6],

$$\boxed{\left.\frac{d}{dt}\int\limits_{V_t}\mu(\mathbf{x},t)\,dV\right|_{V_t\equiv V}\overset{not}{=}\frac{d}{dt}\int\limits_{V_t\equiv V}\mu(\mathbf{x},t)\,dV=\int\limits_{V}\left(\frac{d\mu}{dt}+\mu\nabla\cdot\mathbf{v}\right)dV}\ . \tag{5.17}$$

Recalling the expression of the material derivative of a property (1.15) results in

$$\begin{aligned}\frac{d}{dt}\int\limits_{V_t\equiv V}\mu(\mathbf{x},t)\,dV&=\int\limits_{V}\bigg(\frac{\partial\mu}{\partial t}+\underbrace{\mathbf{v}\cdot\nabla\mu+\mu\nabla\cdot\mathbf{v}}_{\nabla\cdot(\mu\mathbf{v})}\bigg)\,dV=\\&=\int\limits_{V}\frac{\partial\mu}{\partial t}\,dV+\int\limits_{V}\nabla\cdot(\mu\mathbf{v})\,dV=\frac{\partial}{\partial t}\int\limits_{V}\mu\,dV+\int\limits_{V}\nabla\cdot(\mu\mathbf{v})\,dV\ ,\end{aligned} \tag{5.18}$$

where the expression of the local derivative (5.11) has been taken into account. Then, (5.18) produces the expression of the material derivative of a volume integral.

Material derivative of a volume integral

$$\underbrace{\frac{d}{dt}\int\limits_{V_t\equiv V}\mu(\mathbf{x},t)\,dV}_{\substack{\text{material}\\ \text{derivative}}}=\underbrace{\frac{\partial}{\partial t}\int\limits_{V}\mu\,dV}_{\substack{\text{local}\\ \text{derivative}}}+\underbrace{\int\limits_{V}\nabla\cdot(\mu\mathbf{v})\,dV}_{\substack{\text{convective}\\ \text{derivative}}} \tag{5.19}$$

Remark 5.4. The form of the material derivative, given as a sum of a local derivative and a convective derivative, that appears when *differentiating properties* of the continuous medium (see Chapter 1, Section 1.4) also appears here when *differentiating integrals* in the continuous medium. Again, the *convective derivative* is associated with the existence of a velocity (or motion) in the medium and, thus, with the possibility of mass transport.

[6] The expression

$$\frac{d}{dt}\int\limits_{V_t\equiv V}\mu(\mathbf{x},t)\,dV$$

denotes the time derivative of the integral over the material volume V_t (material derivative of the volume integral) *particularized at time t, when the material volume occupies the spatial volume V*.

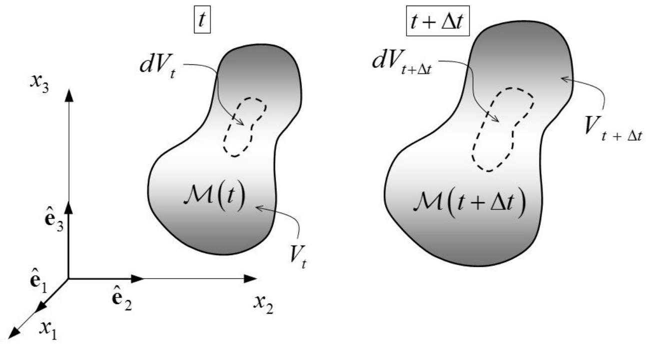

Figure 5.7: Principle of conservation of mass in a continuous medium.

5.4 Conservation of Mass. Mass continuity Equation

Definition 5.4. *Principle of conservation of mass*. The mass of a continuous medium (and, therefore, the mass of any material volume belonging to this medium) is always the same.

Consider a material volume V_t that at times t and $t+\Delta t$ occupies the volumes in space V_t and $V_{t+\Delta t}$, respectively (see Figure 5.7). Consider also the spatial description of the density, $\rho(\mathbf{x},t)$. The mass enclosed by the material volume V at times t and $t+\Delta t$ is, respectively,

$$\mathcal{M}(t) = \int\limits_{V_t} \rho(\mathbf{x},t)\, dV \quad \text{and} \quad \mathcal{M}(t+\Delta t) = \int\limits_{V_{t+\Delta t}} \rho(\mathbf{x},t+\Delta t)\, dV \; . \tag{5.20}$$

By virtue of the principle of conservation of mass, $\mathcal{M}(t) = \mathcal{M}(t+\Delta t)$ must be satisfied.

5.4.1 *Spatial Form of the Principle of Conservation of Mass. Mass Continuity Equation*

The mathematical expression of the principle of conservation of mass of the material volume $\mathcal{M}(t)$ is that the *material derivative* of the integral (5.20) is null,

$$\mathcal{M}'(t) = \frac{d}{dt}\int\limits_{V_t} \rho\, dV = 0 \quad \forall t \; . \tag{5.21}$$

By means of the expression of the material derivative of a volume integral (5.17), the integral (or global) spatial form of the principle of conservation of mass results in

Global spatial form of the principle of conservation of mass

$$\boxed{\frac{d}{dt}\int\limits_{\substack{V_t \\ (\Delta V_t)}} \rho \, dV = \int\limits_{\substack{V_t \\ (\Delta V_t)}} \left(\frac{d\rho}{dt} + \rho \, \nabla \cdot \mathbf{v} \right) dV = 0 \quad \forall \Delta V_t \subset V_t , \, \forall t} \,, \tag{5.22}$$

which must be satisfied for V_t and, also, for any *partial* material volume $\Delta V_t \subset V_t$ that could be considered. In particular, it must be satisfied for each of the elemental material volumes associated with the different particles in the continuous medium that occupy the differential volumes dV_t. Applying (5.22) on each differential volume $dV_t \equiv dV(\mathbf{x},t)$ yields[7]

$$\int\limits_{dV(\mathbf{x},t)} \left(\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} \right) dV = \left(\frac{d\rho(\mathbf{x},t)}{dt} + \rho(\mathbf{x},t) \nabla \cdot \mathbf{v}(\mathbf{x},t) \right) dV(\mathbf{x},t) = 0 \qquad \forall \mathbf{x} \in V_t , \, \forall t$$

$$\Longrightarrow \quad \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 \, dV \quad \forall \mathbf{x} \in V_t , \, \forall t \tag{5.23}$$

Local spatial form of the principle of conservation of mass (mass continuity equation)

$$\boxed{\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 \, dV \quad \forall \mathbf{x} \in V_t , \, \forall t} \tag{5.24}$$

which constitutes the so-called *mass continuity equation*. Replacing the expression of the material derivative of the spatial description of a property (1.15) in (5.24) results in

$$\frac{\partial \rho}{\partial t} + \underbrace{\mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v}}_{\nabla \cdot (\rho \mathbf{v})} = 0 \quad \Longrightarrow \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \,, \tag{5.25}$$

which yields an alternative expression of the mass continuity equation.

[7] This procedure, which allows reducing a global (or integral) expression such as (5.22) to a local (or differential) one such as (5.24), is named in continuum mechanics *localization* process.

$$
\boxed{\left.\begin{array}{l}
\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \\[2mm]
\dfrac{\partial \rho}{\partial t} + \dfrac{\partial (\rho \mathrm{v}_i)}{\partial x_i} = 0 \qquad\qquad i \in \{1,2,3\} \\[2mm]
\dfrac{\partial \rho}{\partial t} + \dfrac{\partial (\rho \mathrm{v}_x)}{\partial x} + \dfrac{\partial (\rho \mathrm{v}_y)}{\partial y} + \dfrac{\partial (\rho \mathrm{v}_z)}{\partial z} = 0
\end{array}\right\} \quad \forall \mathbf{x} \in V_t,\ \forall t} \tag{5.26}
$$

5.4.2 Material Form of the Principle of Conservation of Mass

From (5.22)[8],

$$
\begin{aligned}
&\int_{V_t} \left(\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} \right) dV = \int_{V_t} \left(\frac{d\rho}{dt} + \rho \frac{1}{|\mathbf{F}|} \frac{d|\mathbf{F}|}{dt} \right) dV = \\
&= \int_{V_t} \frac{1}{|\mathbf{F}|} \underbrace{\left(|\mathbf{F}| \frac{d\rho}{dt} + \rho \frac{d|\mathbf{F}|}{dt} \right)}_{\frac{d}{dt}\left(\rho |\mathbf{F}|\right)} dV = \int_{V_t} \frac{1}{|\mathbf{F}|} \frac{d}{dt} \Big(\rho |\mathbf{F}| \Big) \underbrace{dV}_{|\mathbf{F}|\, dV_0} = \\
&= \int_{V_0} \frac{\partial}{\partial t} \Big(\rho(\mathbf{X},t) |\mathbf{F}(\mathbf{X},t)| \Big) dV_0 \qquad \forall \Delta V_0 \subset V_0,\ \forall t\ ,
\end{aligned} \tag{5.27}
$$

where the integration domain is now the volume in the reference configuration, V_0. Given that (5.27) must be satisfied for each and every part ΔV_0 of V_0, a localization process can be applied, which results in[9]

$$
\begin{aligned}
&\frac{\partial}{\partial t} \Big(\rho(\mathbf{X},t) |\mathbf{F}(\mathbf{X},t)| \Big) = 0 \quad \forall \mathbf{X} \in V_0,\ \forall t \\
&\implies \rho(\mathbf{X},t) |\mathbf{F}(\mathbf{X},t)| = \rho(\mathbf{X}) |\mathbf{F}(\mathbf{X})| \quad \forall t \\
&\implies \underbrace{\rho(\mathbf{X},0) |\mathbf{F}|(\mathbf{X},0)}_{\stackrel{not}{=} \rho_0\ |\mathbf{F}|_0} = \underbrace{\rho(\mathbf{X},t) |\mathbf{F}|(\mathbf{X},t)}_{\stackrel{not}{=} \rho_t\ |\mathbf{F}|_t} \implies \rho_0 \underbrace{|\mathbf{F}|_0}_{=1} = \rho_t\ |\mathbf{F}|_t\ .
\end{aligned} \tag{5.28}
$$

$$
\boxed{\begin{array}{c}
\textit{Local material form of the mass conservation principle} \\
\rho_0(\mathbf{X}) = \rho_t(\mathbf{X}) |\mathbf{F}|_t(\mathbf{X}) \quad \forall \mathbf{X} \in V_0,\ \forall t
\end{array}} \tag{5.29}
$$

[8] Here, the expression deduced in Chapter 2, $d|\mathbf{F}|/dt = |\mathbf{F}| \cdot \nabla \cdot \mathbf{v}$, is considered.
[9] The equality $\mathbf{F}(\mathbf{X},0) = \mathbf{1} \implies |\mathbf{F}|_0 = 1$ is used here.

5.5 Balance Equation. Reynolds Transport Theorem

Consider $\mathcal{A}$, an arbitrary (scalar, vector or tensor) property of the continuous medium, and $\Psi(\mathbf{x},t)$, the description of the amount of said property per *unit of mass*. Then, $\rho\Psi(\mathbf{x},t)$ is the amount of this property per *unit of volume*.

5.5.1 Reynolds' Lemma

Consider an arbitrary *material* volume of the continuous medium that at time t occupies the volume in space $V_t \equiv V$. The amount of the generic property $\mathcal{A}$ in the material volume V_t at time t is

$$Q(t) = \int\limits_{V_t \equiv V} \rho\Psi \, dV \ . \tag{5.30}$$

The variation along time of the content of property $\mathcal{A}$ in the material volume V_t is given by the time derivative of $Q(t)$, which using expression (5.17) of the material derivative of a volume integral (with $\mu = \rho\Psi$) results in

$$Q'(t) = \frac{d}{dt} \int\limits_{V_t \equiv V} \underbrace{\rho\Psi}_{\mu} \, dV = \int\limits_V \left(\frac{d(\rho\Psi)}{dt} + \rho\Psi \, \nabla\cdot\mathbf{v} \right) dV \ . \tag{5.31}$$

Considering the expression of the material derivative of a product of functions, grouping terms and introducing the mass continuity equation (5.24) yields

$$\begin{aligned} \frac{d}{dt} \int\limits_{V_t \equiv V} \rho\Psi dV &= \int\limits_V \left(\rho \frac{d\Psi}{dt} + \Psi \frac{d\rho}{dt} + \rho\Psi \, \nabla\cdot\mathbf{v} \right) dV = \\ &= \int\limits_V \left(\rho \frac{d\Psi}{dt} + \Psi \underbrace{\left(\frac{d\rho}{dt} + \rho \nabla\cdot\mathbf{v} \right)}_{=0 \text{ (mass continuity eqn.)}} \right) dV \quad \Longrightarrow \end{aligned} \tag{5.32}$$

$$\boxed{\begin{array}{l} \textit{Reynolds' Lemma} \\ \displaystyle \frac{d}{dt} \int\limits_{V_t \equiv V} \rho\Psi \, dV = \int\limits_V \rho \frac{d\Psi}{dt} \, dV \end{array}} \ . \tag{5.33}$$

5.5.2 Reynolds' Theorem

Consider the arbitrary volume V, fixed in space, shown in Figure 5.8. The amount of property $\mathcal{A}$ in this control volume is

$$Q(t) = \int\limits_V \rho\Psi \, dV \ . \tag{5.34}$$

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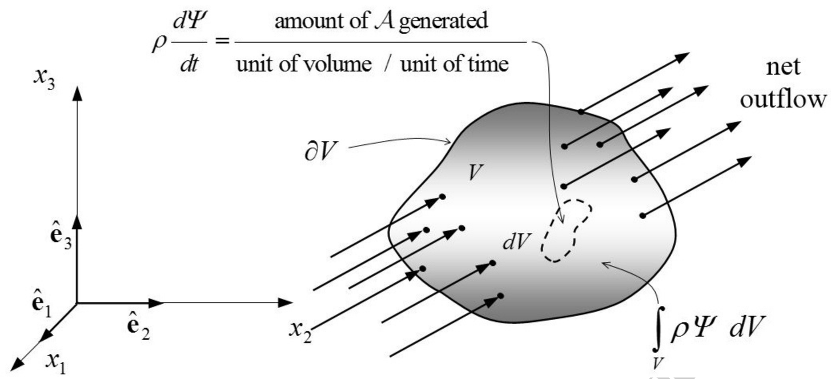

Figure 5.8: Reynolds Transport Theorem.

The variation of the amount of property $\mathcal{A}$ in the material volume V_t, which instantaneously coincides at time t with the control volume V ($V_t \equiv V$), is given by expression (5.19) of the material derivative of a volume integral (with $\mu = \rho\Psi$) and by (5.11),

$$\frac{d}{dt}\int_{V_t \equiv V} \rho\Psi \, dV = \int_V \frac{\partial(\rho\Psi)}{\partial t} \, dV + \int_V \nabla\cdot(\rho\Psi\mathbf{v}) \, dV \,. \tag{5.35}$$

Introducing the Reynolds' Lemma (5.33) and the Divergence Theorem[10] in (5.35) results in

$$\begin{aligned}\frac{d}{dt}\int_{V_t \equiv V} \rho\Psi \, dV &\overset{\substack{\text{Reynolds'}\\ \text{Lemma}}}{=} \int_V \rho \frac{d\Psi}{dt} \, dV = \int_V \frac{\partial(\rho\Psi)}{\partial t} \, dV + \int_V \nabla\cdot(\rho\Psi\mathbf{v}) \, dV = \\ &\overset{\substack{\text{Divergence}\\ \text{Theorem}}}{=} \int_V \frac{\partial(\rho\Psi)}{\partial t} \, dV + \int_{\partial V} \rho\Psi\mathbf{v}\cdot\mathbf{n} \, dS \,,\end{aligned} \tag{5.36}$$

which can be rewritten as follows.

[10] The Divergence Theorem provides the following relation between a volume integral and a surface integral of a tensor $\mathbf{A}$.

$$\int_V \nabla\cdot\mathbf{A} \, dV = \int_{\partial V} \mathbf{n}\cdot\mathbf{A} \, dS \quad \forall V \,,$$

where $\mathbf{n}$ is the outward unit normal vector in the boundary of the volume V.

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Reynolds Transport Theorem

$$\underbrace{\frac{\partial}{\partial t}\int_V \rho\Psi\, dV}_{\substack{\text{variation per unit of}\\ \text{time of the content of}\\ \text{property } \mathcal{A} \text{ in the}\\ \text{control volume } V}} = \underbrace{\int_V \rho\frac{\partial\Psi}{\partial t}\, dV}_{\substack{\text{variation due to the}\\ \text{change in the content of}\\ \text{property } \mathcal{A} \text{ of the parti-}\\ \text{cles in the interior of } V}} - \underbrace{\int_{\partial V} \rho\Psi\,\mathbf{v}\cdot\mathbf{n}\, dS}_{\substack{\text{variation due to the net}\\ \text{convective flux of } \mathcal{A}\\ \text{exiting through the}\\ \text{boundary } \partial V}} \tag{5.37}$$

The local form of the Reynolds Transport Theorem can be obtained by localizing in (5.36),

$$\begin{aligned} &\int_V \rho\frac{d\Psi}{dt}\, dV = \int_V \frac{\partial(\rho\Psi)}{\partial t}\, dV + \int_V \nabla\cdot(\rho\Psi\,\mathbf{v})\, dV \quad \forall\Delta V\subset V \quad \Longrightarrow \\ &\rho\frac{d\Psi}{dt} = \frac{\partial(\rho\Psi)}{\partial t} + \nabla\cdot(\rho\Psi\,\mathbf{v}) \quad \forall\mathbf{x}\in V \quad \Longrightarrow \end{aligned} \tag{5.38}$$

Local form of the Reynolds Transport Theorem

$$\frac{\partial(\rho\Psi)}{\partial t} = \rho\frac{d\Psi}{dt} - \nabla\cdot(\rho\Psi\,\mathbf{v}) \quad \forall\mathbf{x}\in V \tag{5.39}$$

5.6 General Expression of the Balance Equations

Consider a certain property $\mathcal{A}$ of a continuous medium and the amount of this property per unit of mass, $\Psi(\mathbf{x},t)$. In the most general case, it can be assumed that there exists an internal source that generates property $\mathcal{A}$ and that this property can be transported both by motion of mass (convective transport) and by non-convective transport. To this aim, the following terms are defined:

- A source term $k_{\mathcal{A}}(\mathbf{x},t)$ (of the same tensor order than property $\mathcal{A}$) that characterizes the internal generation of the property,

$$k_{\mathcal{A}}(\mathbf{x},t) = \frac{\text{internally generated amount of } \mathcal{A}}{\text{unit of mass / unit of time}}\,. \tag{5.40}$$

- A vector $\mathbf{j}_{\mathcal{A}}(\mathbf{x},t)$ of non-convective flux per unit of surface (a tensor order higher than that of property $\mathcal{A}$) that characterizes the flux of the property due to non-convective mechanisms (see Remark 5.3).

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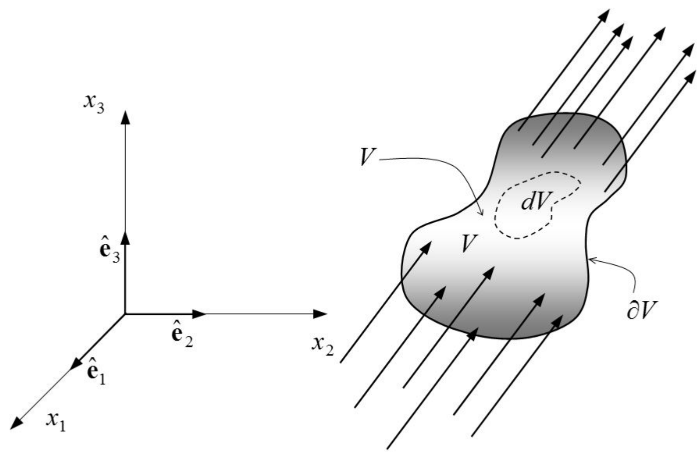

Figure 5.9: An arbitrary control volume used in the definition of the global form of the general balance equation.

Consider an arbitrary control volume V (see Figure 5.9). Then, the variation per unit of time of property $\mathcal{A}$ in volume V will be due to

1) the generation of property $\mathcal{A}$ per unit of time due to the source term,
2) the (net incoming) convective flux of $\mathcal{A}$ through ∂V, and
3) the (net incoming) non-convective flux of $\mathcal{A}$ through ∂V.

That is,

$$
\begin{aligned}
&\int\limits_V \rho k_{\mathcal{A}}(\mathbf{x},t)\, dV = \frac{\text{amount of } \mathcal{A} \text{ generated in } V \text{ due to the internal sources}}{\text{unit of time}} \ , \\
&\int\limits_{\partial V} \rho \Psi\, \mathbf{v} \cdot \mathbf{n}\, dS = \frac{\text{amount of } \mathcal{A} \text{ exiting through } \partial V \text{ per convective flux}}{\text{unit of time}} \ , \\
&\int\limits_{\partial V} \mathbf{j}_{\mathcal{A}} \cdot \mathbf{n}\, dS = \frac{\text{amount of } \mathcal{A} \text{ exiting through } \partial V \text{ per non-convective flux}}{\text{unit of time}} \ ,
\end{aligned}
\tag{5.41}
$$

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and the expression of the *balance of the amount of property $\mathcal{A}$ in the control volume V* results in

Global form of the general balance equation

$$\underbrace{\frac{\partial}{\partial t}\int_V \rho\Psi\, dV}_{\substack{\text{variation of the}\\ \text{amount of } \mathcal{A} \text{ in } V\\ \text{per unit of time}}} = \underbrace{\int_V \rho k_{\mathcal{A}}\, dV}_{\substack{\text{variation due}\\ \text{to internal}\\ \text{generation}}} - \underbrace{\int_{\partial V} \rho\Psi\, \mathbf{v}\cdot\mathbf{n}\, dS}_{\substack{\text{variation due to}\\ \text{the incoming}\\ \text{convective flux}}} - \underbrace{\int_{\partial V} \mathbf{j}_{\mathcal{A}}\cdot\mathbf{n}\, dS}_{\substack{\text{variation due to}\\ \text{the incoming}\\ \text{non-}\\ \text{convective flux}}} \tag{5.42}$$

Using the Divergence Theorem and (5.11), the global form of the general balance equation (5.42) can be written as

$$\begin{gathered}\frac{\partial}{\partial t}\int_V \rho\Psi\, dV = \int_V \rho k_{\mathcal{A}}\, dV - \int_V \nabla\cdot(\rho\Psi\,\mathbf{v})\, dV - \int_V \nabla\cdot\mathbf{j}_{\mathcal{A}}\, dV \quad \Longrightarrow \\ \int_V \left(\frac{\partial}{\partial t}(\rho\Psi) + \nabla\cdot(\rho\Psi\,\mathbf{v})\right) dV = \int_V (\rho k_{\mathcal{A}} - \nabla\cdot\mathbf{j}_{\mathcal{A}})\, dV \quad \forall \Delta V \subset V\end{gathered} \tag{5.43}$$

and localizing in (5.43), the local spatial form of the general balance equation

Local spatial form of the general balance equation

$$\underbrace{\frac{\partial}{\partial t}(\rho\Psi) + \nabla\cdot(\rho\Psi\,\mathbf{v})}_{\rho\frac{d\Psi}{dt}} = \underbrace{\rho\frac{d\Psi}{dt}}_{\substack{\text{variation of the}\\ \text{amount of}\\ \text{property (per}\\ \text{unit of volume}\\ \text{and of time)}}} = \underbrace{\rho k_{\mathcal{A}}}_{\substack{\text{variation}\\ \text{due to}\\ \text{internal}\\ \text{generation}\\ \text{by a source}}} - \underbrace{\nabla\cdot\mathbf{j}_{\mathcal{A}}}_{\substack{\text{variation}\\ \text{due to non-}\\ \text{convective}\\ \text{transport}}} \tag{5.44}$$

is obtained, where the local form of the Reynolds Transport Theorem (5.39) has been taken into account.

Remark 5.5. Expression (5.42) and, especially, expression (5.44),

$$\rho \frac{d\Psi}{dt} = \rho k_{\mathcal{A}} - \nabla \cdot \mathbf{j}_{\mathcal{A}} \ ,$$

exhibit the negative contribution $(-\nabla \cdot \mathbf{j}_{\mathcal{A}})$ of the non-convective flux to the variation in content of the property per unit of volume and of time, $\rho \, d\Psi/dt$. Only when all the flux is convective (by mass transport) can this variation originate solely from the internal generation of this property,

$$\rho \frac{d\Psi}{dt} = \rho k_{\mathcal{A}} \ .$$

Example 5.2 – *Particularize the local spatial form of the general balance equation for the case in which property $\mathcal{A}$ is associated with the mass.*

Solution

If property $\mathcal{A}$ is associated with the mass, $\mathcal{A} \equiv \mathcal{M}$, then:

- The content of $\mathcal{A}$ per unit of mass (mass / unit of mass) is $\Psi = 1$.
- The source term that characterizes the internal generation of mass is $k_{\mathcal{M}} = 0$ since, following the principle of conservation of mass, it is not possible to generate mass.
- The non-convective mass flux vector is $\mathbf{j}_{\mathcal{M}} = \mathbf{0}$ because mass cannot be transported in a non-convective manner.

Therefore, (5.44) results in the balance of mass generation,

$$\rho \frac{d\Psi}{dt} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \ ,$$

which is one of the forms of the mass continuity equation (5.26).

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5.7 Balance of Linear Momentum

Consider a discrete system composed of n particles such that the particle i has a mass m_i, an acceleration $\mathbf{a}_i$ and is subjected to a force $\mathbf{f}_i$ (see Figure 5.10).

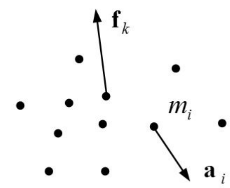

Figure 5.10

Newton's second law states that the force acting on a particle is equal to the mass of this particle times its acceleration. Using the definition of acceleration as the material derivative of the velocity and considering the principle of conservation of mass (the variation of mass of a particle is null) yields[11],

$$\mathbf{f}_i = m_i \mathbf{a}_i = m_i \frac{d\mathbf{v}_i}{dt} = \frac{d}{dt}\left(m_i \mathbf{v}_i\right) \tag{5.45}$$

The *linear momentum of the particle*[12] is defined as the product of its mass by its velocity $(m_i \mathbf{v}_i)$. Then, (5.45) expresses that the force acting on the particle is equal to the variation of the linear momentum of the particle.

Applying now Newton's second law to the discrete system formed by n particles results in

$$\mathbf{R}(t) = \sum_{i=1}^{n} \mathbf{f}_i = \sum_{i=1}^{n} m_i \, \mathbf{a}_i = \sum_{i=1}^{n} m_i \frac{d\mathbf{v}_i}{dt} = \frac{d}{dt} \underbrace{\sum_{i=1}^{n} m_i \mathbf{v}_i}_{\substack{\mathcal{P} = \text{linear} \\ \text{momentum}}} = \frac{d\mathcal{P}(t)}{dt} \,. \tag{5.46}$$

Note that, again, to obtain the last expression in (5.46), the principle of conservation of mass $(dm_i/dt = 0)$ has been used. Equation (5.46) expresses that the *resultant* $\mathbf{R}$ *of all the forces acting on the discrete system of particles is equal to the variation per unit of time of the linear momentum* $\mathcal{P}$ *of the system*. This postulate is denominated the *principle of balance of linear momentum*.

Remark 5.6. If the system is in equilibrium, $\mathbf{R} = \mathbf{0}$. Then,

$$\mathbf{R}(t) = \mathbf{0} \quad \forall t \quad \Longrightarrow \quad \frac{d\mathcal{P}(t)}{dt} = 0 \quad \Longrightarrow \quad \sum_{i=1}^{n} m_i \, \mathbf{v}_i = \mathcal{P} = const. \,,$$

which is known as the *conservation* of linear momentum.

[11] The Einstein notation introduced in (1.1) is not used here.

[12] In mechanics, the names *translational momentum*, *kinetic momentum* or simply *momentum* are also used to refer to the linear momentum.

5.7.1 Global Form of the Balance of Linear Momentum

These concepts, corresponding to classical mechanics, can now be extended to continuum mechanics by defining the linear momentum in a material volume V_t of the continuous medium with mass $\mathcal{M}$ as

$$\mathcal{P}(t) = \int_{\mathcal{M}} \mathbf{v} \underbrace{d\mathcal{M}}_{\rho\, dV} = \int_{V_t} \rho\, \mathbf{v}\, dV \,. \tag{5.47}$$

Definition 5.5. *Principle of balance of linear momentum*. The resultant $\mathbf{R}(t)$ of all the forces acting on a material volume of the continuous medium is equal to the variation per unit of time of its linear momentum,

$$\mathbf{R}(t) = \frac{d\mathcal{P}(t)}{dt} = \frac{d}{dt}\int_{V_t} \rho\, \mathbf{v}\, dV \,.$$

The resultant of all the forces acting on the continuous medium defined above is also known to be (see Figure 5.11)

$$\mathbf{R}(t) = \underbrace{\int_V \rho \mathbf{b}\, dV}_{\substack{\text{body}\\ \text{forces}}} + \underbrace{\int_{\partial V} \mathbf{t}\, dS}_{\substack{\text{surface}\\ \text{forces}}} \quad . \tag{5.48}$$

Applying the principle of balance of linear momentum on the resultant in (5.48) yields the *integral form of the balance of linear momentum.*

Global form of the principle of balance of linear momentum

$$\int_V \rho \mathbf{b}\, dV + \int_{\partial V} \mathbf{t}\, dS = \frac{d}{dt}\int_{V_t \equiv V} \rho \mathbf{v}\, dV \tag{5.49}$$

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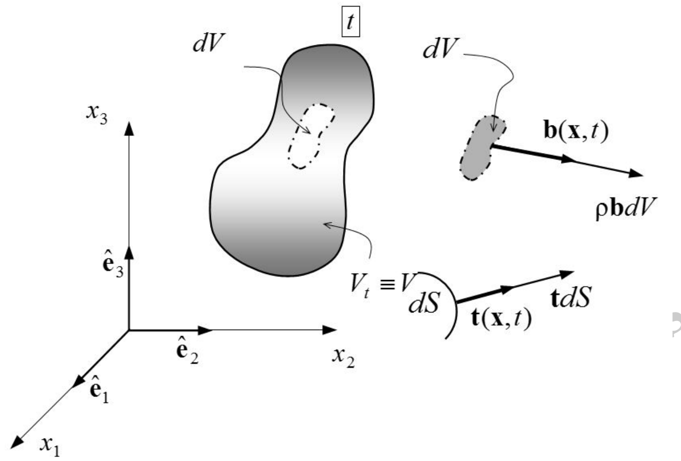

Figure 5.11: Forces acting on a material volume of the continuous medium.

5.7.2 Local Form of the Balance of Linear Momentum

Using Reynolds' Lemma (5.33) on (5.49) and introducing the Divergence Theorem, results in

$$\left.\begin{array}{l} \dfrac{d}{dt}\displaystyle\int_{V_t\equiv V} \rho \mathbf{v}\, dV = \int_V \rho \mathbf{b}\, dV + \int_{\partial V} \underbrace{\mathbf{n}\cdot\boldsymbol{\sigma}}_{\mathbf{t}}\, dS = \int_{V_t\equiv V} \rho \frac{d\mathbf{v}}{dt}\, dV \\ \displaystyle\int_{\partial V} \mathbf{n}\cdot\boldsymbol{\sigma}\, dS \overset{\substack{\text{Divergence}\\ \text{Theorem}}}{=} \int_V \nabla\cdot\boldsymbol{\sigma}\, dV \end{array}\right\} \Longrightarrow \tag{5.50}$$

$$\Longrightarrow \quad \int_V (\nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{b})\, dV + \int_V \rho \frac{d\mathbf{v}}{dt}\, dV \quad \forall \Delta V \subset V \tag{5.51}$$

and, localizing in (5.51), yields the *local spatial form of the balance of linear momentum*, also known as Cauchy's equation[13].

> *Local spatial form of the principle of balance of linear momentum (Cauchy's equation)*
>
> $$\nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{a} \quad \forall \mathbf{x}\in V,\ \forall t \tag{5.52}$$

[13] The Cauchy equation (already stated, but not deduced, in Chapter 4) is, thus, identified as the *local spatial form of the balance of linear momentum.*

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5.8 Balance of Angular Momentum

Consider a discrete system composed of n particles such that for an arbitrary particle i, its position vector is $\mathbf{r}_i$, its mass is m_i, a force $\mathbf{f}_i$ acts on it, and it has a velocity $\mathbf{v}_i$ and an acceleration $\mathbf{a}_i$ (see Figure 5.12). The moment about the origin of the force acting on this particle is $\mathbf{M}_i = \mathbf{r}_i \times \mathbf{f}_i$ and the *moment* about the origin *of the linear momentum*[14] of the particle is $\boldsymbol{\mathcal{L}}_i = \mathbf{r}_i \times m_i \mathbf{v}_i$. Considering Newton's second law, the moment $\mathbf{M}_i$ is[15]

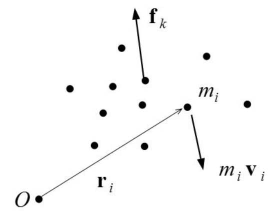

Figure 5.12

$$\mathbf{M}_i = \mathbf{r}_i \times \mathbf{f}_i = \mathbf{r}_i \times m_i \, \mathbf{a}_i = \mathbf{r}_i \times m_i \frac{d\mathbf{v}_i}{dt} \tag{5.53}$$

Extending the previous result to the discrete system formed by n particles, the resultant moment about the origin $\mathbf{M}_O$ of the forces acting on the system of particles is obtained as[16]

$$\left.\begin{aligned} &\mathbf{M}_O(t) = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{f}_i = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \, \mathbf{a}_i = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \frac{d\mathbf{v}_i}{dt} \\ &\frac{d}{dt} \sum_{i=1}^{n} \mathbf{r}_i \times m_i \, \mathbf{v}_i = \underbrace{\sum_{i=1}^{n} \underbrace{\frac{d\mathbf{r}_i}{dt}}_{\mathbf{v}_i} \times m_i \, \mathbf{v}_i}_{=\mathbf{0}} + \sum_{i=1}^{n} \mathbf{r}_i \times m_i \frac{d\mathbf{v}_i}{dt} \end{aligned}\right\} \Longrightarrow$$

$$\Longrightarrow \quad \mathbf{M}_O(t) = \frac{d}{dt} \underbrace{\sum_{i=1}^{n} \mathbf{r}_i \times m_i \, \mathbf{v}_i}_{\substack{\text{Angular} \\ \text{momentum } \boldsymbol{\mathcal{L}}}} = \frac{d\boldsymbol{\mathcal{L}}(t)}{dt} \tag{5.54}$$

Equation (5.54) expresses that *the resultant moment* $\mathbf{M}_O$ *of all the forces acting on the discrete system of particles is equal to the variation per unit of time of the moment of linear momentum* (or angular momentum), $\boldsymbol{\mathcal{L}}$, *of the system*. This postulate is named *principle of balance of angular momentum*.

[15] In mechanics, the *moment of (linear) momentum* is also named *angular momentum* or *rotational momentum*.
[15] The Einstein notation introduced in (1.1) is not used here.
[16] The vector or cross product of a vector times itself is null ($\mathbf{v}_i \times \mathbf{v}_i = \mathbf{0}$).

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Remark 5.7. If the system is in equilibrium, $\mathbf{M}_O = \mathbf{0}$. Then,

$$\mathbf{M}_O(t) = \mathbf{0} \quad \forall t \implies \frac{d\boldsymbol{\mathcal{L}}(t)}{dt} = \mathbf{0} \implies \sum_{i=1}^{n} \mathbf{r}_i \times m_i \mathbf{v}_i = \boldsymbol{\mathcal{L}} = const.,$$

which is known as the *conservation* of angular momentum.

5.8.1 Global Form of the Balance of Angular Momentum

Result (5.54) can be extended to a continuous and infinite system of particles (the continuous medium, see Figure 5.13). In such case, the angular momentum is defined as

$$\boldsymbol{\mathcal{L}} = \int_{\mathcal{M}} \mathbf{r} \times \mathbf{v} \underbrace{d\mathcal{M}}_{\rho\, dV} = \int_{V} \mathbf{r} \times \rho\, \mathbf{v}\, dV \tag{5.55}$$

and the continuous version of the postulate of balance of angular momentum is obtained as follows.

Definition 5.6. *Principle of balance of moment of (linear) momentum or angular momentum.* The resultant moment, about a certain point O in space, of all the actions on a continuous medium is equal to the variation per unit of time of the moment of linear momentum about said point.

$$\mathbf{M}_O(t) = \frac{d\boldsymbol{\mathcal{L}}(t)}{dt} = \frac{d}{dt} \int_{V_t \equiv V} \mathbf{r} \times \rho\, \mathbf{v}\, dV$$

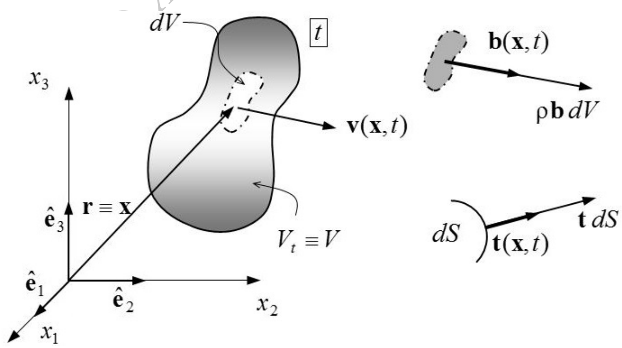

Figure 5.13: Moments acting on a material volume of the continuous medium.

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The resultant moment of the forces acting on the continuous medium (moment of the body forces and moment of the surface forces) is (see Figure 5.13)

$$\mathbf{M}_O(t) = \int_V \mathbf{r} \times \rho\, \mathbf{b}\, dV + \int_{\partial V} \mathbf{r} \times \mathbf{t}\, dS \,, \tag{5.56}$$

then, the *global form of the principle of balance of the angular momentum* results in:

Global form of the principle of balance of angular momentum

$$\frac{d}{dt} \int_{V_t \equiv V} \mathbf{r} \times \rho\, \mathbf{v}\, dV = \int_V \mathbf{r} \times \rho\, \mathbf{b}\, dV + \int_{\partial V} \mathbf{r} \times \mathbf{t}\, dS \tag{5.57}$$

5.8.2 *Local Spatial Form of the Balance of Angular Momentum*

The procedure followed to obtain the local spatial form of the balance equation is detailed below.

Introducing Reynolds' Lemma in (5.57),

$$\begin{aligned} \frac{d}{dt} \int_{V_t \equiv V} \mathbf{r} \times \rho \mathbf{v}\, dV &= \frac{d}{dt} \int_{V_t \equiv V} \rho\, (\mathbf{r} \times \mathbf{v})\, dV = \int_V \rho \frac{d}{dt} (\mathbf{r} \times \mathbf{v})\, dV = \\ &= \int_V \rho \underbrace{\Big(\underbrace{\frac{d\mathbf{r}}{dt}}_{\mathbf{v}} \times \mathbf{v} \Big)}_{=\mathbf{0}} dV + \int_V \rho \left(\mathbf{r} \times \frac{d\mathbf{v}}{dt} \right) dV = \int_V \mathbf{r} \times \rho \frac{d\mathbf{v}}{dt}\, dV \,, \end{aligned} \tag{5.58}$$

and expanding the last term in (5.57),

$$\begin{aligned} \int_{\partial V} \mathbf{r} \times \underbrace{\mathbf{t}}_{\mathbf{n}\cdot\boldsymbol{\sigma}} dS &= \int_{\partial V} \mathbf{r} \times \mathbf{n} \cdot \boldsymbol{\sigma}\, dS = \int_{\partial V} [\mathbf{r}] \times [\mathbf{n} \cdot \boldsymbol{\sigma}]^T dS = \\ &= \int_{\partial V} (\mathbf{r} \times \boldsymbol{\sigma}^T) \cdot \mathbf{n}\, dS \overset{\substack{\text{Divergence} \\ \text{Theorem}}}{=} \int_V \left(\mathbf{r} \times \boldsymbol{\sigma}^T \right) \cdot \nabla\, dV \,, \end{aligned} \tag{5.59}$$

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where the component $\left[\left(\mathbf{r}\times\boldsymbol{\sigma}^T\right)\cdot\nabla\right]_i$ is computed as

$$
\begin{aligned}
&\left[\left(\mathbf{r}\times\boldsymbol{\sigma}^T\right)\cdot\nabla\right]_i \overset{symb}{=} \Big(e_{ijk}\,x_j\,\underbrace{\sigma_{rk}}_{\sigma^T_{kr}}\Big)\frac{\partial}{\partial x_r} = \frac{\partial}{\partial x_r}\left(e_{ijk}\,x_j\,\sigma_{rk}\right) = \\
&= e_{ijk}\underbrace{\frac{\partial x_j}{\partial x_r}}_{\delta_{jr}}\sigma_{rk} + \underbrace{e_{ijk}\,x_j\frac{\partial\sigma_{rk}}{\partial x_r}}_{[\mathbf{r}\times\nabla\cdot\boldsymbol{\sigma}]_i} = \underbrace{e_{ijk}\,\sigma_{jk}}_{m_i} + [\mathbf{r}\times\nabla\cdot\boldsymbol{\sigma}]_i \quad i\in\{1,2,3\}\ .
\end{aligned}
\tag{5.60}
$$

Introducing now (5.60) in (5.59) produces

$$
\boxed{
\begin{aligned}
&\int\limits_{\partial V}\mathbf{r}\times\mathbf{t}\,dS = \int\limits_V\mathbf{m}\,dV + \int\limits_V\left(\mathbf{r}\times\nabla\cdot\boldsymbol{\sigma}\right)dV \\
&m_i = e_{ijk}\,\sigma_{jk} \qquad i,j,k\in\{1,2,3\}
\end{aligned}}
\tag{5.61}
$$

and, finally, replacing (5.58) and (5.61) in (5.57) yields

$$
\int\limits_V\mathbf{r}\times\rho\frac{d\mathbf{v}}{dt}\,dV = \int\limits_V\mathbf{r}\times\rho\mathbf{b}\,dV + \int\limits_V\mathbf{m}\,dV + \int\limits_V\left(\mathbf{r}\times\nabla\cdot\boldsymbol{\sigma}\right)dV\ .
\tag{5.62}
$$

Reorganizing the terms in (5.62) and taking into account Cauchy's equation (5.52) (local spatial form of the balance of linear momentum) results in

$$
\begin{aligned}
&\int\limits_V\mathbf{r}\times\underbrace{\left(\nabla\cdot\boldsymbol{\sigma}+\rho\mathbf{b}-\rho\frac{d\mathbf{v}}{dt}\right)}_{=\mathbf{0}}dV + \int\limits_V\mathbf{m}\,dV = \mathbf{0} \\
&\qquad\Longrightarrow\quad \boxed{\int\limits_V\mathbf{m}\,dV = \mathbf{0}\quad\forall\Delta V\subset V}\ .
\end{aligned}
\tag{5.63}
$$

Then, localizing in (5.63) and considering the value of $\mathbf{m}$ in (5.61), yields

$$
\left.\begin{aligned}
&\mathbf{m}=\mathbf{0}\quad\forall\mathbf{x}\in V \\
&m_i = e_{ijk}\,\sigma_{jk} = 0\quad i\in\{1,2,3\}
\end{aligned}\right\}\Longrightarrow\quad e_{ijk}\,\sigma_{jk}=0\quad i,j,k\in\{1,2,3\}
\tag{5.64}
$$

and particularizing (5.64) for the three possible values of index i:

$$\left.\begin{array}{l} i=1: e_{1jk}\,\sigma_{jk} = \underbrace{e_{123}}_{=1}\,\sigma_{23} + \underbrace{e_{132}}_{=-1}\,\sigma_{32} = \sigma_{23}-\sigma_{32}=0 \;\Rightarrow\; \sigma_{23}=\sigma_{32} \\ i=2: e_{2jk}\,\sigma_{jk} = \underbrace{e_{231}}_{=1}\,\sigma_{31} + \underbrace{e_{213}}_{=-1}\,\sigma_{13} = \sigma_{31}-\sigma_{13}=0 \;\Rightarrow\; \sigma_{31}=\sigma_{13} \\ i=3: e_{3jk}\,\sigma_{jk} = \underbrace{e_{312}}_{=1}\,\sigma_{12} + \underbrace{e_{321}}_{=-1}\,\sigma_{21} = \sigma_{12}-\sigma_{21}=0 \;\Rightarrow\; \sigma_{12}=\sigma_{21} \end{array}\right\} \Longrightarrow$$
$$\Longrightarrow \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T\,, \tag{5.65}$$

which results in the *local spatial form of the balance of angular momentum* translating into the symmetry of the Cauchy stress tensor[17].

$$\boxed{\begin{array}{c}\textit{Local spatial form of the}\\ \textit{principle of balance of angular momentum}\\ \boldsymbol{\sigma}=\boldsymbol{\sigma}^T\end{array}} \tag{5.66}$$

5.9 Power

Definition 5.7. In classical mechanics as well as in continuum mechanics, *power* is defined as a concept, previous to that of energy, that can be quantified as the ability to perform *work per unit of time*. Then, for a system (or continuous medium) the power $W(t)$ entering the system is defined as

$$W(t) = \frac{\text{work performed by the system}}{\text{unit of time}}\,.$$

In some cases, but not in all, the power $W(t)$ is an exact differential of a function $\mathcal{E}(t)$ that, in said cases, receives the name of *energy*,

$$W(t) = \frac{d\mathcal{E}(t)}{dt}\,. \tag{5.67}$$

[17] The symmetry of the Cauchy stress tensor (already stated, but not deduced, in Chapter 4) is, thus, identified as the *local spatial form of the balance of angular momentum.*

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Here, it is assumed that there exist two procedures by which the continuous medium absorbs power from the exterior and performs work per unit of time with this power

- *Mechanical power*, by means of the work performed by the mechanical actions (body and surface forces) acting on the medium.
- *Thermal power*, by means of the heat entering the medium.

5.9.1 Mechanical Power. Balance of Mechanical Energy

Definition 5.8. The *mechanical power entering the continuous medium*, P_e, is the work per unit of time performed by all the (body and surface) forces acting on the medium.

Consider the continuous medium shown in Figure 5.14 is subjected to the action of body forces, characterized by the vector of body forces $\mathbf{b}(\mathbf{x},t)$, and of surface forces, characterized by the traction vector $\mathbf{t}(\mathbf{x},t)$. The expression of the mechanical power entering the system P_e is

$$P_e = \int_V \rho\,\mathbf{b}\cdot\mathbf{v}\,dV + \int_{\partial V} \underbrace{\mathbf{t}}_{\mathbf{n}\cdot\boldsymbol{\sigma}}\cdot\mathbf{v}\,dS = \int_V \rho\,\mathbf{b}\cdot\mathbf{v}\,dV + \int_{\partial V} \mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{v})\,dS\,. \quad (5.68)$$

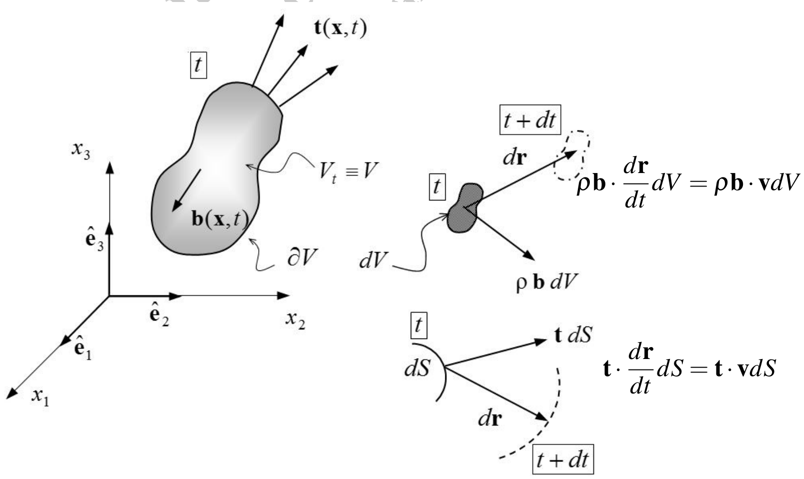

Figure 5.14: Continuous medium subjected to body and surface forces.

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Applying the Divergence Theorem in the last term of (5.68) yields

$$\begin{cases} \displaystyle\int_{\partial V} \mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{v})\,dS = \int_V \nabla\cdot(\boldsymbol{\sigma}\cdot\mathbf{v})\,dV \\ \nabla\cdot(\boldsymbol{\sigma}\cdot\mathbf{v}) = \dfrac{\partial}{\partial x_i}(\sigma_{ij}\,\mathrm{v}_j) = \underbrace{\dfrac{\partial\sigma_{ij}}{\partial x_i}}_{[\nabla\cdot\boldsymbol{\sigma}]_j}\ \mathrm{v}_j + \underbrace{\sigma_{ij}}_{\sigma_{ji}}\underbrace{\dfrac{\partial \mathrm{v}_j}{\partial x_i}}_{[\boldsymbol{l}]_{ji}} = (\nabla\cdot\boldsymbol{\sigma})\cdot\mathbf{v} + \boldsymbol{\sigma}:\boldsymbol{l} \end{cases} \tag{5.69}$$

and, taking into account the identity $\boldsymbol{l} = \mathbf{v}\otimes\nabla = \mathbf{d}+\mathbf{w}$ (see Chapter 2)[18],

$$\boldsymbol{\sigma}:\underbrace{\boldsymbol{l}}_{\mathbf{d}+\mathbf{w}} = \boldsymbol{\sigma}:\mathbf{d} + \underbrace{\boldsymbol{\sigma}:\mathbf{w}}_{=0} = \boldsymbol{\sigma}:\mathbf{d}\,. \tag{5.70}$$

Replacing (5.70) in (5.69) yields

$$\boxed{\int_{\partial V} \mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\mathbf{v})\ dS = \int_V (\nabla\cdot\boldsymbol{\sigma})\cdot\mathbf{v}\,dV + \int_V \boldsymbol{\sigma}:\mathbf{d}\,dV}\ . \tag{5.71}$$

Introducing (5.71) in (5.68), the mechanical power entering the continuous medium results in[19]

$$\begin{aligned} P_e &= \int_V \rho\,\mathbf{b}\cdot\mathbf{v}\,dV + \int_{\partial V}\mathbf{t}\cdot\mathbf{v}\,dS = \int_V \rho\,\mathbf{b}\cdot\mathbf{v}\,dV + \int_V (\nabla\cdot\boldsymbol{\sigma})\cdot\mathbf{v}\,dV + \int_V \boldsymbol{\sigma}:\mathbf{d}\,dV = \\ &= \int_V (\nabla\cdot\boldsymbol{\sigma}+\rho\,\mathbf{b})\cdot\ \mathbf{v}\,dV + \int_V \boldsymbol{\sigma}:\mathbf{d}\,dV = \int_V \rho\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}\,dV + \int_V \boldsymbol{\sigma}:\mathbf{d}\,dV = \\ &\int_V \rho\frac{d}{dt}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)dV + \int_V \boldsymbol{\sigma}:\mathbf{d}\,dV = \int_V \rho\frac{d}{dt}\left(\frac{1}{2}\mathrm{v}^2\right)dV + \int_V \boldsymbol{\sigma}:\mathbf{d}\,dV\,. \end{aligned} \tag{5.72}$$

And applying Reynolds' Lemma (5.33) in (5.72), the mechanical power entering the system results in

Balance of mechanical energy

$$\underbrace{P_e}_{\substack{\text{mechanical}\\\text{power entering}\\\text{the medium}}} = \int_V \rho\,\mathbf{b}\cdot\mathbf{v}\,dV + \int_{\partial V}\mathbf{t}\cdot\mathbf{v}\,dS = \frac{d}{dt}\underbrace{\int_{V_t\equiv V}\frac{1}{2}\rho\mathrm{v}^2\,dV}_{\substack{\mathcal{K}=\text{kinetic}\\\text{energy}}} + \underbrace{\int_V \boldsymbol{\sigma}:\mathbf{d}\,dV}_{\substack{\text{stress}\\\text{power}}} \tag{5.73}$$

[18] The tensor $\boldsymbol{\sigma}$ is symmetric and the tensor $\mathbf{w}$ is antisymmetric. Consequently, their product is null, $\boldsymbol{\sigma}:\mathbf{w}=0$.

[19] The expression $\dfrac{d}{dt}\left(\dfrac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) = \dfrac{1}{2}\dfrac{d\mathbf{v}}{dt}\cdot\mathbf{v} + \dfrac{1}{2}\mathbf{v}\cdot\dfrac{d\mathbf{v}}{dt} = \dfrac{d\mathbf{v}}{dt}\cdot\mathbf{v}$ is used here, in addition to the notation $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2 = \mathrm{v}^2$.

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Equation (5.73) constitutes the continuum mechanics generalization of the balance of mechanical energy in classical mechanics.

Definition 5.9. The *balance of mechanical energy* states that the mechanical energy entering the continuous medium,

$$P_e = \int_V \rho\, \mathbf{b} \cdot \mathbf{v}\, dV + \int_{\partial V} \mathbf{t} \cdot \mathbf{v}\, dS$$

is invested in:

a) modifying the kinetic energy of the particles in the continuous medium,
kinetic energy $\stackrel{not}{=} \mathcal{K} = \int_V \frac{1}{2} \rho\, \mathrm{v}^2 dV \Longrightarrow \frac{d\mathcal{K}}{dt} = \frac{d}{dt} \int_V \frac{1}{2} \rho\, \mathrm{v}^2 dV$.

b) creating stress power,
stress power $\stackrel{def}{=} \int_V \boldsymbol{\sigma} : \mathbf{d}\, dV$.

Remark 5.8. Considering (5.73), the stress power can be defined as the part of the mechanical power entering the system that is not used in modifying the kinetic energy. It can be interpreted as *the work per unit of time* (power) performed by the stresses during the deformation process of the medium.
In a rigid body there is no deformation nor strain rate ($\mathbf{d} = \mathbf{0}$). Therefore, the stresses do not perform mechanical work and the stress power is null. In this case, all the mechanical power entering the system is invested in modifying the kinetic energy of the system and the balance of mechanical energy of a rigid body is recovered.

5.9.2 Thermal Power

Definition 5.10. The *thermal power entering the continuous medium*, Q_e, is the amount of heat per unit of time entering the medium.

The heat entering the medium can be produced by two main causes:

a) Heat entering the medium due to the (*non-convective*) heat flux across the boundary corresponding to the material volume. Note that, since the vol-

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ume is a material volume, the heat flux due to mass transport (convective) is null and, thus, all the heat flux entering the medium will be non-convective.

b) The existence of *heat sources* inside the continuous medium.

- *Non-convective heat flux*

Consider the spatial description of the vector of non-convective heat flux per unit of surface, $\mathbf{q}(\mathbf{x},t)$. Then, the net non-convective heat flux across the boundary of the material volume is (see Figure 5.15)

$$\begin{aligned} \int_{\partial V} \mathbf{q}\cdot\mathbf{n}\, dS &= \frac{\text{amount of heat exiting the medium}}{\text{unit of time}} \\ -\int_{\partial V} \mathbf{q}\cdot\mathbf{n}\, dS &= \frac{\text{amount of heat entering the medium}}{\text{unit of time}} \end{aligned} \tag{5.74}$$

Remark 5.9. A typical example of non-convective flux is heat transfer by *conduction* phenomena. Heat conduction is governed by *Fourier's Law*, which provides the vector of heat flux by (non-convective) conduction $\mathbf{q}(\mathbf{x},t)$ in terms of the temperature $\theta(\mathbf{x},t)$,

$$\left.\begin{array}{c}\text{Fourier's Law of}\\ \text{heat conduction}\end{array}\right\} \quad \mathbf{q}(\mathbf{x},t) = -K\nabla\theta(\mathbf{x},t)\ ,$$

where K is the thermal conductivity, a material property.

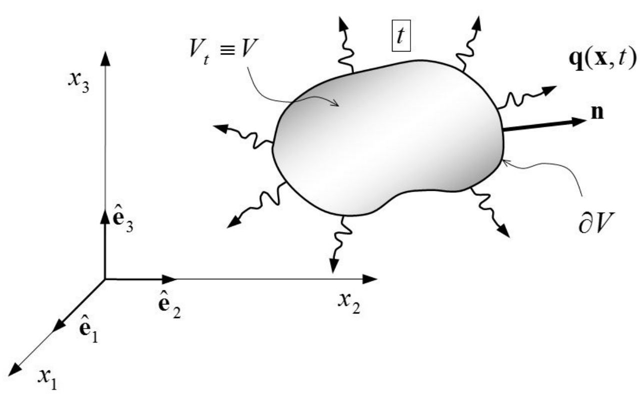

Figure 5.15: Non-convective heat flux.

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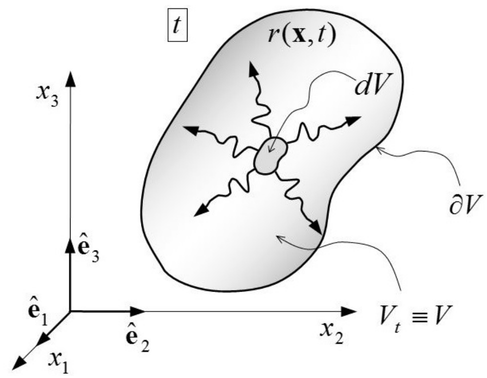

Figure 5.16: Internal heat sources.

- *Internal heat sources*

Heat can be generated (or absorbed) in the interior of the continuous medium due to certain phenomena (chemical reactions, etc.). Consider a scalar function $r(\mathbf{x},t)$ that describes in spatial form the heat generated by the internal sources per unit of mass and unit of time (see Figure 5.16). Then, the heat entering the system, per unit of time, due to the existence of internal heat sources is

$$\int_V \rho r \, dV = \frac{\text{heat generated by the internal sources}}{\text{unit of time}} \, . \tag{5.75}$$

Consequently, the total heat entering the continuous medium per unit of time (or thermal power Q_e) can be expressed as the sum of the contributions of the conduction flux (5.74) and the internal sources (5.75),

$$\boxed{\begin{array}{cc} \text{Heat power entering} & \\ \text{the medium} & Q_e = \int\limits_V \rho r \, dV - \int\limits_{\partial V} \mathbf{q} \cdot \mathbf{n} \, dS \end{array}} \, . \tag{5.76}$$

Then, considering (5.73) and (5.76), the total power entering the continuous medium can be written as follows.

Total power entering the system

$$P_e + Q_e = \frac{d}{dt} \int_{V_t \equiv V} \frac{1}{2} \rho \mathrm{v}^2 \, dV + \int_V \boldsymbol{\sigma} : \mathbf{d} \, dV + \int_V \rho r \, dV - \int_{\partial V} \mathbf{q} \cdot \mathbf{n} \, dS \tag{5.77}$$

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5.10 Energy Balance

5.10.1 Thermodynamic Concepts

- *Thermodynamic system*: a certain amount of continuous matter always formed by the same particles (in the case studied here, a material volume).
- *Thermodynamic variables*: a set of macroscopic variables that characterize the system and intervene in all the physical processes to be studied. They are designated by $\mu_i(\mathbf{x},t) \quad i \in \{1,2,\ldots,n\}$.
- *State, independent or free variables*: a subset of the group of thermodynamic variables in terms of which all the other variables can be expressed.
- *Thermodynamic state*: a thermodynamic state is defined when a certain value is assigned to the state variables and, therefore, to all the thermodynamic variables. In a hyperspace (thermodynamic space) defined by the thermodynamic variables $\mu_i \quad i \in \{1,2,\ldots,n\}$ (see Figure 5.17), a thermodynamic state is represented by a point.
- *Thermodynamic process*: the energetic development of a thermodynamic system that undergoes successive thermodynamic states, changing from an initial state at time t_A to a final state at time t_B (it is a path or continuous segment in the thermodynamic space), see Figure 5.18.
- *Closed cycle*: A thermodynamic process in which the final thermodynamic state coincides with the initial thermodynamic state (all the thermodynamic variables recover their initial value), see Figure 5.19.
- *State function*: any scalar, vector or tensor function $\phi(\mu_1,\ldots,\mu_n)$ of the thermodynamic variables that can be written univocally in terms of these variables.

Consider a thermodynamic space with thermodynamic variables $\mu_i(\mathbf{x},t)$ $i \in \{1,2,\ldots,n\}$ and a function $\phi(\mu_1,\ldots,\mu_n)$ of said variables implicitly defined in terms of a differential form[20]

$$\delta\phi = f_1(\mu_1,\ldots,\mu_n)\,d\mu_1 + \ \ldots \ + f_n(\mu_1,\ldots,\mu_n)\,d\mu_n \,. \tag{5.78}$$

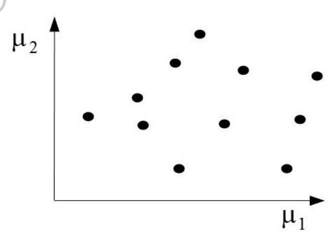

Figure 5.17: Thermodynamic process.

[20] In continuum mechanics thermodynamics it is common to mathematically describe a function $\phi(\mu_1,\ldots,\mu_n)$ of the thermodynamic variables in terms of a differential form $\delta\phi$.

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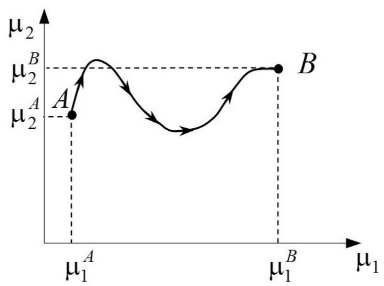

Figure 5.18: Thermodynamic space.

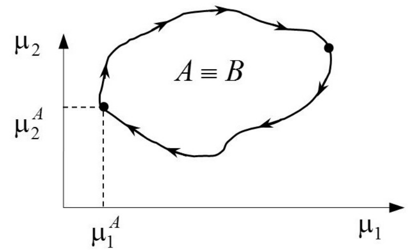

Figure 5.19: Closed cycle.

Consider also a given thermodynamic process $A \rightarrow B$ in the space of the thermodynamic variables. Equation (5.78) provides the value of the function $\phi(\mu_1^B, \dots, \mu_n^B) \stackrel{not}{=} \phi_B$ when its value $\phi(\mu_1^A, \dots, \mu_n^A) \stackrel{not}{=} \phi_A$ and the corresponding path (thermodynamic process) $A \rightarrow B$ are known by means of

$$\phi_B = \phi_A + \int_A^B \delta\phi \ . \tag{5.79}$$

However, (5.79) does not guarantee that the result ϕ_B is independent of the path (thermodynamic process) followed. In mathematical terms, it does not guarantee that the function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by (5.79) is univocal (see Figure 5.20) and, thus, that there exists a single image $\phi\,(\mu_1, \dots, \mu_n)$ corresponding to each point in the thermodynamic space.

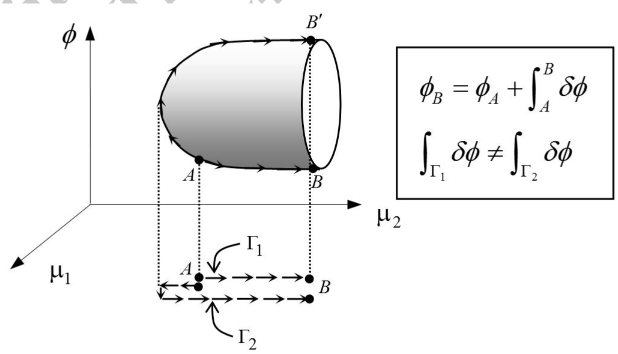

Figure 5.20: Non-univocal function of the thermodynamic variables μ_1 and μ_2.

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Remark 5.10. For a function $\phi(\mu_1,\dots,\mu_n)$, implicitly described in terms of a differential form $\delta\phi$, to be a *state function* (that is, for it to be univocal), said differential form must be an *exact differential* $\delta\phi = d\phi$. In other words, the differential form $\delta\phi$ must be integrable.
The necessary and sufficient condition for a differential form such as (5.78) to be an exact differential is the equality of mixed partial derivatives,

$$\left.\begin{array}{l} \delta\phi = f_1(\mu_1,\dots,\mu_n)\,d\mu_1 + \dots + f_n(\mu_1,\dots,\mu_n)\,d\mu_n \\ \dfrac{\partial f_i(\mu_1,\dots,\mu_n)}{\partial \mu_j} = \dfrac{\partial f_j(\mu_1,\dots,\mu_n)}{\partial \mu_i} \quad \forall i,j \in \{1,\dots,n\} \end{array}\right\} \Leftrightarrow \delta\phi = d\phi\ .$$

If the differential form (5.78) is an exact differential, (5.79) results in

$$\phi_B = \phi_A + \int_A^B d\phi = \phi_A + \Big[\Delta\phi\Big]_A^B \tag{5.80}$$

and the value ϕ_B is independent of the integration path. Then, function ϕ is said to be a *state function that depends only on the values of the state variables* and not on the thermodynamic process.

Remark 5.11. If ϕ is a state function, then $\delta\phi$ is an exact differential and the integral along the complete closed cycle of the differential $\delta\phi$ is null,

$$\int_A^A \delta\phi = \oint d\phi = \underbrace{\Big[\Delta\phi\Big]_A^A}_{=0} = 0\ .$$

Example 5.3 – *Determine whether the function $\phi(\mu_1, \mu_2)$ defined in terms of an exact differential $\delta\phi = 4\mu_2\, d\mu_1 + \mu_1\, d\mu_2$ can be a state function or not.*

Solution

Following (5.78),

$$\begin{array}{l} f_1 \equiv 4\mu_2 \\ f_2 \equiv \mu_1 \end{array} \quad \Longrightarrow \quad \begin{array}{l} \dfrac{\partial f_1}{\partial \mu_2} = 4 \\ \dfrac{\partial f_2}{\partial \mu_1} = 1 \end{array} \quad \Longrightarrow \quad \frac{\partial f_1}{\partial \mu_2} \neq \frac{\partial f_2}{\partial \mu_1}$$

Then, $\delta\phi$ *is not an exact differential* (see Remark 5.10) and ϕ is not a state function.

5.10.2 First Law of Thermodynamics

Experience shows that the mechanical power (5.73) is not an exact differential and, therefore, the mechanical work performed by the system in a closed cycle is not null. The same happens with the thermal power (5.76).

$$\begin{array}{lll} \delta\phi_1 = P_e\, dt & \Longrightarrow & \displaystyle\oint P_e\, dt \neq 0 \\ \delta\phi_2 = Q_e\, dt & \Longrightarrow & \displaystyle\oint Q_e\, dt \neq 0 \end{array} \tag{5.81}$$

However, there exists experimental evidence that proves that the sum of the mechanical and thermal powers, that is, the total power entering the system (5.77) (see Figure 5.21), is, in effect, an exact differential and, thus, a *state function* $\mathcal{E}$ that corresponds to the concept of energy can be defined in terms of it,

$$P_e\, dt + Q_e\, dt = d\mathcal{E} \quad \Rightarrow \quad \mathcal{E}(t) = \int_{t_0}^{t} (P_e + Q_e)\; dt + const. \tag{5.82}$$

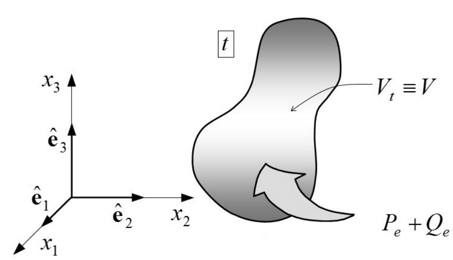

Figure 5.21: Total power entering the system.

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The *first law of thermodynamics* postulates the following:

1) There exists a *state function* $\mathcal{E}$, named *total energy of the system*, such that its variation per unit of time is equal to the sum of the mechanical and thermal powers entering the system.

$$\boxed{\begin{array}{c} \dfrac{d\mathcal{E}}{dt} = P_e + Q_e \\ \underbrace{d\mathcal{E}}_{\substack{\text{Variation of} \\ \text{total energy}}} = \underbrace{P_e\,dt}_{\substack{\text{Mechanical} \\ \text{work}}} + \underbrace{Q_e\,dt}_{\substack{\text{Thermal} \\ \text{work}}} \end{array}} \tag{5.83}$$

2) There exists another *state function* $\mathcal{U}$, named *internal energy of the system*, such that
   a) It is an *extensive property*[21]. Then, a *specific internal energy* $u(\mathbf{x},t)$ (or internal energy per unit of mass) can be defined as

$$\boxed{\mathcal{U} = \int_V \rho u \, dV} \ . \tag{5.84}$$

   b) The variation of the total energy of the system $\mathcal{E}$ is equal to the sum of the variation of the internal energy $\mathcal{U}$ and the variation of the kinetic energy $\mathcal{K}$.

$$\boxed{\underbrace{d\mathcal{E}}_{\substack{\text{Exact} \\ \text{differential}}} = d\mathcal{K} + \underbrace{d\mathcal{U}}_{\substack{\text{Exact} \\ \text{differential}}}} \tag{5.85}$$

Remark 5.12. Note that, since the total energy $\mathcal{E}$ and the internal energy $\mathcal{U}$ of the system have been postulated to be state functions, $d\mathcal{E}$ and $d\mathcal{U}$ in (5.85) are exact differentials. Consequently, the term $d\mathcal{K} = d\mathcal{E} - d\mathcal{U}$ in said equation is also an exact differential (because the difference between exact differentials is also an exact differential) and, thus, is a state function. Then, it is confirmed that (5.85) indirectly postulates the character of state function and, therefore, the *energetic* character of $\mathcal{K}$.

[21] A certain property is extensive when *the complete content of the property is the sum of the content of the property in each of its parts*. An extensive property allows defining the content of this property per unit of mass (*specific value of the property*) or per unit of volume (*density of the property*).

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From (5.83) and considering (5.77),

$$\left.\begin{array}{l}\dfrac{d\mathcal{E}}{dt} = P_e + Q_e = \dfrac{d}{dt}\displaystyle\int_{V_t\equiv V} \frac{1}{2}\rho \mathrm{v}^2 dV + \int_V \boldsymbol{\sigma} : \mathbf{d}\, dV + \int_V \rho r\, dV - \int_{\partial V} \mathbf{q}\cdot\mathbf{n}\, dS \\ \mathcal{K} = \displaystyle\int_V \frac{1}{2}\rho \mathrm{v}^2 dV \end{array}\right\} \Rightarrow$$

$$\frac{d\mathcal{E}}{dt} = \frac{d\mathcal{K}}{dt} + \frac{d\mathcal{U}}{dt} = \underbrace{\frac{d}{dt}\int_V \frac{1}{2}\rho \mathrm{v}^2 dV}_{\dfrac{d\mathcal{K}}{dt}} + \underbrace{\int_V \boldsymbol{\sigma} : \mathbf{d}\, dV + \int_V \rho r\, dV - \int_{\partial V} \mathbf{q}\cdot\mathbf{n}\, dS}_{\dfrac{d\mathcal{U}}{dt}} \tag{5.86}$$

Global form of the internal energy balance

$$\frac{d\mathcal{U}}{dt} = \frac{d}{dt}\int_{V_t\equiv V} \rho u\, dV = \int_V \boldsymbol{\sigma} : \mathbf{d}\, dV + \int_V \rho r\, dV - \int_{\partial V} \mathbf{q}\cdot\mathbf{n}\, dS \tag{5.87}$$

Remark 5.13. From (5.87) it follows that any variation per unit of time of the internal energy $d\mathcal{U}/dt$ is produced by

- a generation of stress power, $\displaystyle\int_V \boldsymbol{\sigma} : \mathbf{d}\, dV$, and
- a variation per unit of time of the content of heat in the medium, $\displaystyle\int_V \rho r\, dV - \int_{\partial V} \mathbf{q}\cdot\mathbf{n}\, dS$.

Applying Reynolds' Lemma (5.33) and the Divergence Theorem on (5.87) yields

$$\frac{d}{dt}\int_{V_t\equiv V} \rho u\, dV = \int_V \rho \frac{du}{dt} dV = \int_V \boldsymbol{\sigma} : \mathbf{d}\, dV + \int_V \rho r\, dV - \int_V \nabla\cdot\mathbf{q}\, dV \quad \forall \Delta V \subset V\,. \tag{5.88}$$

Finally, localizing in (5.88) results in the *local spatial form of the internal energy balance*.

> *Local spatial form of the internal energy balance (energy equation)*
> $$\rho \frac{du}{dt} = \boldsymbol{\sigma} : \mathbf{d} + (\rho r - \nabla \cdot \mathbf{q}) \qquad \forall \mathbf{x} \in V, \quad \forall t \tag{5.89}$$

5.11 Reversible and Irreversible Processes

The first law of thermodynamics leads to a balance equation that must be fulfilled for all the physical processes that take place in reality,

$$P_e + Q_e = \frac{d\mathcal{E}}{dt} = \frac{d\mathcal{U}}{dt} + \frac{d\mathcal{K}}{dt} \,. \tag{5.90}$$

In particular, if an *isolated system*[22] is considered, the time variation of the total energy of the system will be null ($d\mathcal{E}/dt = 0 \Rightarrow$ the total energy is conserved). Therefore, the energy balance equation (5.90), established by the first law of thermodynamics, imposes that any variation of internal energy $d\mathcal{U}/dt$ must be compensated with a variation of kinetic energy $d\mathcal{K}/dt$ of equal value but of opposite sign, and vice-versa (see Figure 5.22).

What the first law of thermodynamics does not establish is whether this (kinetic and internal) energy exchange in an isolated system can take place equally in both directions or not ($d\mathcal{U}/dt = -d\mathcal{K}/dt > 0$ or $d\mathcal{U}/dt = -d\mathcal{K}/dt < 0$). That is, it does not establish any restriction that indicates if an imaginary and arbitrary

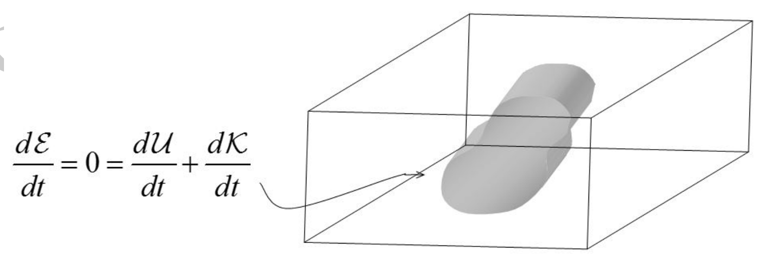

Figure 5.22: Isolated thermodynamic system.

[22] An *isolated thermodynamic system* is a system that cannot exchange energy with its exterior. In a strict sense, the only perfectly isolated system is the universe, although one can think of quasi-isolated or imperfectly isolated smaller systems.

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process that implies an energy exchange in a certain direction is physically possible or not. It only establishes the fulfillment of the energy balance (5.90) in the event that the process takes place.

However, experience shows that certain processes that could be imagined theoretically never take place in reality. Suppose, for example, the isolated system in Figure 5.23 consisting of

– a rigid (non-deformable) wheel that spins with angular velocity ω, and

– a brake that can be applied on the wheel at a certain instant of time.

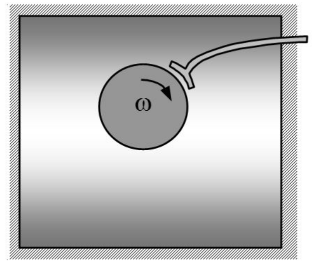

Figure 5.23

Consider now the following two processes:

1) At a certain instant of time the brake acts, the rotation speed of the wheel ω decreases and, thus, so does its kinetic energy ($d\mathcal{K} < 0$). On the other hand, due to the friction between the brake and the wheel, heat is generated and there is an increase of the internal energy ($d\mathcal{U} > 0$). Experience shows that this process, in which the internal energy increases at the expense of decreasing the kinetic energy[23], can take place in reality and, therefore, is a *physically feasible* process.

2) Maintaining the brake disabled, at a certain instant of time the wheel spontaneously increases its rotation speed ω and, thus, its kinetic energy increases ($d\mathcal{K} > 0$). According to the first law of thermodynamics, the internal energy of the system will decrease ($d\mathcal{U} < 0$). However, experience shows that this (spontaneous) increase of speed *never* takes place, and neither does the decrease in the amount of heat of the system (which would be reflected in a decrease in temperature).

The conclusion to this observation is that *the second process considered in the example is not a feasible physical process*. More generally, only thermodynamic processes that tend to increase the internal energy and decrease the kinetic energy, and not the other way round, are feasible for the system under consideration.

It is concluded, then, that *the first law of thermodynamics is only applicable when a particular physical process is feasible*, and the need to determine when a particular physical process is feasible, or if a physical process is feasible in one direction, in both or in none, is noted. The answer to this problem is provided by the *second law of thermodynamics*.

[23] The wheel, being a non-deformable medium, has null stress power (see Remark 5.8) and all the variation of internal energy of the system derives from a variation of its heat content (see Remark 5.13).

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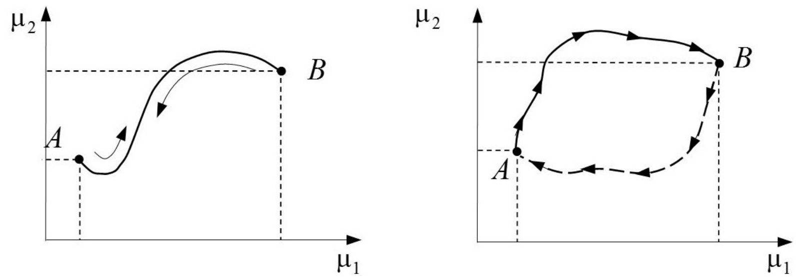

Figure 5.24: Reversible (left) and irreversible (right) processes.

The previous considerations lead to the classification, from a thermodynamic point of view, of the possible physical processes in *feasible and non-feasible processes* and, in addition, suggest classifying the feasible processes into *reversible and irreversible processes*.

Definition 5.11. A thermodynamic process $A \to B$ is a *reversible process* when it is possible to return from the final thermodynamic state B to the initial thermodynamic state A along the same path (see Figure 5.24).
A thermodynamic process $A \to B$ is an *irreversible process* when it is not possible to return from the final thermodynamic state B to the initial thermodynamic state A, *along the same path* (even if a different path can be followed, see Figure 5.24).

In general, within a same thermodynamic process there will exist reversible and irreversible sections.

5.12 Second Law of Thermodynamics. Entropy

5.12.1 Second Law of Thermodynamics. Global form

The second law of thermodynamic postulates the following:

1) There exists a state function named *absolute temperature* $\theta(\mathbf{x},t)$ that is intensive[24] and strictly positive $(\theta > 0)$.

[24] A certain property is intensive when *the complete content of the property is not the sum of the content of the property in each of its parts*. Contrary to what happens with extensive properties, in this case the content of the property cannot be defined per unit of mass (*specific value of the property*) or per unit of volume (*density of the property*). Temperature is a paradigmatic example of intensive property.

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2) There exists a state function named *entropy* S with the following characteristics:
   a) It is an *extensive* variable. This implies that there exists a specific entropy (entropy per unit of mass) s such that

$$s = \frac{\text{entropy}}{\text{unit of mass}} \quad \Longrightarrow \quad S = \int_V \rho s \, dV \ . \tag{5.91}$$

   b) The inequality

Integral form of the second law of thermodynamics

$$\boxed{\frac{dS}{dt} = \frac{d}{dt} \int_{V_t \equiv V} \rho s \, dV \ \geq \ \int_V \rho \frac{r}{\theta} \, dV - \int_{\partial V} \frac{\mathbf{q}}{\theta} \cdot \mathbf{n} \, dS} \tag{5.92}$$

   is satisfied, where:
   – The sign $=$ corresponds to *reversible* processes.
   – The sign $>$ corresponds to *irreversible* processes.
   – The sign $<$ cannot occur and indicates that the corresponding process is *not feasible*.

5.12.2 Physical Interpretation of the Second Law of Thermodynamics

As discussed Section 5.9.2, the magnitude *heat in the system* is characterized by

a) A source term (or generation of heat per unit of mass and unit of time) $r(\mathbf{x},t)$, defined in the interior of the material volume.
b) The non-convective flux (heat flux by conduction) across the boundary of the material surface, defined in terms of a non-convective flux vector per unit of surface $\mathbf{q}(\mathbf{x},t)$.

These terms allow computing the amount of heat per unit of time entering a material volume V_t, which at a certain instant of time occupies the spatial volume $V_t \equiv V$ with outward unit normal vector $\mathbf{n}$, as

$$\boxed{Q_e = \int_V \rho r \, dV - \int_{\partial V} \mathbf{q} \cdot \mathbf{n} \, dS} \ . \tag{5.93}$$

Consider now a new magnitude defined as *heat per unit of absolute temperature* in the system. If $\theta(\mathbf{x},t)$ is the absolute temperature, the amount of said magnitude will be characterized by

a) A source term r/θ corresponding to the generation of *heat per unit of absolute temperature*, per unit of mass and unit of time.

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b) A non-convective flux vector $\mathbf{q}/\theta$ of the *heat per unit of absolute temperature*.

Magnitude	**Source term**	**Non-convective flux vector**
$\dfrac{\text{heat}}{\text{unit of time}}$	r	$\mathbf{q}$
$\dfrac{\text{heat/unit of absolute temperature}}{\text{unit of time}}$	$\dfrac{r}{\theta}$	$\dfrac{\mathbf{q}}{\theta}$

Similarly to (5.93), the new source term r/θ and non-convective flux vector $\mathbf{q}/\theta$ allow computing the amount of *heat per unit of absolute temperature* entering the material volume per unit of time as

$$\boxed{\frac{(\text{heat/unit of temperature})\text{ entering }V}{\text{unit of time}} = \int_V \rho\frac{r}{\theta}\,dV - \int_{\partial V}\frac{\mathbf{q}}{\theta}\cdot\mathbf{n}\,dS} \ . \tag{5.94}$$

Observing now (5.94), the second term in this expression is identified as the magnitude defined in (5.92). This circumstance allows interpreting the second law of thermodynamics establishing that *the generation of entropy per unit of time in a continuous medium is always larger than or equal to the amount of heat per unit of temperature entering the system per unit of time*.

Global form of the second law of thermodynamics

$$\frac{dS}{dt} \geq \underbrace{\int_V \rho\frac{r}{\theta}\,dV - \int_{\partial V}\frac{\mathbf{q}}{\theta}\cdot\mathbf{n}\,dS}_{\substack{\text{amount of the property}\\ \text{“heat / unit of absolute temperature”}\\ \text{entering the domain V per unit of time}}} \tag{5.95}$$

Consider now the decomposition of the total entropy of the system S into two distinct components:

- $S^{(i)}$: *entropy generated* (produced) *internally by the continuous medium*. Its generation rate is $dS^{(i)}/dt$.
- $S^{(e)}$: *entropy generated by the interaction of the continuous medium with its exterior*. Its variation rate is $dS^{(e)}/dt$.

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Then, the following is naturally satisfied.

$$\boxed{\frac{dS}{dt} = \frac{dS^{(e)}}{dt} + \frac{dS^{(i)}}{dt}} \tag{5.96}$$

Now, if one establishes that the variation rate of the entropy generated by the interaction with the exterior *coincides* with the magnitude *heat per unit of absolute temperature* in (5.93),

$$\boxed{\frac{dS^{(e)}}{dt} = \int_V \rho \frac{r}{\theta} dV - \int_{\partial V} \frac{\mathbf{q}}{\theta} \cdot \mathbf{n}\, dS} \tag{5.97}$$

and, taking into account (5.95) to (5.97), the variation per unit of time of the internally generated entropy results in

$$\boxed{\frac{dS^{(i)}}{dt} = \frac{dS}{dt} - \frac{dS^{(e)}}{dt} = \frac{dS}{dt} - \left(\int_V \rho \frac{r}{\theta} dV - \int_{\partial V} \frac{\mathbf{q}}{\theta} \cdot \mathbf{n}\, dS \right) \geq 0} \, . \tag{5.98}$$

Remark 5.14. According to (5.98), the internally generated entropy $S^{(i)}$ of the system (continuous medium) never decreases ($dS^{(i)}/dt \geq 0$). In a perfectly isolated system (strictly speaking, only the universe is a perfectly isolated system) there is no interaction with the exterior and the variation of entropy due to interaction with the exterior is null, ($dS^{(e)}/dt = 0$). In this case, the second law of thermodynamics establishes that

$$\frac{dS^{(i)}}{dt} = \frac{dS}{dt} \geq 0$$

or, in other words, *the total entropy of a perfectly isolated system never decreases*. This is the starting point of some alternative formulations of the second law of thermodynamics.

5.12.3 Reformulation of the Second Law of Thermodynamics

In view of the considerations in Section 5.12.2, the second law of thermodynamics can be reformulated as follows:

1) There exists a state function named absolute temperature such that it is always strictly positive,

$$\theta(\mathbf{x},t) > 0 \, . \tag{5.99}$$

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2) There exists a state function named entropy that is an extensive variable and, thus, can be defined in terms of a specific entropy (or entropy per unit of mass) $s(\mathbf{x},t)$ as

$$S(t) = \int_V \rho s \, dV \ . \tag{5.100}$$

3) Entropy can be generated internally, $S^{(i)}$, or produced by interaction with the exterior, $S^{(e)}$. Both components of the entropy are extensive variables and their content in a material volume V can be defined in terms of their respective specific values $s^{(i)}$ and $s^{(e)}$,

$$S^{(i)} = \int_V \rho s^{(i)} \, dV \quad \text{and} \quad S^{(e)} = \int_V \rho s^{(e)} \, dV \tag{5.101}$$

$$S = S^{(i)} + S^{(e)} \quad \Longrightarrow \quad \frac{dS}{dt} = \frac{dS^{(i)}}{dt} + \frac{dS^{(e)}}{dt} \tag{5.102}$$

and introducing Reynolds' Lemma (5.33) in (5.102) yields

$$\begin{aligned} \frac{dS^{(i)}}{dt} &= \frac{d}{dt} \int_{V_t \equiv V} \rho s^{(i)} \, dV = \int_V \rho \frac{ds^{(i)}}{dt} \, dV \ , \\ \frac{dS^{(e)}}{dt} &= \frac{d}{dt} \int_{V_t \equiv V} \rho s^{(e)} \, dV = \int_V \rho \frac{ds^{(e)}}{dt} \, dV \ . \end{aligned} \tag{5.103}$$

4) The variation of external entropy (generated by the interaction with the exterior) is associated with the variation of the magnitude *heat per unit of absolute temperature*, and is defined as

$$\frac{dS^{(e)}}{dt} = \int_V \rho \frac{r}{\theta} \, dV - \int_{\partial V} \frac{\mathbf{q}}{\theta} \cdot \mathbf{n} \, dS \ . \tag{5.104}$$

5) *The internally generated entropy never diminishes*. Based on the variation of its content during the thermodynamic process, the following situations are defined:

$$\frac{dS^{(i)}}{dt} \geq 0 \ \rightarrow \ \begin{cases} = 0 \text{ reversible process} \\ > 0 \text{ irreversible process} \\ < 0 \text{ non-feasible process} \end{cases} \tag{5.105}$$

5.12.4 Local Form of the Second Law of Thermodynamics. Clausius-Planck Equation

Using (5.101) to (5.104), expression (5.105) is rewritten as

$$
\begin{gathered}
\frac{dS^{(i)}}{dt} = \frac{dS}{dt} - \frac{dS^{(e)}}{dt} \geq 0 \\
\frac{d}{dt}\int\limits_{V_t \equiv V} \rho s^{(i)}\, dV = \frac{d}{dt}\int\limits_{V_t \equiv V} \rho s\, dV - \left(\int\limits_{V} \rho \frac{r}{\theta}\, dV - \int\limits_{\partial V} \frac{\mathbf{q}}{\theta}\cdot \mathbf{n}\, dS \right) \geq 0
\end{gathered}
\tag{5.106}
$$

Applying Reynolds' Lemma (5.33) (on the first and second integral of the left-hand term in (5.106)) and the Divergence Theorem (on the last term) yields

$$
\int\limits_{V} \rho \frac{ds^{(i)}}{dt}\, dV = \int\limits_{V} \rho \frac{ds}{dt}\, dV - \left(\int\limits_{V} \rho \frac{r}{\theta}\, dV - \int\limits_{V} \nabla \cdot \left(\frac{\mathbf{q}}{\theta} \right) dV \right) \geq 0 \quad \forall \Delta V \subset V
\tag{5.107}
$$

and localizing in (5.107), the local form of the second law of thermodynamics or Clausius-Duhem equation is obtained.

Local form of the second law of thermodynamics (Clausius-Duhem inequality)

$$
\boxed{\rho \frac{ds^{(i)}}{dt} = \rho \frac{ds}{dt} - \left(\rho \frac{r}{\theta} - \nabla \cdot \left(\frac{\mathbf{q}}{\theta} \right) \right) \geq 0 \qquad \forall \mathbf{x} \in V, \quad \forall t}
\tag{5.108}
$$

Where, again, in (5.108) the sign

- $=$ corresponds to *reversible* processes,
- $>$ corresponds to *irreversible* processes, and
- $<$ indicates that the corresponding process is *not feasible*.

Equation (5.108) can be rewritten as follows.

$$
\left.
\begin{gathered}
\nabla \cdot \left(\frac{\mathbf{q}}{\theta} \right) = \frac{1}{\theta} \nabla \cdot \mathbf{q} - \frac{1}{\theta^2}\, \mathbf{q} \cdot \nabla \theta \\
\rho \underbrace{\frac{ds^{(i)}}{dt}}_{\overset{not}{=} \dot{s}^{(i)}} = \rho \underbrace{\frac{ds}{dt}}_{\overset{not}{=} \dot{s}} - \rho \frac{r}{\theta} + \frac{1}{\theta} \nabla \cdot \mathbf{q} - \frac{1}{\theta^2}\, \mathbf{q} \cdot \nabla \theta \geq 0
\end{gathered}
\right\} \Rightarrow
\tag{5.109}
$$

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$$\boxed{\overset{\bullet}{s}^{(i)} = \underbrace{\overset{\bullet}{s} - \frac{r}{\theta} + \frac{1}{\rho\theta}\nabla\cdot\mathbf{q}}_{\overset{\bullet}{s}^{(i)}_{local}} \quad \underbrace{-\frac{1}{\rho\theta^2}\mathbf{q}\cdot\nabla\theta}_{\overset{\bullet}{s}^{(i)}_{cond}} \geq 0} \tag{5.110}$$

Then, a much stronger (more restrictive) formulation of the second law of thermodynamics can be posed. This formulation postulates that the internally generated entropy, $\overset{\bullet}{s}^{(i)}$, can be generated locally, $\overset{\bullet}{s}^{(i)}_{local}$, or by heat conduction, $\overset{\bullet}{s}^{(i)}_{cond}$, and that both contributions to the generation of entropy must be non-negative.

Local internal generation of entropy
(Clausius-Planck inequality)

$$\overset{\bullet}{s}^{(i)}_{local} = \overset{\bullet}{s} - \frac{r}{\theta} + \frac{1}{\rho\theta}\nabla\cdot\mathbf{q} \geq 0 \tag{5.111}$$

Internal generation of entropy by heat conduction

$$\overset{\bullet}{s}^{(i)}_{cond} = -\frac{1}{\rho\theta^2}\mathbf{q}\cdot\nabla\theta \geq 0 \tag{5.112}$$

Remark 5.15. Equation (5.112) can be interpreted in the following manner: since the density, ρ, and the absolute temperature, θ, are positive magnitudes, said equation can be written as

$$\mathbf{q}\cdot\nabla\theta \leq 0\,,$$

which establishes that the non-convective heat flux, $\mathbf{q}$, and the temperature gradient, $\nabla\theta$, are vectors that have opposite directions (their dot product is negative). In other words, (5.112) is the mathematical expression of the experimentally verified fact that *heat flows by conduction from the hottest to the coldest parts in the medium* (see Figure 5.24), characterizing as *non-feasible* those processes in which the contrary occurs.

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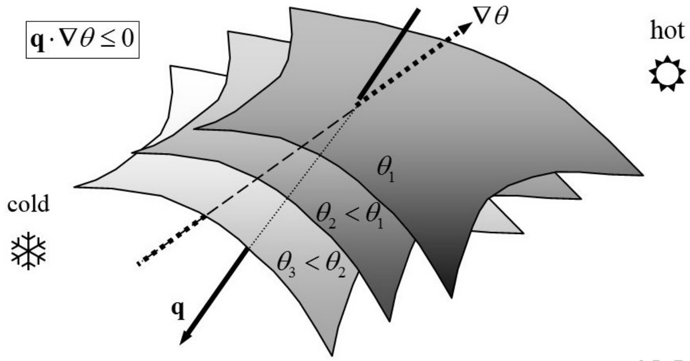

Figure 5.25: Heat flux is opposed to the thermal gradient.

Remark 5.16. In the context of Fourier's Law of heat conduction, $q=-K\,\nabla\theta$ (see Remark 5.9), expression (5.112) can be written as

$$\left.\begin{array}{l}\mathbf{q}\cdot\nabla\theta\leq 0\\ \mathbf{q}=-K\nabla\theta\end{array}\right\}\Longrightarrow\quad -K\,|\nabla\theta|^2\leq 0\quad\Longrightarrow\quad K\geq 0$$

which reveals that negative values of the thermal conductivity K lack physical meaning.

5.12.5 Alternative Forms of the Second Law of Thermodynamics

Alternative expressions of the Clausius-Planck equation (5.111) in combination with the local form of the energy balance equation (5.89) are often used in continuum mechanics.

- *Clausius-Planck equation in terms of the specific internal energy*

A common form of expressing the Clausius-Planck equation is doing so in terms of the specific internal energy $u(\mathbf{x},t)$ in (5.84). This expression is obtained using the local spatial form of the energy balance equation (5.89),

$$\rho\frac{du}{dt}\overset{not}{=}\rho\dot{u}=\boldsymbol{\sigma}:\mathbf{d}+\rho r-\nabla\cdot\mathbf{q}\quad\Longrightarrow\quad\rho r-\nabla\cdot\mathbf{q}=\rho\dot{u}-\boldsymbol{\sigma}:\mathbf{d}\,,\qquad(5.113)$$

and, replacing it in the Clausius-Planck equation (5.111),

$$\rho\theta\dot{s}^{(i)}_{local}=\rho\theta\dot{s}-(\rho r-\nabla\cdot\mathbf{q})=\rho\theta\dot{s}-\rho\dot{u}+\boldsymbol{\sigma}:\mathbf{d}\geq 0\,.\qquad(5.114)$$

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$$\boxed{\begin{array}{c}\textit{Clausius-Planck equation in terms of the internal energy} \\ -\rho\,(\dot{u}-\theta\dot{s})+\boldsymbol{\sigma}:\mathbf{d}\geq 0\end{array}} \tag{5.115}$$

- *Clausius-Planck equation in terms of the Helmholtz free energy*

Another possibility is to express the Clausius-Planck equation in terms of the (specific) Helmholtz free energy $\psi\,(\mathbf{x},t)$, which is defined in terms of the internal energy, the entropy and the temperature as

$$\boxed{\psi \overset{def}{=} u - s\theta} \,. \tag{5.116}$$

Differentiating (5.116) with respect to time results in

$$\dot{\psi}=\dot{u}-s\dot{\theta}-\dot{s}\theta \quad \Longrightarrow \quad \dot{u}-\theta\dot{s}=\dot{\psi}+s\dot{\theta} \tag{5.117}$$

and, replacing (5.117) in (5.115), yields the Clausius-Planck equation in terms of the Helmholtz free energy,

$$\rho\theta\dot{s}^{(i)}_{local}=-\rho\,(\dot{u}-\theta\dot{s})+\boldsymbol{\sigma}:\mathbf{d}=-\rho\left(\dot{\psi}+s\dot{\theta}\right)+\boldsymbol{\sigma}:\mathbf{d}\geq 0\,. \tag{5.118}$$

$$\boxed{\begin{array}{c}\textit{Clausius-Planck equation in terms of the free energy} \\ -\rho\left(\dot{\psi}+s\dot{\theta}\right)+\boldsymbol{\sigma}:\mathbf{d}\geq 0\end{array}} \tag{5.119}$$

For the infinitesimal strain case, $\mathbf{d}=\dot{\boldsymbol{\varepsilon}}$ (see Chapter 2, Remark 2.22), and replacing in (5.119) results in

$$\boxed{\begin{array}{c}\textit{Clausius-Planck equation (infinitesimal strain)} \\ -\rho\left(\dot{\psi}+s\dot{\theta}\right)+\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}\geq 0\end{array}} \,. \tag{5.120}$$

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5.13 Continuum Mechanics Equations. Constitutive Equations

At this point it is convenient to summarize the set of (local) differential equations provided by the balance principles.

1) *Conservation of mass*. Mass continuity equation.

$$\left.\begin{aligned} &\frac{d\rho}{dt}+\rho\nabla\cdot\mathbf{v}=0 \\ &\frac{d\rho}{dt}+\rho\frac{\partial \mathrm{v}_i}{\partial x_i}=0 \end{aligned}\right\} \rightarrow \text{ 1 equation} \tag{5.121}$$

2) *Balance of linear momentum*. Cauchy's equation.

$$\left.\begin{aligned} &\nabla\cdot\boldsymbol{\sigma}+\rho\mathbf{b}=\rho\frac{d\mathbf{v}}{dt} \\ &\frac{\partial\sigma_{ji}}{\partial x_j}+\rho b_i=\rho\frac{d\mathrm{v}_i}{dt} \quad i\in\{1,2,3\} \end{aligned}\right\} \rightarrow \text{ 3 equations} \tag{5.122}$$

3) *Balance of angular momentum*. Symmetry of the stress tensor.

$$\left.\begin{aligned} &\boldsymbol{\sigma}=\boldsymbol{\sigma}^T \\ &\sigma_{12}=\sigma_{21}\ ;\quad \sigma_{13}=\sigma_{31}\ ;\quad \sigma_{23}=\sigma_{32} \end{aligned}\right\} \rightarrow \text{ 3 equations} \tag{5.123}$$

4) *Energy balance*. First law of thermodynamics.

$$\left.\begin{aligned} &\rho\frac{du}{dt}=\boldsymbol{\sigma}:\mathbf{d}+(\rho r-\nabla\cdot\mathbf{q}) \\ &\rho\frac{du}{dt}=\sigma_{ij}\,d_{ij}+\left(\rho r-\frac{\partial q_i}{\partial x_i}\right) \end{aligned}\right\} \rightarrow \text{ 1 equation} \tag{5.124}$$

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5) *Second law of thermodynamics*. Clausius-Planck and heat flux inequalities.

$$
\left.\begin{aligned}
&-\rho\left(\dot{u}-\theta\dot{s}\right)+\boldsymbol{\sigma}:\mathbf{d}\geq 0\\
&-\rho\left(\dot{u}-\theta\dot{s}\right)+\sigma_{ij}d_{ij}\geq 0
\end{aligned}\right\}\rightarrow\ 1\text{ restriction}
$$

$$
\left.\begin{aligned}
&-\frac{1}{\rho\theta^2}\mathbf{q}\cdot\nabla\theta\geq 0\\
&-\frac{1}{\rho\theta^2}q_i\frac{\partial\theta}{\partial x_i}\geq 0
\end{aligned}\right\}\rightarrow\ 1\text{ restriction}
\tag{5.125}
$$

These add up to a total of *8 partial differential equations* (PDEs) and two restrictions. Counting the number of unknowns that intervene in these equations results in[25]

$$
\left.\begin{aligned}
&\rho\ \rightarrow 1\text{ unknown}\\
&\mathbf{v}\ \rightarrow 3\text{ unknowns}\\
&\boldsymbol{\sigma}\ \rightarrow 9\text{ unknowns}\\
&u\ \rightarrow 1\text{ unknown}\\
&\mathbf{q}\ \rightarrow 3\text{ unknowns}\\
&\theta\ \rightarrow 1\text{ unknown}\\
&s\ \rightarrow 1\text{ unknown}
\end{aligned}\right\}19\text{ unknowns}
$$

Therefore, it is obvious that additional equations are needed to solve the problem. These equations, which receive the generic name of *constitutive equations* and are specific to the material that constitutes the continuous medium, are

6) *Fourier's law of heat conduction*.

$$
\left.\begin{aligned}
&\mathbf{q}=-K\,\nabla\theta\\
&q_i=-K\,\frac{\partial\theta}{\partial x_i}\qquad i\in\{1,2,3\}
\end{aligned}\right\}\rightarrow\ 3\text{ equations}
\tag{5.126}
$$

[25] The six components of the strain rate tensor $\mathbf{d}$ in (5.124) and (5.125) are not considered unknowns because they are assumed to be implicitly calculable in terms of the velocity $\mathbf{v}$ by means of the relation $\mathbf{d}(\mathbf{v})=\nabla^s\mathbf{v}$ (see Chapter 2, Section 2.13.2).

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7) *Constitutive equations (per se)*[26].

$$
\begin{array}{ll}
\left.\begin{array}{c}\text{Thermo-}\\ \text{mechanical}\\ \text{constitutive}\\ \text{equations}\end{array}\right\} & f_i\left(\boldsymbol{\sigma},\boldsymbol{\varepsilon}\left(\mathbf{v}\right),\theta,\boldsymbol{\mu}\right)=0 \quad i\in\{1,\ldots,6\} \rightarrow 6 \text{ equations} \\
\left.\begin{array}{c}\text{Entropy}\\ \text{constitutive}\\ \text{equation}\end{array}\right\} & s=s\left(\boldsymbol{\varepsilon}\left(\mathbf{v}\right),\theta,\boldsymbol{\mu}\right)=0 \qquad\qquad\qquad \rightarrow 1 \text{ equation}
\end{array}
\tag{5.127}
$$

where $\boldsymbol{\mu}=\left\{\mu_1,\ldots,\mu_p\right\}$ are a set of new thermodynamic variables (p new unknowns) introduced by the thermo-mechanical constitutive equations.

8) *Thermodynamic equations of state*.

$$
\left.\begin{array}{ll}
\left.\begin{array}{c}\text{Caloric}\\ \text{eqn. of state}\end{array}\right\} & u=g\left(\rho,\boldsymbol{\varepsilon}\left(\mathbf{v}\right),\theta,\boldsymbol{\mu}\right) \\
\left.\begin{array}{c}\text{Kinetic}\\ \text{eqns. of state}\end{array}\right\} & F_i\left(\rho,\theta,\boldsymbol{\mu}\right)=0 \quad i\in\{1,2,\ldots,p\}
\end{array}\right\} \rightarrow (1+p) \text{ eqns.}
\tag{5.128}
$$

There is now a set of $(1+p)$ equations and $(1+p)$ unknowns that, with the adequate boundary conditions, constitute a mathematically well-defined problem.

Remark 5.17. The mass continuity equation, Cauchy's equation, the symmetry of the stress tensor, the energy balance and the inequalities of the second law of thermodynamics (equations (5.121) to (5.125)) are valid and general for all the continuous medium, regardless of the material that constitutes the medium, and for any range of displacements and strains. Conversely, the constitutive equations (5.126) to (5.128) are specific to the material or the type of continuous medium being studied (solid, fluid, gas) and differentiate them from one another.

[26] The strains $\boldsymbol{\varepsilon}$ often intervene in the thermo-mechanical constitutive equations. However, these are not considered as additional unknowns because they are assumed to be implicitly calculable in terms of the equation of motion which, in turn, can be calculated by integration of the velocity field, $\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}\left(\mathbf{v}\right)$ (see Chapters 1 and 2).

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5.13.1 Uncoupled Thermo-Mechanical Problem

To solve the general problem in continuum mechanics, a system of partial differential equations must be solved, which involve the $(1+p)$ equations and the $(1+p)$ unknowns discussed in the previous section. However, under certain circumstances or hypotheses, the general problem can be decomposed into two smaller problems (each of them involving a smaller number of equations and unknowns), named *mechanical problem* and *thermal problem*, and that can be solved independently (uncoupled) from one another.

For example, consider the temperature distribution $\theta(\mathbf{x},t)$ is known *a priori*, or that it does not intervene in a relevant manner in the thermo-mechanical constitutive equations (5.127), and that, in addition, said constitutive equations do not involve new thermodynamic variables ($\boldsymbol{\mu} = \{\emptyset\}$). In this case, the following set of equations are considered[27]

$$
\left.\begin{array}{lll}
\text{Mass continuity equation:} & \dfrac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 & \text{(1 eqn)} \\[2ex]
\text{Cauchy's equation:} & \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \dfrac{d\mathbf{v}}{dt} & \text{(3 eqn)} \\[2ex]
\text{Mechanical constitutive equations:} & \begin{array}{l} f_i(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v})) = 0 \\ \quad i \in \{1, \ldots, 6\} \end{array} & \text{(6 eqn)}
\end{array}\right\} \rightarrow 10 \text{ equations}, \tag{5.129}
$$

which involve the following unknowns.

$$
\left.\begin{array}{l}
\rho(\mathbf{x},t) \rightarrow 1 \text{ unknown} \\
\mathbf{v}(\mathbf{x},t) \rightarrow 3 \text{ unknowns} \\
\boldsymbol{\sigma}(\mathbf{x},t) \rightarrow 6 \text{ unknowns}
\end{array}\right\} \; 19 \text{ unknowns} \tag{5.130}
$$

The problem defined by equations (5.129) and (5.130) constitutes the so-called *mechanical problem*, which involves the variables (5.130) (named *mechanical variables*) that, moreover, are the real interest in many engineering problems.

The mechanical problem constitutes, in this case, a system of *reduced* differential equations, with respect to the general problem, and can be solved *independently* of the rest of equations of said problem.

[27] For simplicity, it is assumed that the symmetry of the stress tensor (5.123) is already imposed. Then this equation is eliminated from the set of equations and the number of unknowns of $\boldsymbol{\sigma}$ is reduced from 9 to 6 components.

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PROBLEMS

Problem 5.1 – *Justify whether the following statements are true or false.*

a) The mass flux across a closed material surface is null only when the motion is stationary.

b) The mass flux across a closed control surface is null when this flux is stationary.

Solution

a) The statement is false because a material surface is always constituted by the same particles and, therefore, cannot be crossed by any particle throughout its motion. For this reason, the mass flux across a material surface is always null, independently of the motion being stationary or not.

b) The statement is true because the application of the mass continuity equation on a stationary flux implies

$$\left.\begin{array}{ll} \text{Mass continuity equation} \Longrightarrow & \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \\ \text{Stationary flux} \qquad \Longrightarrow & \dfrac{\partial \rho}{\partial t} = 0 \end{array}\right\} \Longrightarrow \quad \nabla \cdot (\rho \mathbf{v}) = 0 \; .$$

Resulting, thus, what had to be proven,

$$\nabla \cdot (\rho \mathbf{v}) = 0 \quad \Longrightarrow \quad \int\limits_{V} \nabla \cdot (\rho \mathbf{v}) \; dV = \int\limits_{\partial V} \rho \mathbf{v} \cdot \mathbf{n} \, dS = 0 \; .$$

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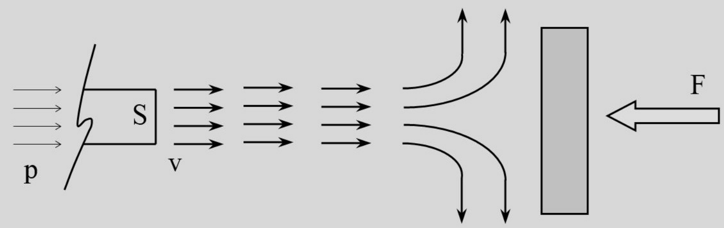
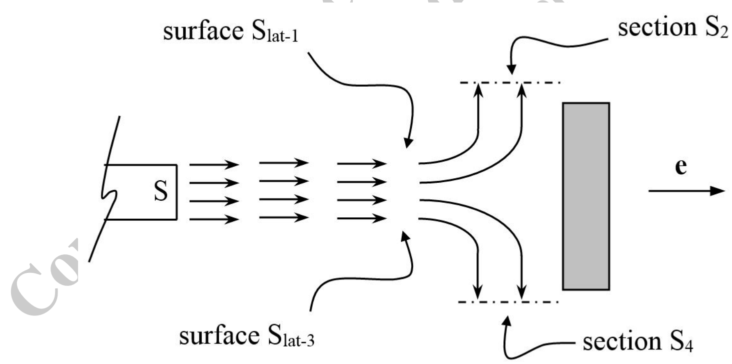
Problem 5.2 – *A water jet with cross-section* S*, pressure* p *and velocity* **v***, impacts perpendicularly on a disc as indicated in the figure below. Determine the force* F *in steady-state regime that must be exerted on the disc for it to remain in a fixed position (consider the atmospheric pressure is negligible).*



Solution

Taking into account the Reynolds Transport Theorem (5.39) and that the problem is in steady-state regime, the forces acting on the fluid are

$$\sum \mathbf{F}_{ext/f} = \frac{d}{dt} \int_V \rho \mathbf{v}\, dV = \int_V \frac{\partial}{\partial t} (\rho \mathbf{v})\, dV + \int_{\partial V} \rho \mathbf{v} (\mathbf{n} \cdot \mathbf{v})\, dS = \int_S \rho \mathbf{v} (\mathbf{n} \cdot \mathbf{v})\, dS\,.$$



Note that the velocity vector of the fluid along the surfaces S_{lat-1} and S_{lat-3} is perpendicular to the outward unit normal vector of the volume that encloses the fluid, therefore, $\mathbf{v} \cdot \mathbf{n} = 0$. The same happens in the walls of the disc.
The vectors **v** and **n** in sections S_2 and S_4 are not perpendicular but, because there exists symmetry and **v** is perpendicular to **F**, they do not contribute components to the horizontal forces. Therefore, the only forces acting on the fluid are

$$\sum \mathbf{F}_{ext/f} = \int_{\partial V} \rho \mathbf{v} (\mathbf{n} \cdot \mathbf{v})\, dS = \int_S \rho \mathrm{v} \mathbf{e} (-\mathbf{e} \cdot \mathrm{v} \mathbf{e})\, dS = -\rho \mathrm{v}^2 \mathrm{S} \mathbf{e}\,.$$

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On the other hand, the external force, the pressure of the water jet and the atmospheric pressure (which is negligible) also act on the fluid,

$$\sum \mathbf{F}_{ext/f} = -\mathrm{F}\mathbf{e} + \text{ atmospheric pressure forces } + \mathrm{pS}\mathbf{e} = -\mathrm{F}\mathbf{e} + \mathrm{pS}\mathbf{e} \,.$$

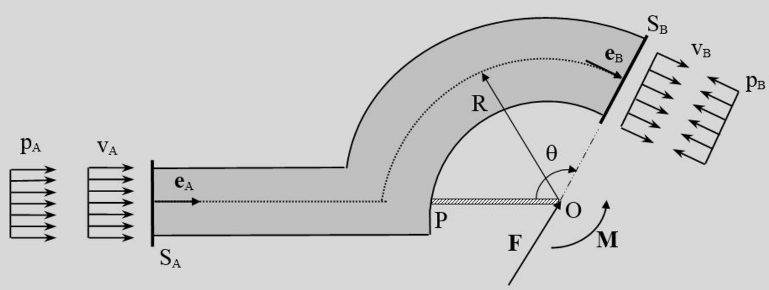
Equating both expressions and isolating the value of the module of the force **F** finally results in

$$\boxed{\mathrm{F} = \rho \mathrm{v}^2 \mathrm{S} + \mathrm{pS}} \,.$$

Problem 5.3 – *A volume flow rate Q circulates, in steady-state regime, through a pipe from end A (with cross-section S_A) to end B (with cross-section $S_B < S_A$). The pipe is secured at point O by a rigid element $P-O$.*

Determine:

a) The entry and exit velocities v_A and v_B in terms of the flow rate.
*b) The values of the angle θ that maximize and minimize the reaction force **F** at O, and the corresponding values of said reaction force.*
*c) The values of the angle θ that maximize and minimize the reaction moment **M** about O, and the corresponding values of said reaction moment.*
d) The power W of the pump needed to provide the flow rate Q.



Hypotheses:

1) The water is a perfect fluid $(\sigma_{ij} = -\mathrm{p}\,\delta_{ij})$ and incompressible.
2) The weight of the pipe and the water are negligible.

Solution

a) The incompressible character of water implies that the density is constant for a same particle and, therefore, $d\rho/dt = 0$. Introducing this into the mass continuity equation (5.24), results in

$$\nabla \cdot \mathbf{v} = 0 \quad \Longleftrightarrow \quad \int_V \nabla \cdot \mathbf{v}\, dV = 0 \qquad \forall V \, . \tag{1}$$

The adequate integration volume must now be defined. To this aim, a control volume such that its boundary is a closed surface must be found ($S = \partial V$) to be able to apply the Divergence Theorem,

$$\int_V \nabla \cdot \mathbf{v}\, dV = \int_{\partial V} \mathbf{n} \cdot \mathbf{v}\, dS \qquad \forall V \tag{2}$$

where $\mathbf{n}$ is the outward unit normal vector in the boundary of the volume V. Then, by means of [1] and [2], the conclusion is reached that the net outflow across the contour of the control volume is null,

$$\int_{\partial V} \mathbf{n} \cdot \mathbf{v}\, dS = 0 \qquad \forall V \, .$$

The volume the defined by the water contained inside the pipe between the cross-sections S_A and S_B is taken as control volume. Consider, in addition, the unit vectors $\mathbf{e}_A$ and $\mathbf{e}_B$ perpendicular to said cross-sections, respectively, and in the direction of the flow of water. Then, the following expression is deduced. Note that the extended integral on the boundary ∂V is applied only on cross-sections S_A and S_B since $\mathbf{n} \cdot \mathbf{v} = 0$ on the walls of the pipe, that is, $\mathbf{n}$ and $\mathbf{v}$ are perpendicular to one another.

$$\int_{\partial V} \mathbf{n} \cdot \mathbf{v}\, dS = \int_{S_A} \mathbf{n} \cdot \mathbf{v}\, dS + \int_{S_B} \mathbf{n} \cdot \mathbf{v}\, dS = \int_{S_A} (-\mathbf{e}_A) \cdot \mathrm{v}_A \mathbf{e}_A\, dS + \int_{S_B} \mathbf{e}_B \cdot \mathrm{v}_B \mathbf{e}_B\, dS = 0$$

$$\Longrightarrow \quad -\mathrm{v}_A S_A + \mathrm{v}_B S_B = 0 \quad \Longrightarrow \quad \mathrm{v}_A S_A = \mathrm{v}_B S_B = Q$$

It is verified, thus, that the flow rate at the entrance and exit of the pipe are the same,

$$\boxed{\mathrm{v}_A = \frac{Q}{S_A} \quad ; \quad \mathrm{v}_B = \frac{Q}{S_B}} \, . \tag{3}$$

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b) The balance of linear momentum equation (5.49) must be applied to find the value of the force $\mathbf{F}$,

$$\mathbf{R} = \int_V \rho \mathbf{b}\, dV + \int_{\partial V} \mathbf{t}\, dS = \frac{d}{dt}\int_V \rho \mathbf{v}\, dV \quad , \qquad [4]$$

where $\mathbf{R}$ is the total resultant of the forces acting on the fluid. On the other hand, expanding the right-hand term in [4] by means of the Reynolds Transport Theorem (5.39), yields

$$\frac{d}{dt}\int_V \rho \mathbf{v}\, dV = \frac{\partial}{\partial t}\int_V \rho \mathbf{v}\, dV + \int_{\partial V} \rho \mathbf{v}\,(\mathbf{n}\cdot\mathbf{v})\ dS \ . \qquad [5]$$

The problem is being solved for a steady-state regime, i.e., the local derivative of any property is null. In addition, the flow is known to exist solely through sections S_A and S_B since $\mathbf{n}$ and $\mathbf{v}$ are perpendicular to one another on the walls of the pipe. Therefore, according to [4] and [5],

$$\begin{aligned}
\mathbf{R} &= \int_{S_A} \rho \mathbf{v}\,(\mathbf{n}\cdot\mathbf{v})\ dS + \int_{S_B} \rho \mathbf{v}\,(\mathbf{n}\cdot\mathbf{v})\ dS = \\
&= \int_{S_A} \rho \mathrm{v}_A \mathbf{e}_A\,(-\mathbf{e}_A\cdot \mathrm{v}_A\mathbf{e}_A)\ dS + \int_{S_B} \rho \mathrm{v}_B \mathbf{e}_B\,(\mathbf{e}_B\cdot \mathrm{v}_B\mathbf{e}_B)\ dS
\end{aligned}$$

$$\mathbf{R} = -\rho \mathrm{v}_A^2 S_A\, \mathbf{e}_A + \rho \mathrm{v}_B^2 S_B\, \mathbf{e}_B . \qquad [6]$$

Introducing [3] in [6] allows expressing the resultant force $\mathbf{R}$ in terms of Q,

$$\mathbf{R} = -\rho Q^2 \left(-\frac{1}{S_A}\,\mathbf{e}_A + \frac{1}{S_B}\,\mathbf{e}_B \right) \ .$$

Now the different forces that compose $\mathbf{R}$ must be analyzed. According to the statement of the problem, body forces can be neglected ($\mathbf{b} = 0$). Therefore, only surface forces must be taken into account, that is, the forces applied on the boundary of the control volume (S_A, S_B and S_{lat}, where this last one corresponds to the lateral surface of the walls),

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$$\mathbf{R} = \int_V \rho \mathbf{b} \, dV + \int_{\partial V} \mathbf{t} \, dS = \int_{\partial V} \mathbf{t} \, dS = \int_{S_A} \mathbf{t} \, dS + \int_{S_B} \mathbf{t} \, dS + \int_{S_{lat}} \mathbf{t} \, dS =$$

$$= \int_{S_A} \mathrm{p}_A \mathbf{e}_A \, dS + \int_{S_B} \mathrm{p}_B \left(-\mathbf{e}_B\right) \, dS + \mathbf{R}_{p/f} \, .$$

Here, $\mathbf{R}_{p/f}$ represents the forces exerted on the fluid by the walls of the pipe, which initially are unknown but can be obtained using [6] as follows.

$$\mathbf{R}_{p/f} = \mathbf{R} - \int_{S_A} \mathrm{p}_A \mathbf{e}_A \, dS - \int_{S_B} \mathrm{p}_B \left(-\mathbf{e}_B\right) \, dS$$

$$\mathbf{R}_{p/f} = -\rho \mathrm{v}_A^2 \, S_A \, \mathbf{e}_A + \rho \mathrm{v}_B^2 \, S_B \, \mathbf{e}_B - \mathrm{p}_A \, S_A \, \mathbf{e}_A + \mathrm{p}_B \, S_B \, \mathbf{e}_B$$

$$\mathbf{R}_{p/f} = -\left(\rho \mathrm{v}_A^2 + \mathrm{p}_A\right) S_A \, \mathbf{e}_A - \left(\rho \mathrm{v}_B^2 + \mathrm{p}_B\right) S_B \, \mathbf{e}_B \qquad [7]$$

Introducing [3], $\mathbf{R}_{p/f}$ can be expressed in terms of Q,

$$\mathbf{R}_{p/f} = -\left(\rho \frac{Q^2}{S_A} + \mathrm{p}_A S_A\right) \mathbf{e}_A - \left(\rho \frac{Q^2}{S_B} + \mathrm{p}_B S_B\right) \mathbf{e}_B \, .$$

Now the relation between $\mathbf{R}_{p/f}$ and the unknown being sought, $\mathbf{F}$, must be found. To this aim, the *action and reaction law* is considered, and the pipe and the rigid element $P-O$ are regarded as a single body. Under these conditions, the force exerted by the fluid on the pipe is

$$\mathbf{R}_{f/p} = -\mathbf{R}_{p/f} \, .$$

Since it is the only action on the body, and taking into account that the weight of the pipe is negligible, this force must be compensated by an exterior action $\mathbf{F}$ for the body to be in equilibrium.

$$\mathbf{R}_{f/p} + \mathbf{F} = \mathbf{0} \qquad \Longrightarrow \qquad \mathbf{F} = -\mathbf{R}_{f/p} = \mathbf{R}_{p/f}$$

Introducing [7], the value of $\mathbf{F}$ is finally obtained as

$$\mathbf{F} = -\left(\rho \mathrm{v}_A^2 + \mathrm{p}_A\right) S_A \, \mathbf{e}_A + \left(\rho \mathrm{v}_B^2 + \mathrm{p}_B\right) S_B \, \mathbf{e}_B \, .$$

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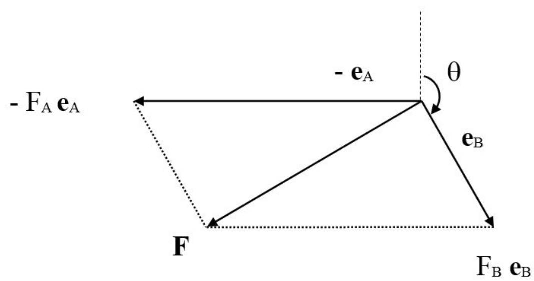
Using [3], the force $\mathbf{F}$ is expressed in terms of Q,

$$\mathbf{F} = -\left(\rho \frac{Q^2}{S_A} + \mathrm{p}_A S_A\right)\mathbf{e}_A + \left(\rho \frac{Q^2}{S_B} + \mathrm{p}_B S_B\right)\mathbf{e}_B \ . \qquad [8]$$

There are two possible ways of obtaining the maximum and minimum of $|\mathbf{F}|$ in terms of θ:

1) Determine the expression of $|\mathbf{F}|$ and search for its extremes by imposing that its derivative is zero (this option not recommended).
2) Direct method, in which the two vectors acting in the value of $\mathbf{F}$ are analyzed (this option developed below).

According to [7], the value of $\mathbf{F}$ depends on the positive scalar values F_A and F_B, which multiply the vectors $(-\mathbf{e}_A)$ and $\mathbf{e}_B$, respectively.



The vector $(-\mathbf{e}_A)$ is fixed and does not depend on θ but $\mathbf{e}_B$ does vary with θ. The scalars F_A and F_B are constant values. Therefore, the maximum and minimum values of $\mathbf{F}$ will be obtained when F_A and F_B either completely add or subtract one another, respectively. That is, when the vectors $(-\mathbf{e}_A)$ and $\mathbf{e}_B$ are parallel to each other. Taking into account [3] and [8], the maximum and minimum values are found to be:

− Minimum value of $\mathbf{F}$

$$\boxed{\theta = \frac{\pi}{2}}$$

$$\boxed{|\mathbf{F}|_{\min} = \rho Q^2 \left(\frac{1}{S_B} - \frac{1}{S_A}\right) + \mathrm{p}_B S_B - \mathrm{p}_A S_A}$$

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− Maximum value of **F**

$$\boxed{\theta = \frac{3\pi}{2}}$$

$$\boxed{|\mathbf{F}|_{\min} = \rho Q^2 \left(\frac{1}{S_B} + \frac{1}{S_A} \right) + \mathrm{p}_B S_B + \mathrm{p}_A S_A}$$

c) The balance of angular momentum equation (5.57) must be applied to find the moment **M** about point O,

$$\mathbf{M}_{liq} = \int_V \mathbf{r} \times \rho \mathbf{b} \, dV + \int_{\partial V} \mathbf{r} \times \mathbf{t} \, dS = \frac{d}{dt} \int_V \mathbf{r} \times \rho \mathbf{v} \, dV \ , \qquad [9]$$

where $\mathbf{M}_{liq}$ is the resultant moment of the moments acting on the fluid. On the other hand, expanding the right-hand term in [9] by means of the Reynolds Transport Theorem (5.39), yields

$$\frac{d}{dt} \int_V \mathbf{r} \times \rho \mathbf{v} \, dV = \frac{\partial}{\partial t} \int_V \mathbf{r} \times \rho \mathbf{v} \, dV + \int_{\partial V} (\mathbf{r} \times \rho \mathbf{v}) (\mathbf{n} \cdot \mathbf{v}) \, dS \ . \qquad [10]$$

As in b), because the problem is in steady-state regime, the local derivative is null. Again, **n** and **v** are perpendicular to one another on the walls of the pipe and, thus, considering [9] and [10], results in the expression

$$\mathbf{M}_{liq} = \int_{S_A} (\mathbf{r} \times \rho \mathbf{v}) (\mathbf{n} \cdot \mathbf{v}) \ dS + \int_{S_B} (\mathbf{r} \times \rho \mathbf{v}) (\mathbf{n} \cdot \mathbf{v}) \ dS \ , \qquad [11]$$

where the following must be taken into account:

1. The solution to each integral can be determined considering the resultant of the velocities in the middle point of each cross-section since the velocity distributions are uniform and parallel in both cases.
2. For cross-section S_A, the resultant of the velocity vector applied on the center of the cross-section acts on point O and, therefore, does not generate any moment because the cross product of the position vector at the center of S_A and the velocity vector are null.
3. For cross-section S_B, vectors **r** and **v** belong to the plane of the paper and, thus, their cross product has the direction of the vector $(-\mathbf{e}_z)$. In addition, they are perpendicular to each other, so the module of their cross product is the product of their modules.

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Applying these considerations to [11] yields

$$\mathbf{M}_{liq} = \int_{S_B} R\rho\, \mathrm{v}_B \,(-\mathbf{e}_z)\,(\mathbf{e}_B \cdot \mathrm{v}_B\, \mathbf{e}_B)\; dS$$

$$\mathbf{M}_{liq} = -\rho\, \mathrm{v}_B^2 R\, S_B\, \mathbf{e}_z = -\rho \frac{Q^2}{S_B} R\, \mathbf{e}_z \qquad [12]$$

The following step consists in studying the contributions of the body forces, which in this case are null ($\mathbf{b} = \mathbf{0}$), and of the surface forces.

$$\begin{aligned}
\mathbf{M}_{liq} &= \int_V \mathbf{r} \times \rho \mathbf{b}\, dV + \int_{\partial V} \mathbf{r} \times \mathbf{t}\, dS = \int_{\partial V} \mathbf{r} \times \mathbf{t}\, dS = \\
&= \int_{S_A} \mathbf{r} \times \mathbf{t}\, dS + \int_{S_B} \mathbf{r} \times \mathbf{t}\, dS - \int_{S_{lat}} \mathbf{r} \times \mathbf{t}\, dS = \\
&= \mathbf{0} + \int_{S_B} R\, \mathrm{p}_B\, \mathbf{e}_z\, dS + \mathbf{M}_{p/f} = R\, \mathrm{p}_B\, S_B\, \mathbf{e}_z + \mathbf{M}_{p/f}\,,
\end{aligned}$$

where $\mathbf{M}_{p/f}$ is the moment exerted by the pipe on the fluid. To determine its expression, [12] is used,

$$\mathbf{M}_{p/f} = \mathbf{M}_{liq} - R\, \mathrm{p}_B\, S_B\, \mathbf{e}_z = -\rho\, \mathrm{v}_B^2 R\, S_B\, \mathbf{e}_z - R\, \mathrm{p}_B\, S_B\, \mathbf{e}_z\,,$$

$$\mathbf{M}_{p/f} = -R\, S_B \left(\rho\, \mathrm{v}_B^2 + \mathrm{p}_B\right) \mathbf{e}_z = -R \left(\rho\, \frac{Q^2}{S_B} + \mathrm{p}_B\, S_B\right) \mathbf{e}_z. \qquad [13]$$

Introducing the *action and reaction law* will allow obtaining the moment exerted by the fluid on the pipe,

$$\mathbf{M}_{p/f} = -\mathbf{M}_{f/p}\,.$$

Considering the pipe and the rigid element $P - O$ as a single body in equilibrium and neglecting the weight of the pipe,

$$\mathbf{M}_{f/p} + \mathbf{M} = \mathbf{0} \qquad \Longrightarrow \qquad \mathbf{M} = -\mathbf{M}_{f/p} = \mathbf{M}_{p/f}\,.$$

Finally, the value of the moment $\mathbf{M}$ is obtained, using [13].

$$\boxed{\mathbf{M} = -R\, S_B \left(\rho \mathrm{v}_B^2 + \mathrm{p}_B\right) \mathbf{e}_z = -R \left(\rho \frac{Q^2}{S_B} + \mathrm{p}_B S_B\right) \mathbf{e}_z}$$

Note that this result does not depend on the angle θ and, therefore, its module will have a constant value.

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d) To determine the value of the power W needed to provide a volume flow rate Q the *balance of mechanical energy equation* (5.73) is used.

$$W = \frac{d}{dt}\int_V \frac{1}{2}\rho \mathrm{v}^2 \, dV + \int_V \boldsymbol{\sigma} : \mathbf{d} \, dV \qquad [14]$$

The stress power in an incompressible perfect fluid is null,

$$\int_V \boldsymbol{\sigma} : \mathbf{d} \, dV = 0 \, .$$

This is proven as follows.

$$\begin{aligned}
\boldsymbol{\sigma} : \mathbf{d} &= -\mathrm{p}\, \mathbf{1} : \mathbf{d} = -\mathrm{p}\,\mathrm{Tr}\,(\mathbf{d}) = -\mathrm{p}\,\mathrm{Tr}\left(\frac{1}{2}\left(\boldsymbol{l} + \boldsymbol{l}^T\right)\right) = \\
&= -\mathrm{p}\,\mathrm{Tr}\,(\boldsymbol{l}) = -\mathrm{p}\,\mathrm{Tr}\begin{bmatrix} \dfrac{\partial \mathrm{v}_x}{\partial x} & \dfrac{\partial \mathrm{v}_x}{\partial y} & \dfrac{\partial \mathrm{v}_x}{\partial z} \\ \dfrac{\partial \mathrm{v}_y}{\partial x} & \dfrac{\partial \mathrm{v}_y}{\partial y} & \dfrac{\partial \mathrm{v}_y}{\partial z} \\ \dfrac{\partial \mathrm{v}_z}{\partial x} & \dfrac{\partial \mathrm{v}_z}{\partial y} & \dfrac{\partial \mathrm{v}_z}{\partial z} \end{bmatrix} = \\
&= -\mathrm{p}\left(\frac{\partial \mathrm{v}_x}{\partial x} + \frac{\partial \mathrm{v}_y}{\partial y} + \frac{\partial \mathrm{v}_z}{\partial z}\right) = -\mathrm{p}\, \nabla \cdot \mathbf{v} = 0 \, ,
\end{aligned}$$

where [1] has been applied in relation to the incompressibility condition, to conclude that the divergence of the velocity is null.
Applying the Reynolds Transport Theorem (5.39) on the term of the material derivative of the kinetic energy in [14] results in

$$W = \frac{d}{dt}\int_V \frac{1}{2}\rho \mathrm{v}^2 \, dV = \frac{\partial}{\partial t}\int_V \frac{1}{2}\rho \mathrm{v}^2 \, dV + \int_{\partial V} \frac{1}{2}\rho \mathrm{v}^2 \left(\mathbf{n} \cdot \mathbf{v}\right) \, dS \, .$$

And, again, considering the problem is in steady-state regime and that **n** and **v** are perpendicular to one another on the walls of the pipe, the expression of the incoming power W is determined.

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$$
\begin{aligned}
W &= \int\limits_{S_A} \frac{1}{2}\rho \mathrm{v}^2 \left(\mathbf{n}\cdot\mathbf{v}\right)\, dS + \int\limits_{S_B} \frac{1}{2}\rho \mathrm{v}^2 \left(\mathbf{n}\cdot\mathbf{v}\right)\, dS = \\
&= \int\limits_{S_A} \frac{1}{2}\rho \mathrm{v}_A^2 \left(-\mathrm{v}_A\right)\, dS + \int\limits_{S_B} \frac{1}{2}\rho \mathrm{v}_B^2 \left(\mathrm{v}_B\right)\, dS = \frac{1}{2}\rho \mathrm{v}_A^3\, S_A + \frac{1}{2}\rho \mathrm{v}_B^3\, S_B
\end{aligned}
$$

Then, by means of [3], the final result is obtained.

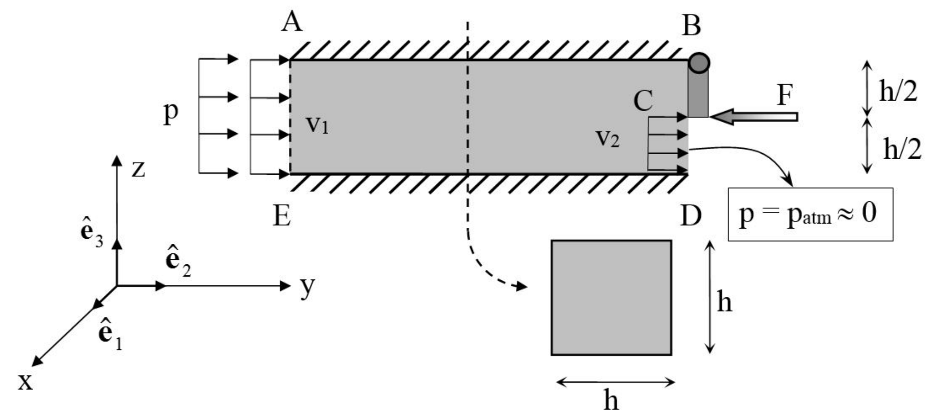
$$
\boxed{W = \frac{1}{2}\rho Q^3 \left(\frac{1}{S_B^2} - \frac{1}{S_A^2}\right)}
$$

EXERCISES

5.1 – *Justify why the following statements are true.*

- *a) In an incompressible flow, the volume flow rate across a control surface is null.*
- *b) In a steady-state flow, the mass flux across a closed control surface is null.*
- *c) In an incompressible fluid in steady-state regime, the density is uniform only when the density at the initial time is uniform.*

5.2 – *The figure below shows the longitudinal cross-section of a square pipe. Water flows through this pipe, entering through section AE and exiting through section CD. The exit section includes a floodgate BC that can rotate around hinge B and is maintained in vertical position by the action of force F.*



Determine:

- *a) The exit velocity* v_2 *in terms of the entrance velocity* v_1 *(justify the expression used).*
- *b) The resultant force and moment at point B of the actions exerted on the fluid by the interior of the pipe.*
- *c) The resultant force and moment at point B of the actions exerted by the fluid on floodgate BC.*
- *d) The value of the force F and the reactions the pipe exerts on floodgate BC.*

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e) The power of the pump needed to maintain the flow.

Additional hypotheses:

1) Steady-state regime

2) Incompressible fluid

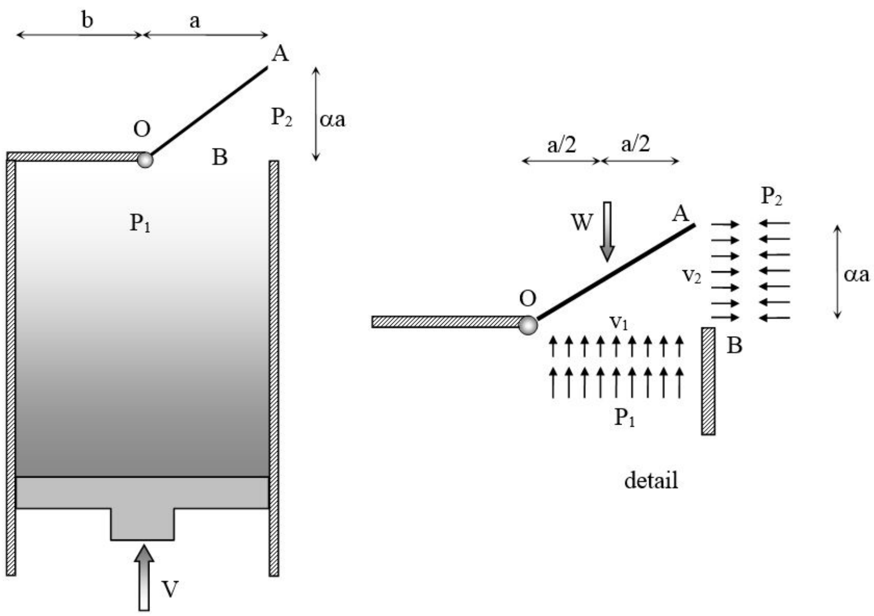
3) The pressures acting on the lateral walls of the pipe are assumed constant and equal to the entrance pressure p*.*

4) The exit pressure is equal to the atmospheric pressure, which is negligible.

5) Perfect fluid: $\sigma_{ij} = -\mathrm{p}\delta_{ij}$

6) The weights of the fluid and the floodgate are negligible.

5.3 – *The figure below shows the longitudinal cross-section of a pump used to inject an incompressible fluid, fitted with a retention valve OA whose weight, per unit of width (normal to the plane of the figure), is W. Consider a steady-state motion, driven by the velocity of the piston V and the internal uniform pressure* P_1*. The external uniform pressure is* P_2*.*



Determine:

a) The uniform velocities v_1 *and* v_2 *in terms of V (justify the expression used).*

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b) The resultant force, per unit of width, exerted by the fluid on the valve OA.

c) The resultant moment about O, per unit of width, exerted by the fluid on the valve OA.

d) The value of W needed for the valve OA to maintain its position (as shown in the figure) during the injection process.

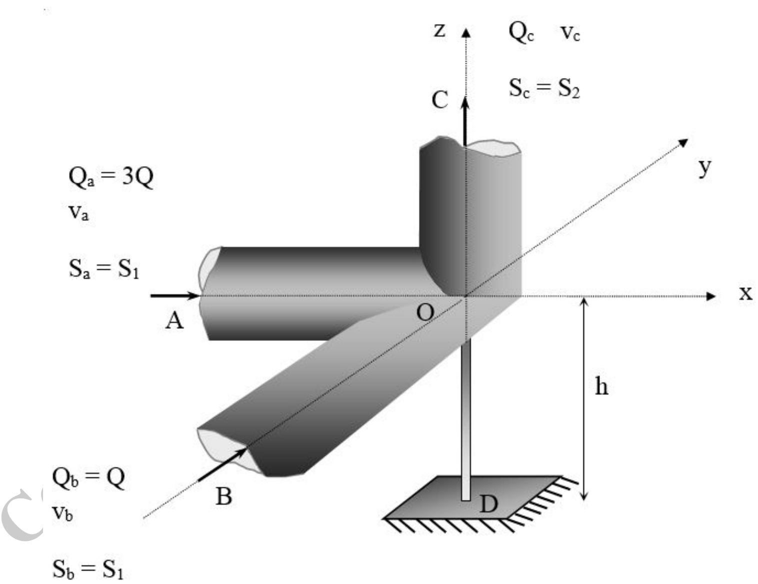
Additional hypotheses:

1) The body forces of the fluid are negligible.

2) Perfect fluid: $\sigma_{ij} = -\mathrm{p}\delta_{ij}$

Perform the analysis by linear meter.

5.4 – *A perfect and incompressible fluid flows through the pipe junction shown in the figure below. The junction is held in place by a rigid element O – D.*



Determine:

a) The entrance velocities $(\mathrm{v}_A$ *and* $\mathrm{v}_B)$ *and the exit velocity* (v_C) *in terms of the volume flow rate Q (justify the expression used).*

b) The resultant force and moment at O of the actions exerted on the fluid by the interior of the pipes in the junction.

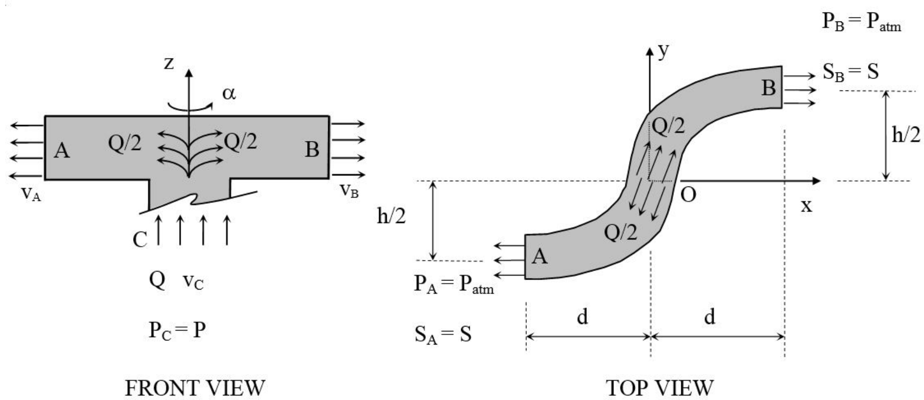
c) The reaction force and moment at D of the rigid element.

d) The power W of the pump needed to provide the volume flow rates indicated in the figure.

Additional hypotheses:

1) *The weights of the fluid and the pipes are negligible.*

5.5 – *The front and top cross-sections of an irrigation sprinkler are shown in the figure below. A volume flow rate Q of water enters through section C at a pressure P and exits through sections A and B at an atmospheric pressure P_{atm}. The flow is assumed to be in steady-state regime.*



Determine:

a) *The entrance and exit velocities (justify the expression used).*
b) *The resultant force and moment at point O of the actions exerted on the fluid by the interior walls of the sprinkler.*
c) *The reaction that must be exerted on point O to avoid the sprinkler from moving in the vertical direction.*
d) *The angular acceleration of the sprinkler's rotation* α. *To this aim, assume that* I_0 *and* I_1 *are, respectively, the central moments of inertia about point O of the empty sprinkler and the sprinkler full of water.*
e) *The power needed to provide a volume flow rate* $2Q$, *considering that* W^* *is the power of the pump needed to provide a volume flow rate Q.*

Additional hypotheses:

1) *Incompressible fluid*
2) *Perfect fluid:* $\sigma_{ij} = -\text{p}\delta_{ij}$
3) *The weights of the sprinkler and the water inside it are negligible.*
4) $S_A = S_B = S$ *and* $S_C = S_*$
5) $m = I\alpha$

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Chapter 6
Linear Elasticity

6.1 Hypothesis of the Linear Theory of Elasticity

The linear theory of elasticity can be considered a simplification of the general theory of elasticity, but a close enough approximation for most engineering applications. The simplifying hypotheses of the linear theory of elasticity are

a) **Infinitesimal strains**. The displacements and its gradients are small, see Chapter 2.

- *Small displacements*. The material configuration (corresponding to the reference time t_0) is indistinguishable from the spatial one (corresponding to the present time t) and, consequently, the spatial and material coordinates cannot be distinguished from each other either, see Figure 6.1.

$$\mathbf{x} = \mathbf{X} + \underbrace{\mathbf{u}}_{\approx \mathbf{0}} \quad \Longrightarrow \quad \mathbf{x} \approx \mathbf{X} \tag{6.1}$$

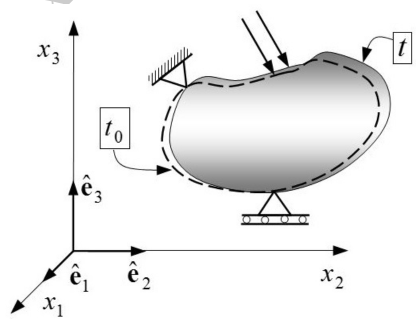

Figure 6.1: Small displacements are considered in the linear theory of elasticity.

From (6.1), one can write

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{1} \quad \Longrightarrow \quad |\mathbf{F}| \approx 1 \ . \tag{6.2}$$

Remark 6.1. As a consequence of (6.1), *there is no difference between the spatial and material descriptions* of a property,

$$\mathbf{x} = \mathbf{X} \quad \Longrightarrow \quad \gamma(\mathbf{x},t) = \gamma(\mathbf{X},t) = \Gamma(\mathbf{X},t) = \Gamma(\mathbf{x},t) \ ,$$

and all references to the spatial and material descriptions (in addition to any associated concepts such as local derivative, material derivative, etc.) no longer make sense in infinitesimal elasticity.
Likewise, the spatial Nabla differential operator (∇) is indistinguishable from the material one $(\bar{\nabla})$,

$$\frac{\partial(\bullet)}{\partial \mathbf{X}} = \frac{\partial(\bullet)}{\partial \mathbf{x}} \quad \Longrightarrow \quad \nabla(\bullet) = \bar{\nabla}(\bullet) \ .$$

Remark 6.2. As a consequence of (6.2) and the principle of conservation of mass, the density in the present configuration $\rho_t \equiv \rho(\mathbf{X},t)$ coincides with the one in the reference configuration $\rho_0 \equiv \rho(\mathbf{X},0)$ (which is assumed to be known),

$$\rho_0 = \rho_t |\mathbf{F}| \approx \rho_t \ ,$$

and, therefore, *the density is not an unknown in linear elasticity problems*.

- *Small displacement gradients*. As a consequence, no distinction is made between the material strain tensor $\mathbf{E}(\mathbf{X},t)$ and the spatial strain tensor $\mathbf{e}(\mathbf{x},t)$, which collapse into the infinitesimal strain tensor $\boldsymbol{\varepsilon}(\mathbf{x},t)$.

$$\mathbf{E}(\mathbf{X},t) \approx \mathbf{e}(\mathbf{x},t) = \boldsymbol{\varepsilon}(\mathbf{x},t)$$

$$\begin{cases} \boldsymbol{\varepsilon} = \nabla^S \mathbf{u} = \dfrac{1}{2}(\mathbf{u} \otimes \nabla + \nabla \otimes \mathbf{u}) \\ \varepsilon_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) \qquad i,j \in \{1,2,3\} \end{cases} \tag{6.3}$$

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b) **Existence of a neutral state**. The existence of a neutral state in which the strains and stresses are null is accepted. Usually, the neutral state is understood to occur in the reference configuration.

$$\begin{cases} \boldsymbol{\varepsilon}(\mathbf{x},t_0) = \mathbf{0} \\ \boldsymbol{\sigma}(\mathbf{x},t_0) = \mathbf{0} \end{cases} \tag{6.4}$$

c) **The deformation process is considered (in principle) to be isothermal**[1] **and adiabatic**.

Definition 6.1. *Isothermal processes* are those that take place at a temperature $\theta(\mathbf{x},t)$ that is constant along time,

$$\theta(\mathbf{x},t) \equiv \theta(\mathbf{x}) \ .$$

Adiabatic processes are those that take place without heat generation at any point and instant of time.

Heat generated inside a domain V per unit of time:

$$Q_e = \int_V \rho r \, dV - \int_{\partial V} \mathbf{q}\cdot\mathbf{n} \, dS = 0 \qquad \forall \Delta V \subset V$$

$$\Longrightarrow \quad \rho r - \nabla\cdot\mathbf{q} = 0 \qquad \forall \mathbf{x} \quad \forall t$$

Slow deformation processes are commonly considered to be adiabatic.

6.2 Linear Elastic Constitutive Equation. Generalized Hooke's Law

Hooke's law for unidimensional problems establishes the *proportionality* between the stress, σ, and the strain, ε, by means of the constant named elastic modulus, E,

$$\sigma = E\varepsilon \ . \tag{6.5}$$

In the theory of elasticity, this proportionality is generalized to the multidimensional case by assuming the *linearity* of the relation between the components of the stress tensor $\boldsymbol{\sigma}$ and those of the strain tensor $\boldsymbol{\varepsilon}$ in the expression known as *generalized Hooke's law*,

[1] The restriction to *isothermal* processes disappears in the linear theory of *thermoelasticity*, which will be addressed in Section 6.6.

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$$\begin{array}{c}\text{Generalized}\\ \text{Hooke's law}\end{array} \begin{cases} \boldsymbol{\sigma}(\mathbf{x},t) = \mathbb{C} : \boldsymbol{\varepsilon}(\mathbf{x},t) \\ \sigma_{ij} = \mathbb{C}_{ijkl}\varepsilon_{kl} \qquad i,j \in \{1,2,3\} \end{cases} \tag{6.6}$$

which constitutes the constitutive equation of a linear elastic material.

The fourth-order tensor $\mathbb{C}$ (denoted as *tensor of elastic constants*) has $3^4 = 81$ components. However, due to the symmetry of the tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, it must exhibit certain symmetries in relation to the exchange of its indexes. These are:

$$\begin{array}{l} \left.\begin{array}{l}\mathbb{C}_{ijkl} = \mathbb{C}_{jikl}\\ \mathbb{C}_{ijkl} = \mathbb{C}_{ijlk}\end{array}\right\} \rightarrow \text{ minor symmetries} \\ \mathbb{C}_{ijkl} = \mathbb{C}_{klij} \qquad \rightarrow \text{ major symmetries} \end{array} \tag{6.7}$$

Consequently, the number of different constants in the tensor of elastic constants $\mathbb{C}$ is reduced to 21.

> *Remark 6.3.* An essential characteristic of the elastic behavior (which is verified in (6.5)) is that the stresses at a certain point and time, $\boldsymbol{\sigma}(\mathbf{x},t)$, depend (only) on the strains at said point and time, $\boldsymbol{\varepsilon}(\mathbf{x},t)$, and not on the history of previous strains.

6.2.1 Elastic Potential

Consider the *specific internal energy* $u(\mathbf{x},t)$ (internal energy per unit of mass) and the *density of internal energy* $\hat{u}(\mathbf{x},t)$ (internal energy per unit of volume), which related through

$$\begin{array}{c} \hat{u}(\mathbf{x},t) = \rho_0\, u(\mathbf{x},t) \ , \\ \rho \dfrac{du}{dt} \approx \rho_0 \dfrac{du}{dt} = \dfrac{d\overbrace{(\rho_0\, u)}^{\hat{u}}}{dt} = \dfrac{d\hat{u}}{dt} \ , \end{array} \tag{6.8}$$

where $\rho_0 \approx \rho$ (see Remark 6.2) has been taken into account. Consider now the energy equation in its local form[2],

$$\rho_0 \frac{du}{dt} = \frac{d\hat{u}}{dt} = \boldsymbol{\sigma} : \mathbf{d} + \rho_0\, r - \nabla \cdot \mathbf{q} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \implies \frac{d\hat{u}}{dt} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \ , \tag{6.9}$$

[2] The identity $\mathbf{d} = \dot{\boldsymbol{\varepsilon}}$, characteristic of the infinitesimal strain case, is considered here.

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where the adiabatic nature of the deformation process $(\rho_0\, r - \nabla \cdot \mathbf{q} = 0)$ has been considered. Then, the global (integral) form of the energy equation in (6.9) is obtained by integrating over the material volume V.

Global form of the energy equation in linear elasticity

$$\begin{aligned}
&\frac{d\mathcal{U}}{dt} = \frac{d}{dt}\int\limits_{V_t \equiv V} \hat{u}\, dV = \int\limits_V \frac{d\hat{u}}{dt}\, dV = \int\limits_V \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}\, dV \\
&\text{with } \mathcal{U}(t) = \int\limits_V \hat{u}(\mathbf{x},t)\, dV
\end{aligned} \qquad (6.10)$$

Here, $\mathcal{U}(t)$ is the internal energy of the material volume considered.

Remark 6.4. The stress power (in the case of linear elasticity) is an exact differential,

$$\text{stress power } = \int\limits_V \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}\, dV = \frac{d\mathcal{U}}{dt}\,.$$

Replacing now (6.6) in (6.9),

$$\begin{aligned}
\frac{d\hat{u}}{dt} \stackrel{not}{=} \dot{\hat{u}} &= \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} = \dot{\varepsilon}_{ij}\, \sigma_{ij} = \dot{\varepsilon}_{ij}\, \mathbb{C}_{ijkl}\, \varepsilon_{kl} = \frac{1}{2}\Big(\dot{\varepsilon}_{ij}\, \mathbb{C}_{ijkl}\, \varepsilon_{kl} + \overbrace{\dot{\varepsilon}_{ij}\, \mathbb{C}_{ijkl}\, \varepsilon_{kl}}^{\substack{i \leftrightarrow k \\ j \leftrightarrow l}}\Big) = \\
&= \frac{1}{2}\Big(\dot{\varepsilon}_{ij}\, \mathbb{C}_{ijkl}\, \varepsilon_{kl} + \dot{\varepsilon}_{kl}\, \mathbb{C}_{klij}\, \varepsilon_{ij}\Big) = \frac{1}{2}\left(\dot{\varepsilon}_{ij}\, \mathbb{C}_{ijkl}\, \varepsilon_{kl} + \varepsilon_{ij}\, \mathbb{C}_{ijkl}\, \dot{\varepsilon}_{kl}\right) = \\
&= \frac{1}{2}\frac{d}{dt}\left(\varepsilon_{ij}\, \mathbb{C}_{ijkl}\, \varepsilon_{kl}\right) = \frac{1}{2}\frac{d}{dt}\left(\boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon}\right)\,,
\end{aligned} \qquad (6.11)$$

where the symmetries in (6.7) have been taken into account. Integrating the expression obtained and imposing the condition that the density of internal energy $\hat{u}(\mathbf{x},t_0)$ in the neutral state be null[3] (for $t = t_0 \Rightarrow \boldsymbol{\varepsilon}(\mathbf{x},t_0) = \mathbf{0}$) produces the density of internal energy.

[3] The condition $\hat{u}(\mathbf{x},t_0) = 0$ can be introduced without loss of generality.

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$$
\left.\begin{array}{l}
\hat{u}(\mathbf{x},t)=\dfrac{1}{2}\left(\boldsymbol{\varepsilon}(\mathbf{x},t):\mathbb{C}:\boldsymbol{\varepsilon}(\mathbf{x},t)\right)+a(\mathbf{x}) \\
\hat{u}(\mathbf{x},t_0)=0 \qquad \forall \mathbf{x}
\end{array}\right\} \Longrightarrow
$$

$$
\Longrightarrow \frac{1}{2}\underbrace{\boldsymbol{\varepsilon}(\mathbf{x},t_0)}_{=\mathbf{0}}:\mathbb{C}:\boldsymbol{\varepsilon}(\mathbf{x},t_0)+a(\mathbf{x})=a(\mathbf{x})=0 \qquad \forall \mathbf{x} \tag{6.12}
$$

$$
\boxed{\left.\begin{array}{c}\text{Density of}\\ \text{internal energy}\end{array}\right\}\hat{u}(\boldsymbol{\varepsilon})=\frac{1}{2}\left(\boldsymbol{\varepsilon}:\mathbb{C}:\boldsymbol{\varepsilon}\right)=\frac{1}{2}\varepsilon_{ij}\,\mathbb{C}_{ijkl}\,\varepsilon_{kl}} \tag{6.13}
$$

Now, (6.13) is differentiated with respect to $\boldsymbol{\varepsilon}$, considering once more the symmetries in (6.7).

$$
\begin{cases}
\dfrac{\partial \hat{u}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}=\dfrac{1}{2}\mathbb{C}:\boldsymbol{\varepsilon}+\dfrac{1}{2}\boldsymbol{\varepsilon}:\mathbb{C}=\dfrac{1}{2}\mathbb{C}:\boldsymbol{\varepsilon}+\dfrac{1}{2}\mathbb{C}:\boldsymbol{\varepsilon}=\mathbb{C}:\boldsymbol{\varepsilon}=\boldsymbol{\sigma} \\
\dfrac{\partial \hat{u}(\boldsymbol{\varepsilon})}{\partial \varepsilon_{ij}}=\dfrac{1}{2}\mathbb{C}_{ijkl}\,\varepsilon_{kl}+\dfrac{1}{2}\varepsilon_{kl}\,\mathbb{C}_{klij}=\dfrac{1}{2}\mathbb{C}_{ijkl}\,\varepsilon_{kl}+\dfrac{1}{2}\mathbb{C}_{ijkl}\,\varepsilon_{kl}=\mathbb{C}_{ijkl}\,\varepsilon_{kl}=\sigma_{ij}
\end{cases} \tag{6.14}
$$

$$
\Longrightarrow \quad \begin{cases}
\dfrac{\partial \hat{u}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}=\boldsymbol{\sigma} \\
\dfrac{\partial \hat{u}(\boldsymbol{\varepsilon})}{\partial \varepsilon_{ij}}=\sigma_{ij} \qquad i,j\in\{1,2,3\}
\end{cases} \tag{6.15}
$$

Equation (6.15) qualifies the density of internal energy $\hat{u}(\boldsymbol{\varepsilon})$ as a potential for the stresses (which are obtained by differentiation of this potential), named elastic potential.

$$
\boxed{\text{Elastic potential}\begin{cases}
\hat{u}(\boldsymbol{\varepsilon})=\dfrac{1}{2}\boldsymbol{\varepsilon}:\underbrace{\mathbb{C}:\boldsymbol{\varepsilon}}_{=\boldsymbol{\sigma}}=\dfrac{1}{2}\boldsymbol{\sigma}:\boldsymbol{\varepsilon} \\
\dfrac{\partial \hat{u}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}=\boldsymbol{\sigma}
\end{cases}} \tag{6.16}
$$

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6.3 Isotropy. Lamé's Constants. Hooke's Law for Isotropic Linear Elasticity

Definition 6.2. An *isotropic material* is that which has the same properties in all directions.

The elastic properties of a linear elastic material are contained in the tensor of elastic constants $\mathbb{C}$ defined in (6.6) and (6.7). Consequently, the components of this tensor must be independent of the orientation of the Cartesian system used[4]. Consider, for example, the systems $\{x_1, x_2, x_3\}$ and $\{x_1{}', x_2{}', x_3{}'\}$ in Figure 6.2, the constitutive equation for these two systems is written as

$$\begin{aligned} \{x_1, x_2, x_3\} &\implies [\boldsymbol{\sigma}] = [\mathbb{C}] : [\boldsymbol{\varepsilon}] \\ \{x_1{}', x_2{}', x_3{}'\} &\implies [\boldsymbol{\sigma}]' = [\mathbb{C}]' : [\boldsymbol{\varepsilon}]' \end{aligned} \tag{6.17}$$

and, for the case of an isotropic material, the components of $\mathbb{C}$ in both systems must be the same ($[\mathbb{C}] = [\mathbb{C}]'$). Therefore, the aforementioned definition of isotropy, which has a physical character, translates into the *isotropic character*, in the mathematical sense, of the tensor of elastic constants $\mathbb{C}$.

$$\text{Tensor of elastic constants} \begin{cases} \mathbb{C} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I} \\ \mathbb{C}_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left[\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right] \\ \qquad\qquad\qquad i, j, k, l \in \{1, 2, 3\} \end{cases} \tag{6.18}$$

Here, λ and μ are known as *Lamé's constants*, which characterize the elastic behavior of the material and must be obtained experimentally.

Remark 6.5. The isotropy condition reduces the number of elastic constants of the material from 21 to 2.

[4] A tensor is isotropic if it maintains its components in any Cartesian coordinate system. The most general expression of a *fourth-order isotropic tensor* is $\mathbb{C} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}$, $\forall \lambda, \mu$. Here, the fourth-order symmetric (isotropic) unit tensor $\mathbf{I}$ is defined by means of its components as $[\mathbf{I}]_{ijkl} = \left[\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right] / 2$.

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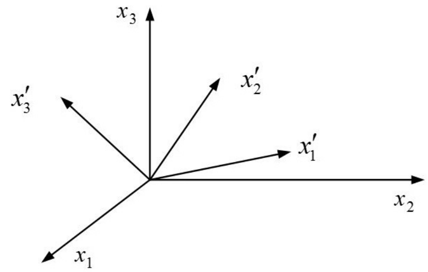


Figure 6.2: Representation of the Cartesian systems $\{x_1,x_2,x_3\}$ and $\{x_1{}',x_2{}',x_3{}'\}$.

Introducing (6.18) in (6.6) results in the *isotropic linear elastic constitutive equation*,

$$\sigma_{ij} = \mathbb{C}_{ijkl}\,\varepsilon_{kl} = \lambda\,\delta_{ij}\underbrace{\delta_{kl}\,\varepsilon_{kl}}_{\varepsilon_{ll}} + 2\mu\underbrace{\left(\frac{1}{2}\underbrace{\delta_{ik}\,\delta_{jl}\,\varepsilon_{kl}}_{\varepsilon_{ij}} + \frac{1}{2}\underbrace{\delta_{il}\,\delta_{jk}\,\varepsilon_{kl}}_{\varepsilon_{ji}=\varepsilon_{ij}}\right)}_{\varepsilon_{ij}} . \tag{6.19}$$

Constitutive eqn. for a linear elastic material. Hooke's law.	$\begin{cases} \boldsymbol{\sigma} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\,\mathbf{1} + 2\mu\boldsymbol{\varepsilon} \\ \sigma_{ij} = \lambda\,\delta_{ij}\varepsilon_{ll} + 2\mu\varepsilon_{ij} \qquad i,j\in\{1,2,3\} \end{cases}$	(6.20)

6.3.1 Inversion of Hooke's Law. Young's Modulus. Poisson's Ratio

The constitutive equation (6.20) provides the stresses in terms of the strains. To obtain its inverse expression, the following procedure is followed.

a) The trace of (6.20) is obtained,

$$\left.\begin{array}{l} \mathrm{Tr}(\boldsymbol{\sigma}) = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\underbrace{\mathrm{Tr}(\mathbf{1})}_{3} + 2\mu\,\mathrm{Tr}(\boldsymbol{\varepsilon}) = (3\lambda+2\mu)\,\mathrm{Tr}(\boldsymbol{\varepsilon}) \\ (i=j) \Longrightarrow \sigma_{ii} = \lambda\,\varepsilon_{ll}\underbrace{\delta_{ii}}_{3} + 2\mu\varepsilon_{ii} = (3\lambda+2\mu)\,\varepsilon_{ll} \end{array}\right\} \Longrightarrow$$

$$\Longrightarrow \mathrm{Tr}(\boldsymbol{\varepsilon}) = \frac{1}{(3\lambda+2\mu)}\,\mathrm{Tr}(\boldsymbol{\sigma}) \ . \tag{6.21}$$

b) $\boldsymbol{\varepsilon}$ is isolated from (6.20) and introduced in (6.21),

$$\boldsymbol{\varepsilon} = -\frac{1}{2\mu}\lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\,\mathbf{1} + \frac{1}{2\mu}\boldsymbol{\sigma} = -\frac{\lambda}{2\mu\,(3\lambda+2\mu)}\,\mathrm{Tr}(\boldsymbol{\sigma})\,\mathbf{1} + \frac{1}{2\mu}\boldsymbol{\sigma} . \tag{6.22}$$

The new elastic properties E (*Young's modulus*) and ν (*Poisson's ratio*) are defined as follows.

$$
\left.\begin{array}{ll}
\text{Young's modulus or tensile (elastic) modulus} & E = \dfrac{\mu\,(3\lambda + 2\mu)}{\lambda + \mu} \\[2ex]
\text{Poisson's ratio} & \nu = \dfrac{\lambda}{2\,(\lambda + \mu)}
\end{array}\right\} \Longrightarrow
$$

$$
\Longrightarrow \begin{cases}
\lambda = \dfrac{\nu E}{(1+\nu)\,(1-2\nu)} \\[2ex]
\mu = \dfrac{E}{2\,(1+\nu)} = G \qquad \text{shear (elastic) modulus}
\end{cases}
\tag{6.23}
$$

Equation (6.22) can be expressed in terms of E and ν, resulting in the *inverse Hooke's law*.

$$
\text{Inverse constitutive equation for an isotropic linear elastic material}
\begin{cases}
\boldsymbol{\varepsilon} = -\dfrac{\nu}{E}\,\mathrm{Tr}\,(\boldsymbol{\sigma})\,\mathbf{1} + \dfrac{1+\nu}{E}\,\boldsymbol{\sigma} \\[2ex]
\varepsilon_{ij} = -\dfrac{\nu}{E}\,\sigma_{ll}\,\delta_{ij} + \dfrac{1+\nu}{E}\,\sigma_{ij} \\
\qquad\qquad\qquad i,j \in \{1,2,3\}
\end{cases}
\tag{6.24}
$$

Finally, (6.24) is rewritten, using engineering notation for the components of the strain and stress tensors.

$$
\begin{array}{ll}
\varepsilon_x = \dfrac{1}{E}\,(\sigma_x - \nu\,(\sigma_y + \sigma_z)) & \gamma_{xy} = \dfrac{1}{G}\,\tau_{xy} \\[2ex]
\varepsilon_y = \dfrac{1}{E}\,(\sigma_y - \nu\,(\sigma_x + \sigma_z)) & \gamma_{xz} = \dfrac{1}{G}\,\tau_{xz} \\[2ex]
\varepsilon_z = \dfrac{1}{E}\,(\sigma_z - \nu\,(\sigma_x + \sigma_y)) & \gamma_{yz} = \dfrac{1}{G}\,\tau_{yz}
\end{array}
\tag{6.25}
$$

Example 6.1 – *Consider an uniaxial tensile test of a rectangular cuboid composed of an isotropic linear elastic material with Young's modulus E and shear modulus G, such that its uniform stress state results in*

$$\sigma_x \neq 0 \quad \text{and} \qquad \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \; .$$

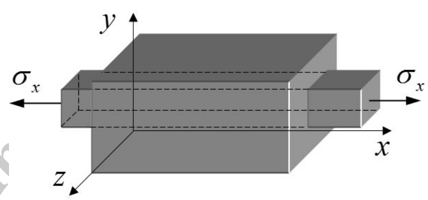
Obtain the strains in engineering notation.

Solution

From (6.25) one obtains

$$\sigma_y = \sigma_z = 0 \Longrightarrow \begin{cases} \varepsilon_x = \dfrac{\sigma_x}{E} \\ \varepsilon_y = -\nu \dfrac{\sigma_x}{E} \\ \varepsilon_z = -\nu \dfrac{\sigma_x}{E} \end{cases} \qquad \tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \Longrightarrow \begin{cases} \gamma_{xy} = \dfrac{\tau_{xy}}{G} = 0 \\ \gamma_{xz} = \dfrac{\tau_{xz}}{G} = 0 \\ \gamma_{yz} = \dfrac{\tau_{yz}}{G} = 0 \end{cases}$$

Therefore, due to these strains, the rectangular cuboid subjected to an uniaxial tensile test, shown in the figure below, stretches in the x-direction and contracts in the y- and z-directions.



6.4 Hooke's Law in Spherical and Deviatoric Components

Consider the decomposition of the stress tensor $\boldsymbol{\sigma}$ and the deformation tensor $\boldsymbol{\varepsilon}$ in their spherical and deviatoric parts,

$$\boldsymbol{\sigma} = \frac{1}{3}\,\mathrm{Tr}\,(\boldsymbol{\sigma})\,\mathbf{1} + \boldsymbol{\sigma}' = \sigma_m \mathbf{1} + \boldsymbol{\sigma}' \; , \tag{6.26}$$

$$\boldsymbol{\varepsilon} = \frac{1}{3}\,\mathrm{Tr}\,(\boldsymbol{\varepsilon})\,\mathbf{1} + \boldsymbol{\varepsilon}' = \frac{1}{3} e \mathbf{1} + \boldsymbol{\varepsilon}' \; . \tag{6.27}$$

The volumetric strain $e = \mathrm{Tr}(\boldsymbol{\varepsilon})$ is obtained by computing the trace of (6.24).

$$
\begin{aligned}
e = \mathrm{Tr}(\boldsymbol{\varepsilon}) &= -\frac{\nu}{E}\,\mathrm{Tr}(\boldsymbol{\sigma})\underbrace{\mathrm{Tr}(\mathbf{1})}_{3} + \frac{1+\nu}{E}\,\mathrm{Tr}(\boldsymbol{\sigma}) = \frac{1-2\nu}{E}\underbrace{\mathrm{Tr}(\boldsymbol{\sigma})}_{3\sigma_m} = \\
&= \frac{3(1-2\nu)}{E}\sigma_m
\end{aligned}
\tag{6.28}
$$

$$
\Longrightarrow
\begin{cases}
\sigma_m = \dfrac{E}{3(1-2\nu)}e = K\,e \\
K \overset{def}{=} \lambda + \dfrac{2}{3}\mu = \dfrac{E}{3(1-2\nu)} = \text{bulk modulus}
\end{cases}
\tag{6.29}
$$

Introducing (6.26), (6.27) and (6.29) in (6.24), results in

$$
\begin{aligned}
\boldsymbol{\varepsilon} &= -\frac{\nu}{E}3\sigma_m\mathbf{1} + \frac{1+\nu}{E}(\sigma_m\mathbf{1} + \boldsymbol{\sigma}') = \frac{1-2\nu}{E}\underbrace{\sigma_m}_{\frac{E}{3(1-2\nu)}e}\mathbf{1} + \frac{1+\nu}{E}\boldsymbol{\sigma}' = \\
&= \frac{1}{3}e\mathbf{1} + \frac{1+\nu}{E}\boldsymbol{\sigma}' \quad \Longrightarrow \boldsymbol{\varepsilon} = \frac{1}{3}e\mathbf{1} + \boldsymbol{\varepsilon}' = \frac{1}{3}e\mathbf{1} + \frac{1+\nu}{E}\boldsymbol{\sigma}' \\
&\Longrightarrow \boldsymbol{\varepsilon}' = \frac{1+\nu}{E}\boldsymbol{\sigma}' = \frac{1}{2\mu}\boldsymbol{\sigma}' = \frac{1}{2G}\boldsymbol{\sigma}'\,.
\end{aligned}
\tag{6.30}
$$

Equations (6.29) and (6.30) relate the spherical part (characterized by the mean stress σ_m and the volumetric strain e) and the deviatoric part ($\boldsymbol{\sigma}'$ and $\boldsymbol{\varepsilon}'$) of the stress and strain tensors as follows.

$$
\boxed{
\begin{array}{lll}
\sigma_m = Ke & & \text{Spherical part} \\
\left.\begin{array}{ll} \boldsymbol{\sigma}' = 2G\boldsymbol{\varepsilon}' & \\ \sigma'_{ij} = 2G\varepsilon'_{ij} & i,j \in \{1,2,3\} \end{array}\right\} & & \text{Deviatoric part}
\end{array}
}
\tag{6.31}
$$

> *Remark 6.6.* Note the proportionality between σ_m and e as well as between σ'_{ij} and ε'_{ij} (component to component), see Figure 6.3.

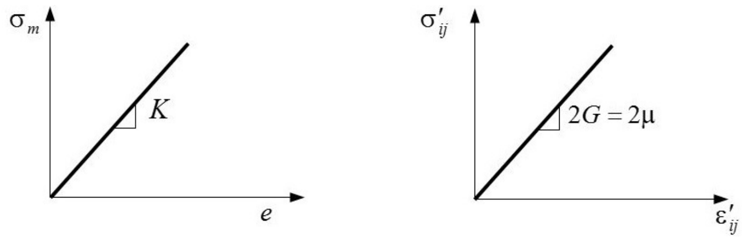


Figure 6.3: Hooke's law in spherical and deviatoric components.

6.5 Limits in the Values of the Elastic Properties

Thermodynamic considerations allow proving that the tensor of elastic constants $\mathbb{C}$ is positive-definite[5], and, thus,

$$\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} > 0 \qquad \forall \boldsymbol{\varepsilon} \neq \mathbf{0} \, . \tag{6.32}$$

Remark 6.7. As a consequence of (6.32), the elastic potential is always null or positive,

$$\hat{u}(\boldsymbol{\varepsilon}) = \frac{1}{2} \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} \geq 0 \, .$$

Remark 6.8. The elastic potential has *a minimum at the neutral state*, that is, for $\boldsymbol{\varepsilon} = \mathbf{0}$ (see Figure 6.4). In effect, from (6.15),

$$\hat{u}(\boldsymbol{\varepsilon}) = \frac{1}{2} \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} \, , \quad \boldsymbol{\sigma} = \frac{\partial \hat{u}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} = \mathbb{C} : \boldsymbol{\varepsilon} \quad \text{and} \quad \frac{\partial^2 \hat{u}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} = \mathbb{C} \, .$$

Then, for $\boldsymbol{\varepsilon} = \mathbf{0}$,

$$\left. \frac{\partial \hat{u}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \right|_{\boldsymbol{\varepsilon}=\mathbf{0}} = \mathbf{0} \qquad \Longrightarrow \quad \hat{u}(\boldsymbol{\varepsilon}) \text{ has an extreme (maximum-minimum) at } \boldsymbol{\varepsilon} = \mathbf{0}.$$

$$\left. \frac{\partial^2 \hat{u}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}} \right|_{\boldsymbol{\varepsilon}=\mathbf{0}} = \underbrace{\mathbb{C}}_{\substack{\text{positive-} \\ \text{definite}}} \qquad \Longrightarrow \quad \text{The extreme is a minimum.}$$

[5] A fourth-order symmetric tensor $\mathbf{A}$ is defined positive-definite if for all second-order tensor $\mathbf{x} \neq \mathbf{0}$ the expression $\mathbf{x} : \mathbf{A} : \mathbf{x} = x_{ij} \, A_{ijkl} \, x_{kl} > 0$ is satisfied and, in addition, $\mathbf{x} : \mathbf{A} : \mathbf{x} = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$.

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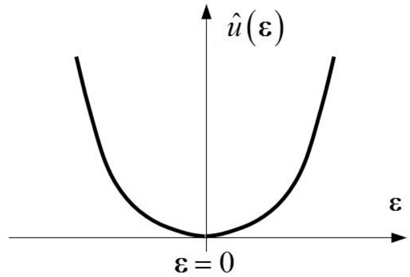

Figure 6.4: Elastic potential.

Consider the expression of the elastic potential (6.16) and the constitutive equation (6.20), then,

$$\begin{aligned} \hat{u}(\boldsymbol{\varepsilon}) &= \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \frac{1}{2}\big(\lambda \operatorname{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}\big) : \boldsymbol{\varepsilon} = \\ &= \frac{1}{2}\lambda \operatorname{Tr}(\boldsymbol{\varepsilon}) \underbrace{\mathbf{1} : \boldsymbol{\varepsilon}}_{\operatorname{Tr}(\boldsymbol{\varepsilon})} + \mu\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} = \frac{1}{2}\lambda \operatorname{Tr}^2(\boldsymbol{\varepsilon}) + \mu\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} . \end{aligned} \qquad (6.33)$$

Expression (6.33) can also be written in terms of the spherical and deviatoric components of strain[6],

$$\boxed{\hat{u}(\boldsymbol{\varepsilon}) = \frac{1}{2}\lambda\big(\underbrace{\operatorname{Tr}(\boldsymbol{\varepsilon})}_{e}\big)^2 + \mu\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} = \frac{1}{2}\lambda e^2 + \mu\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} .} \qquad (6.34)$$

Here, the double contraction of the infinitesimal strain tensor is

$$\begin{aligned} \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} &= \left(\frac{1}{3}e\mathbf{1} + \boldsymbol{\varepsilon}'\right) : \left(\frac{1}{3}e\mathbf{1} + \boldsymbol{\varepsilon}'\right) = \frac{1}{9}e^2 \underbrace{\mathbf{1} : \mathbf{1}}_{3} + \frac{2}{3}e \underbrace{\mathbf{1} : \boldsymbol{\varepsilon}'}_{\operatorname{Tr}(\boldsymbol{\varepsilon}') = 0} + \boldsymbol{\varepsilon}' : \boldsymbol{\varepsilon}' = \\ &= \frac{1}{3}e^2 + \boldsymbol{\varepsilon}' : \boldsymbol{\varepsilon}' . \end{aligned} \qquad (6.35)$$

Replacing (6.35) in (6.34),

$$\hat{u}(\boldsymbol{\varepsilon}) = \frac{1}{2}\lambda e^2 + \frac{1}{3}\mu e^2 + \mu\boldsymbol{\varepsilon}' : \boldsymbol{\varepsilon}' = \frac{1}{2}\underbrace{\left(\lambda + \frac{2}{3}\mu\right)}_{K} e^2 + \mu\boldsymbol{\varepsilon}' : \boldsymbol{\varepsilon}' . \qquad (6.36)$$

[6] The trace of a deviatoric tensor is always null, $\operatorname{Tr}(\boldsymbol{\varepsilon}') = 0$.

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$$\boxed{\hat{u}(\boldsymbol{\varepsilon}) = \frac{1}{2}Ke^2 + \mu\boldsymbol{\varepsilon}' : \boldsymbol{\varepsilon}' \geq 0} \tag{6.37}$$

Consider now an isotropic linear elastic material, characterized by a certain value of its elastic properties. Equation (6.37) must be satisfied for any deformation process. Consider two particular types:

a) A pure spherical deformation process

$$\left.\begin{array}{l} \boldsymbol{\varepsilon}^{(1)} = \dfrac{1}{3}e\,\mathbf{1} \\ \boldsymbol{\varepsilon}'^{(1)} = \mathbf{0} \end{array}\right\} \implies \hat{u}^{(1)} = \frac{1}{2}Ke^2 \geq 0 \implies K > 0 \tag{6.38}$$

b) A pure deviatoric deformation process[7]

$$\left.\begin{array}{l} \boldsymbol{\varepsilon}^{(2)} = \boldsymbol{\varepsilon}' \\ e^{(2)} = \mathbf{0} \end{array}\right\} \implies \hat{u}^{(2)} = \mu\boldsymbol{\varepsilon}' : \boldsymbol{\varepsilon}' \geq 0 \implies \mu > 0 \tag{6.39}$$

Equations (6.38) and (6.39) lead to

$$K = \frac{E}{3(1-2\nu)} > 0 \qquad \text{and} \qquad \mu = G = \frac{E}{2(1+\nu)} > 0 \tag{6.40}$$

which are the limits in the values of the elastic constants K and G. Experience proves that the Poisson's ratio ν is always non-negative[8] and, therefore

$$\boxed{\begin{array}{l} \left.\begin{array}{r} \dfrac{E}{2(1+\nu)} > 0 \\ \nu \geq 0 \end{array}\right\} \implies E > 0\,, \\ \left.\begin{array}{r} \dfrac{E}{3(1-2\nu)} > 0 \\ E \geq 0 \end{array}\right\} \implies 0 \leq \nu \leq \dfrac{1}{2}\,. \end{array}} \tag{6.41}$$

---

[7] The double contraction or double dot product of a tensor by itself is always equal or greater than zero: $\boldsymbol{\varepsilon}' : \boldsymbol{\varepsilon}' = \varepsilon'_{ij}\varepsilon'_{ij} \geq 0$.

[8] In rare cases, a material can have a negative Poisson's ratio. Such materials are named auxetic materials.

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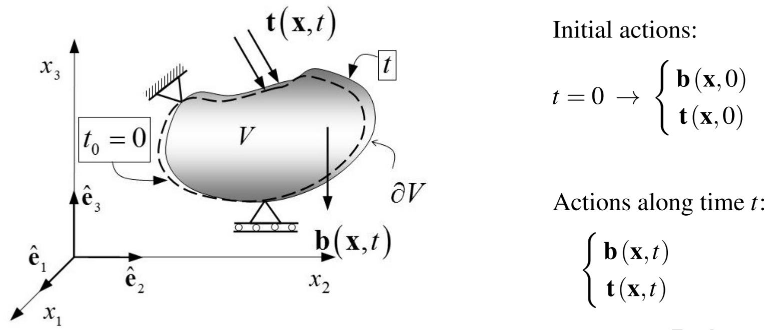

Figure 6.5: Linear elastic problem.

6.6 The Linear Elastic Problem

Consider the linear elastic solid[9] in Figure 6.5, which is subjected to certain actions characterized by the vector of body forces $\mathbf{b}(\mathbf{x},t)$ in the interior of the volume V and the traction vector $\mathbf{t}(\mathbf{x},t)$ on the boundary ∂V. The set of equations that allow determining the evolution along time of the displacements $\mathbf{u}(\mathbf{x},t)$, strains $\boldsymbol{\varepsilon}(\mathbf{x},t)$ and stresses $\boldsymbol{\sigma}(\mathbf{x},t)$ is named *linear elastic problem*.

6.6.1 Governing Equations

The linear elastic problem is governed by the following equations:

a) Cauchy's equation (balance of linear momentum)

$$\boxed{\begin{array}{ll} \nabla \cdot \boldsymbol{\sigma}(\mathbf{x},t) + \rho_0 \mathbf{b}(\mathbf{x},t) = \rho_0 \dfrac{\partial^2 \mathbf{u}(\mathbf{x},t)}{\partial t^2} & \\ & \text{(3 equations)} \\ \dfrac{\partial \sigma_{ij}}{\partial x_i} + \rho_0 b_j = \rho_0 \dfrac{\partial^2 u_j}{\partial t^2} \quad j \in \{1,2,3\} & \end{array}} \tag{6.42}$$

b) Constitutive equation (isotropic linear elastic)[10]

$$\boxed{\begin{array}{ll} \boldsymbol{\sigma}(\mathbf{x},t) = \lambda \, \mathrm{Tr}(\boldsymbol{\varepsilon}(\mathbf{x},t)) \mathbf{1} + 2\mu \boldsymbol{\varepsilon}(\mathbf{x},t) & \\ & \text{(6 equations)} \\ \sigma_{ij} = \lambda \delta_{ij} \varepsilon_{ll} + 2\mu \varepsilon_{ij} \quad i,j \in \{1,2,3\} & \end{array}} \tag{6.43}$$

[9] Here, *linear elastic solid* refers to a continuous medium constituted by a material that obeys the linear elastic constitutive equation.
[10] The symmetry of the stress and strain tensors entails that only six of the nine equations are different from one another. In addition, when listing the unknowns, only the *different* components of these tensors will be considered.

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c) Geometric equation (compatibility relation between infinitesimal strains and displacements)

$$\boxed{\begin{array}{ll}\boldsymbol{\varepsilon}(\mathbf{x},t)=\nabla^S\mathbf{u}(\mathbf{x},t)=\dfrac{1}{2}(\mathbf{u}\otimes\nabla+\nabla\otimes\mathbf{u}) & \\ \varepsilon_{ij}=\dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j}+\dfrac{\partial u_j}{\partial x_i}\right)\quad i,j\in\{1,2,3\} & \end{array}\quad \text{(6 equations)}} \tag{6.44}$$

These equations involve the following unknowns:

$$\begin{array}{ll}\bullet\ \mathbf{u}(\mathbf{x},t) & \text{(3 unknowns)}\\ \bullet\ \boldsymbol{\varepsilon}(\mathbf{x},t) & \text{(6 unknowns)}\\ \bullet\ \boldsymbol{\sigma}(\mathbf{x},t) & \text{(6 unknowns)}\end{array} \tag{6.45}$$

and constitute a system of partial differential equations (PDEs). The system is composed of 15 differential equations with the 15 unknowns listed in (6.45). These are of the type $(\bullet)(x,y,z,t)$, and, thus, must be solved in the $\mathbb{R}^3\times\mathbb{R}^+$ space. The problem is well defined when the adequate boundary conditions are provided.

6.6.2 Boundary Conditions

6.6.2.1 Boundary Conditions in Space

Consider the boundary $\Gamma\equiv\partial V$ of the solid is divided into three parts, Γ_u, Γ_σ and $\Gamma_{u\sigma}$, characterized by (see Figure 6.6)

$$\begin{aligned}&\Gamma_u\bigcup\Gamma_\sigma\bigcup\Gamma_{u\sigma}=\Gamma\equiv\partial V\ ,\\ &\Gamma_u\bigcap\Gamma_\sigma=\Gamma_u\bigcap\Gamma_{u\sigma}=\Gamma_{u\sigma}\bigcap\Gamma_\sigma=\{\emptyset\}\ .\end{aligned} \tag{6.46}$$

These allow defining the *boundary conditions in space*, that is, those conditions that affect the spatial arguments (x,y,z) of the unknowns (6.45) of the problem.

- Boundary Γ_u: *prescribed displacements*

$$\boxed{\left.\begin{array}{l}\mathbf{u}(\mathbf{x},t)=\mathbf{u}^*(\mathbf{x},t)\\ u_i(\mathbf{x},t)=u_i^*(\mathbf{x},t)\quad i\in\{1,2,3\}\end{array}\right\}\quad\forall\mathbf{x}\in\Gamma_u\quad\forall t} \tag{6.47}$$

- Boundary Γ_σ: *prescribed tractions*

$$\boxed{\left.\begin{array}{l}\boldsymbol{\sigma}(\mathbf{x},t)\cdot\mathbf{n}=\mathbf{t}^*(\mathbf{x},t)\\ \sigma_{ij}(\mathbf{x},t)\cdot n_j=t_i^*(\mathbf{x},t)\quad i,j\in\{1,2,3\}\end{array}\right\}\quad\forall\mathbf{x}\in\Gamma_\sigma\quad\forall t} \tag{6.48}$$

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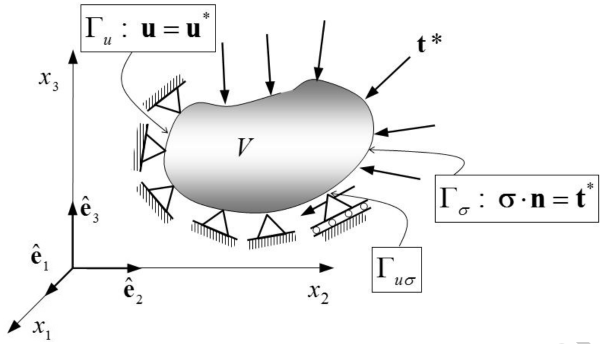


Figure 6.6: Boundary conditions in space.

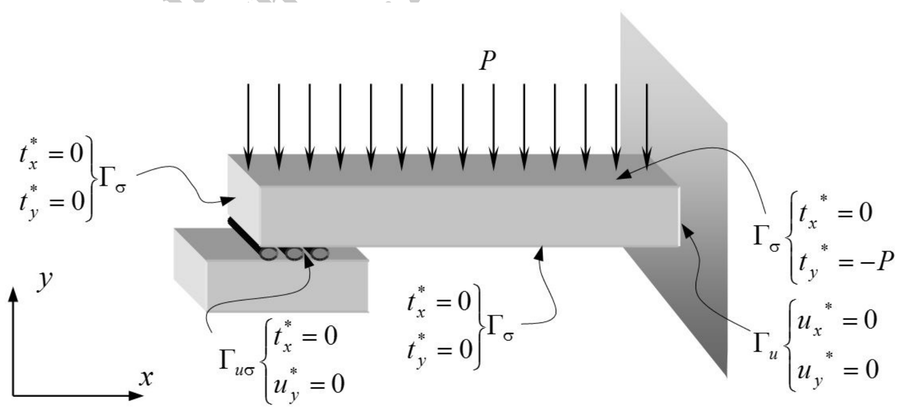
- Boundary $\Gamma_{u\sigma}$: *prescribed displacements and tractions*[11]

$$\boxed{\begin{array}{l|l} u_i(\mathbf{x},t) = u_i^*(\mathbf{x},t) & \\ \sigma_{jk}(\mathbf{x},t) \cdot n_k = t_j^*(\mathbf{x},t) & \big(i,j,k \in \{1,2,3\},\ i \neq j\big) \quad \forall \mathbf{x} \in \Gamma_{u\sigma} \quad \forall t \end{array}} \tag{6.49}$$

Example 6.2 – *Exemplification of the boundary conditions in space.*

Solution

The different types of boundary conditions in space are illustrated in the following figure of a beam.



[11] In $\Gamma_{u\sigma}$ certain components (components i) have prescribed displacements while the others (components j) have the traction vector prescribed.

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6.6.2.2 Boundary Conditions in Time: Initial Conditions

In general, at the initial or reference time $t = 0$ the displacements and velocities are known.

$$\boxed{\left.\begin{array}{l}\mathbf{u}(\mathbf{x},0) = \mathbf{0} \\ \left.\dfrac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t}\right|_{t=0} \stackrel{not}{=} \dot{u}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x})\end{array}\right\} \quad \forall \mathbf{x} \in V} \tag{6.50}$$

6.6.3 *Quasi-Static Problem*

The system of equations (6.42) to (6.50) can be visualized, from a mechanical point of view, as a system of *actions* or data (the body forces $\mathbf{b}(\mathbf{x},t)$, the traction vector $\mathbf{t}^*(\mathbf{x},t)$, the prescribed displacements $\mathbf{u}^*(\mathbf{x},t)$ and the initial velocities $\mathbf{v}_0(\mathbf{x})$) that, introduced into a mathematical *model* composed of the differential equations given in Section 6.6.1 and the boundary conditions described in Section 6.6.2, provides the *response* or solution in the form of the displacement field $\mathbf{u}(\mathbf{x},t)$, the deformation field $\boldsymbol{\varepsilon}(\mathbf{x},t)$ and the stress field $\boldsymbol{\sigma}(\mathbf{x},t)$.

$$\boxed{\underbrace{\left.\begin{array}{c}\mathbf{b}(\mathbf{x},t) \\ \mathbf{u}^*(\mathbf{x},t) \\ \mathbf{t}^*(\mathbf{x},t) \\ \mathbf{v}_0(\mathbf{x})\end{array}\right\}}_{\text{Actions} \stackrel{not}{=} \mathbb{A}(\mathbf{x},t)} \Rightarrow \left\langle \begin{array}{c} MATHEMATICAL \\ MODEL: \\ PDEs + BCs \end{array}\right\rangle \Rightarrow \underbrace{\left\{\begin{array}{c}\mathbf{u}(\mathbf{x},t) \\ \boldsymbol{\varepsilon}(\mathbf{x},t) \\ \boldsymbol{\sigma}(\mathbf{x},t)\end{array}\right.}_{\text{Responses} \stackrel{not}{=} \mathbb{R}(\mathbf{x},t)}} \tag{6.51}$$

In the most general case[12], both the actions and the responses depend on time (see Figure 6.7) and the system of PDEs must be integrated over both the space

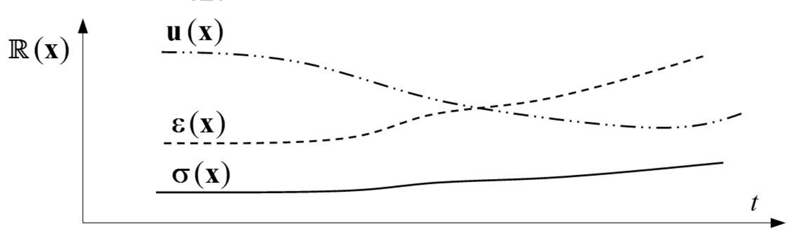

Figure 6.7: Evolution of the response along time.

[12] In this case (general problem), the problem is named *dynamic problem*.

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and the time variables $\left(\mathbb{R}^3 \times \mathbb{R}^+\right)$. However, in certain cases, the integration space can be reduced in one dimension, the one corresponding to time. This is the case for the so-called quasi-static problems.

Definition 6.3. A *quasi-static linear elastic problem* is a linear elastic problem in which the acceleration is considered to be negligible,

$$\mathbf{a} = \frac{\partial^2 \mathbf{u}(\mathbf{x},t)}{\partial t^2} \approx \mathbf{0} \, .$$

This hypothesis is acceptable when the *actions are applied slowly*. In such case, the variation of the actions $\mathbb{A}$ along time is slow $\left(\partial^2 \mathbb{A}/\partial t^2 \approx \mathbf{0}\right)$ and, due to the continuous dependency of the results on the data, the variation of the response $\mathbb{R}$ along time is also small $\left(\partial^2 \mathbb{R}/\partial t^2 \approx \mathbf{0}\right)$. Consequently, the second derivative of the response is considered negligible and, in particular,

$$\frac{\partial^2 \mathbf{u}(\mathbf{x},t)}{\partial t^2} \approx \mathbf{0} \, .$$

The governing differential equations are reduced to the following in the case of a quasi-static problem:

a) *Cauchy's equation*, also known as *equilibrium equation*.

$$\boxed{\nabla \cdot \boldsymbol{\sigma}(\mathbf{x},t) + \rho_0 \mathbf{b}(\mathbf{x},t) = \rho_0 \frac{\partial^2 \mathbf{u}(\mathbf{x},t)}{\partial t^2} = \mathbf{0}} \qquad (6.52)$$

b) *Constitutive equation*

$$\boxed{\boldsymbol{\sigma}(\mathbf{x},t) = \lambda \operatorname{Tr}(\boldsymbol{\varepsilon}(\mathbf{x},t))\,\mathbf{1} + 2\mu \boldsymbol{\varepsilon}(\mathbf{x},t)} \qquad (6.53)$$

c) *Geometric equation*, which *no longer involves any time derivative*.

$$\boxed{\boldsymbol{\varepsilon}(\mathbf{x},t) = \nabla^S \mathbf{u}(\mathbf{x},t) = \frac{1}{2}(\mathbf{u} \otimes \nabla + \nabla \otimes \mathbf{u})} \qquad (6.54)$$

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The system of differential equations only needs to be integrated in space (solved in $\mathbb{R}^3$) with the boundary conditions in space of Section 6.6.2.1. Moreover, *time merely serves as a parameter describing the evolution of the actions*, which are usually described in terms of the *load factor* or *pseudo-time* $\lambda(t)$.

$$\boxed{\underbrace{\left.\begin{matrix}\mathbf{b}(\mathbf{x},\lambda)\\ \mathbf{u}^*(\mathbf{x},\lambda)\\ \mathbf{t}^*(\mathbf{x},\lambda)\end{matrix}\right\}}_{\text{Actions}\overset{not}{=}\mathbb{A}(\mathbf{x},\lambda)} \Rightarrow \left\langle\begin{matrix} MATHEMATICAL \\ MODEL: \\ PDEs+BCs\end{matrix}\right\rangle \Rightarrow \underbrace{\left\{\begin{matrix}\mathbf{u}(\mathbf{x},\lambda)\\ \boldsymbol{\varepsilon}(\mathbf{x},\lambda)\\ \boldsymbol{\sigma}(\mathbf{x},\lambda)\end{matrix}\right.}_{\text{Responses}\overset{not}{=}\mathbb{R}(\mathbf{x},\lambda)}} \tag{6.55}$$

In other words, for each value of the actions (characterized by a fixed value of λ^*), $\mathbb{A}(\mathbf{x}\lambda^*)$, a response $\mathbb{R}(\mathbf{x},\lambda^*)$ is obtained. Varying the value of λ^* produces a family of actions and its corresponding family of responses.

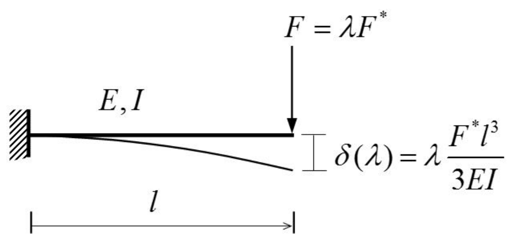
Example 6.3 – *Application to a typical problem of strength of materials.*

Solution

Consider a cantilever beam subjected to a force $F(t)$ at its free end. Under the quasi-static problem hypothesis, and considering a parametrized action of the type λF^*, the response (deflection at its free end) can be computed as

$$\delta(\lambda) = \lambda \frac{F^* l^3}{3EI} .$$

This is the classical solution obtained in strength of materials.
Now, if the evolution along time of $\lambda(t)$ can take any form, *the value of* $\delta(t) = \delta(\lambda(t))$ *corresponding to each instant of time only depends on the corresponding value of* λ.



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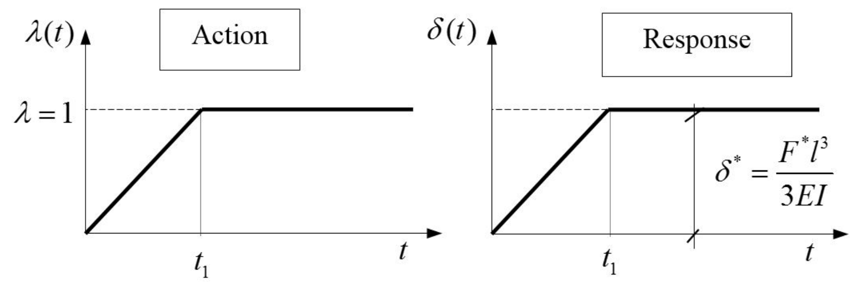

6.7 Solution to the Linear Elastic Problem

The linear elastic problem can be typically solved following two different approaches:

a) Displacement formulation
b) Stress formulation

Their names are directly related to which is the main unknown being considered in each formulation (displacements or stresses, respectively).

> *Remark 6.9.* At present, the displacement formulation has greater application because most *numerical* methods used to solve the linear elastic problem are based on this approach.

6.7.1 Displacement Formulation: Navier's Equation

Consider the equations that constitute the linear elastic problem:

$$\begin{array}{ll} \nabla\cdot\boldsymbol{\sigma}+\rho_0\mathbf{b}=\rho_0\dfrac{\partial^2\mathbf{u}}{\partial t^2} & \text{Cauchy's equation} \\ \boldsymbol{\sigma}=\lambda\,\mathrm{Tr}\,(\boldsymbol{\varepsilon})\,\mathbf{1}+2\mu\boldsymbol{\varepsilon} & \text{Constitutive equation} \\ \boldsymbol{\varepsilon}=\nabla^S\mathbf{u}=\dfrac{1}{2}\left(\mathbf{u}\otimes\nabla+\nabla\otimes\mathbf{u}\right) & \text{Geometric equation} \end{array} \tag{6.56}$$

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$$\boxed{\left.\begin{aligned} &\Gamma_u : \mathbf{u} = \mathbf{u}^* \\ &\Gamma_\sigma : \mathbf{t}^* = \boldsymbol{\sigma} \cdot \mathbf{n} \end{aligned}\right\} \quad \text{Boundary conditions in space}} \tag{6.57}$$

$$\boxed{\left.\begin{aligned} &\mathbf{u}(\mathbf{x},0) = \mathbf{0} \\ &\dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{v}_0 \end{aligned}\right\} \quad \text{Initial conditions}} \tag{6.58}$$

The aim is to pose a reduced system in which only the displacement field $\mathbf{u}(\mathbf{x},t)$ intervenes as an unknown. The first step consists in replacing the constitutive equation in the Cauchy's equation, both given in (6.56).

$$\begin{aligned} &\nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{b} = \nabla \cdot \Big(\lambda \operatorname{Tr}(\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}\Big) + \rho_0 \mathbf{b} = \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} \\ &\Longrightarrow \quad \lambda \nabla \cdot \big(\operatorname{Tr}(\boldsymbol{\varepsilon})\mathbf{1}\big) + 2\mu \nabla \cdot \boldsymbol{\varepsilon} + \rho_0 \mathbf{b} = \rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} \end{aligned} \tag{6.59}$$

Consider the following identities[13].

$$\begin{aligned} \left[\nabla \cdot \boldsymbol{\varepsilon}\right]_i &= \frac{\partial \varepsilon_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right) = \frac{1}{2}\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{2}\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) = \\ &= \frac{1}{2}\left[\nabla^2 \mathbf{u}\right]_i + \frac{1}{2}\frac{\partial}{\partial x_i}(\nabla \cdot \mathbf{u}) = \left[\frac{1}{2}\nabla^2 \mathbf{u} + \frac{1}{2}\nabla(\nabla \cdot \mathbf{u})\right]_i \qquad i \in \{1,2,3\} \\ &\boxed{\nabla \cdot \boldsymbol{\varepsilon} = \frac{1}{2}\nabla(\nabla \cdot \mathbf{u}) + \frac{1}{2}\nabla^2 \mathbf{u}} \end{aligned} \tag{6.60}$$

$$\begin{aligned} \left[\nabla \cdot \big(\operatorname{Tr}(\boldsymbol{\varepsilon})\mathbf{1}\big)\right]_i &= \frac{\partial}{\partial x_j}(\varepsilon_{ll}\,\delta_{ij}) = \frac{\partial}{\partial x_j}\left(\frac{\partial u_l}{\partial x_l}\delta_{ij}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial u_l}{\partial x_l}\right) = \\ &= \frac{\partial}{\partial x_i}(\nabla \cdot \mathbf{u}) = \left[\nabla(\nabla \cdot \mathbf{u})\right]_i \qquad i \in \{1,2,3\} \\ &\boxed{\nabla \cdot \big(\operatorname{Tr}(\boldsymbol{\varepsilon})\mathbf{1}\big) = \nabla(\nabla \cdot \mathbf{u})} \end{aligned} \tag{6.61}$$

[13] The Laplace operator of a vector $\mathbf{v}$ is defined as $\left[\nabla^2 \mathbf{v}\right]_i \stackrel{def}{=} \partial^2 v_i / (\partial x_j \partial x_j)$.

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Equation (6.59) can be rewritten by replacing the expressions in the identities (6.60) and (6.61), resulting in

$$\boxed{\begin{array}{ll}\text{Navier's} \\ \text{equation}\end{array} \begin{cases} (\lambda+\mu)\,\nabla(\nabla\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}+\rho_0\mathbf{b}=\rho_0\dfrac{\partial^2\mathbf{u}}{\partial t^2} \\ (\lambda+\mu)\,u_{j,ji}+\mu\,u_{i,jj}+\rho_0\,b_i=\rho_0\,\ddot{u}_i \quad i\in\{1,2,3\}\end{cases}} \tag{6.62}$$

which constitutes a system of second-order PDEs in displacements $\mathbf{u}(\mathbf{x},t)$ (that must be, thus, integrated in $\mathbb{R}^3\times\mathbb{R}^+$), and receives the name of *Navier's equation*.

The boundary conditions can also be written in terms of the displacements as follows. Replacing the constitutive equation of (6.56) in the boundary conditions in Γ_σ of (6.57) results in

$$\begin{aligned} \mathbf{t}^* &= \boldsymbol{\sigma}\cdot\mathbf{n} = \big(\lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\,\mathbf{1}+2\mu\boldsymbol{\varepsilon}\big)\cdot\mathbf{n} = \lambda\,(\mathrm{Tr}(\boldsymbol{\varepsilon}))\,\mathbf{n}+2\mu\boldsymbol{\varepsilon}\cdot\mathbf{n} = \\ &= \lambda\,(\nabla\cdot\mathbf{u})\,\mathbf{n}+2\mu\,\big(\nabla^S\cdot\mathbf{u}\big)\cdot\mathbf{n} = \lambda\,(\nabla\cdot\mathbf{u})\,\mathbf{n}+\mu\,(\mathbf{u}\otimes\nabla+\nabla\otimes\mathbf{u})\cdot\mathbf{n} \end{aligned} \tag{6.63}$$

and the boundary conditions in space (6.57) expressed in terms of the displacements are obtained.

$$\boxed{\begin{array}{l} \left.\begin{array}{r} \mathbf{u}=\mathbf{u}^* \\ u_i=u_i^* \quad i\in\{1,2,3\}\end{array}\right\} \text{ in } \Gamma_u \\ \left.\begin{array}{l} \lambda\,(\nabla\cdot\mathbf{u})\,\mathbf{n}+\mu\,(\mathbf{u}\otimes\nabla+\nabla\otimes\mathbf{u})\cdot\mathbf{n}=\mathbf{t}^* \\ \lambda\,u_{l,l}\,n_i+\mu\,(u_{i,j}\,n_j+u_{j,i}\,n_j)=t_i^* \quad i,j\in\{1,2,3\}\end{array}\right\} \text{ in } \Gamma_\sigma \end{array}} \tag{6.64}$$

The initial conditions (6.58) remain unchanged. Integrating the system (6.62) yields the displacement field $\mathbf{u}(\mathbf{x},t)$. Differentiation of this field and substitution in the geometric equation of (6.56) produces the strain field $\boldsymbol{\varepsilon}(\mathbf{x},t)$, and, finally, replacing the strain in the constitutive equation results in the stress field $\boldsymbol{\sigma}(\mathbf{x},t)$.

6.7.1.1 Navier's Equation in Cylindrical and Spherical Coordinates

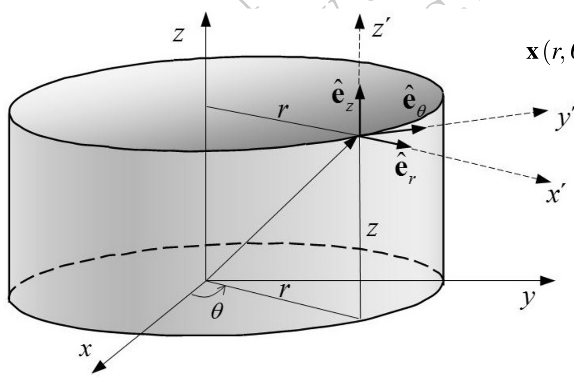
Navier's equation (6.62) is expressed in compact or index notation and is independent of the coordinate system considered. The components of this equation are expressed as follows in the cylindrical and spherical coordinate systems (see section 2.15).

Cylindrical coordinates

$$
\begin{aligned}
&(\lambda+2\mu)\,\frac{\partial e}{\partial r}-\frac{2\mu}{r}\frac{\partial \omega_z}{\partial \theta}+2\mu\frac{\partial \omega_\theta}{\partial z}+\rho b_r=\rho\,\frac{\partial^2 u_r}{\partial t^2}\\
&(\lambda+2\mu)\,\frac{1}{r}\frac{\partial e}{\partial \theta}-2\mu\frac{\partial \omega_r}{\partial z}+2\mu\frac{\partial \omega_z}{\partial r}+\rho b_\theta=\rho\,\frac{\partial^2 u_\theta}{\partial t^2}\\
&(\lambda+2\mu)\,\frac{\partial e}{\partial z}-\frac{2\mu}{r}\frac{\partial\,(r\omega_\theta)}{\partial r}+\frac{2\mu}{r}\frac{\partial \omega_r}{\partial \theta}+\rho b_z=\rho\,\frac{\partial^2 u_z}{\partial t^2}
\end{aligned}
\tag{6.65}
$$

where

$$
\begin{aligned}
&\omega_r=-\Omega_{\theta z}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}-\frac{\partial u_\theta}{\partial z}\right)\\
&\omega_\theta=-\Omega_{zr}=\frac{1}{2}\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)\\
&\omega_z=-\Omega_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial\,(r u_\theta)}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\\
&e=\frac{1}{r}\frac{\partial\,(r u_r)}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}
\end{aligned}
$$



$$
\mathbf{x}\,(r,\theta,z)\stackrel{not}{\equiv}\begin{bmatrix} x=r\,\cos\theta\\ y=r\,\sin\theta\\ z=z\end{bmatrix}
$$

Figure 6.8: Cylindrical coordinates.

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Spherical coordinates

$$
\begin{aligned}
&(\lambda+2\mu)\,\frac{\partial e}{\partial r}-\frac{2\mu}{r\sin\theta}\frac{\partial\left(\omega_\phi\sin\theta\right)}{\partial\theta}+\frac{2\mu}{r\sin\theta}\frac{\partial\omega_\theta}{\partial\phi}+\rho b_r=\rho\frac{\partial^2u_r}{\partial t^2}\\
&(\lambda+2\mu)\,\frac{1}{r}\frac{\partial e}{\partial\theta}-\frac{2\mu}{r\sin\theta}\frac{\partial\omega_r}{\partial\phi}+\frac{2\mu}{r\sin\theta}\frac{\partial\left(r\omega_\phi\sin\theta\right)}{\partial r}+\rho b_\theta=\rho\frac{\partial^2u_\theta}{\partial t^2}\\
&(\lambda+2\mu)\,\frac{1}{r\sin\theta}\frac{\partial e}{\partial\phi}-\frac{2\mu}{r}\frac{\partial\left(r\omega_\theta\right)}{\partial r}+\frac{2\mu}{r}\frac{\partial\omega_r}{\partial\theta}+\rho b_\phi=\rho\frac{\partial^2u_\phi}{\partial t^2}
\end{aligned}
\qquad (6.66)
$$

where

$$
\begin{aligned}
&\omega_r=-\Omega_{\theta\phi}=\frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial\left(u_\phi\sin\theta\right)}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right)\\
&\omega_\theta=-\Omega_{\phi r}=\frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{1}{r}\frac{\partial\left(ru_\phi\right)}{\partial r}\right)\\
&\omega_z=-\Omega_{r\theta}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial\left(ru_\theta\right)}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\\
&e=\frac{1}{r^2\sin\theta}\left(\frac{\partial\left(r^2u_r\sin\theta\right)}{\partial r}+\frac{\partial\left(ru_\theta\sin\theta\right)}{\partial\theta}+\frac{\partial\left(ru_\phi\right)}{\partial\phi}\right)
\end{aligned}
$$

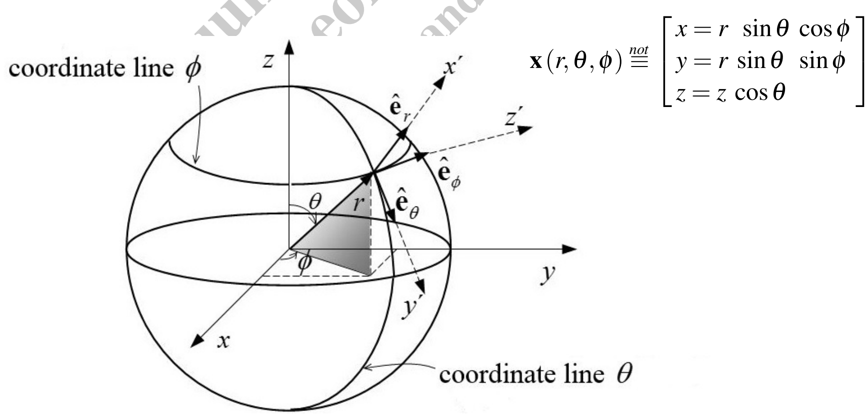

Figure 6.9: Spherical coordinates.

6.7.2 Stress Formulation: Beltrami-Michell Equation

This formulation is solely valid for the quasi-static case discussed in Section 6.6.3. Consider, thus, the equations that constitute the quasi-static linear elastic problem:

$$\boxed{\begin{array}{ll} \nabla\cdot\boldsymbol{\sigma}+\rho_0\mathbf{b}=\mathbf{0} & \text{Equilibrium equation} \\ \boldsymbol{\varepsilon}=-\dfrac{\nu}{E}\,\mathrm{Tr}\left(\boldsymbol{\sigma}\right)\mathbf{1}+\dfrac{1+\nu}{E}\boldsymbol{\sigma} & \text{Inverse constitutive equation} \\ \boldsymbol{\varepsilon}=\nabla^S\mathbf{u}=\dfrac{1}{2}\left(\mathbf{u}\otimes\nabla+\nabla\otimes\mathbf{u}\right) & \text{Geometric equation} \end{array}} \qquad (6.67)$$

$$\boxed{\left.\begin{array}{l} \Gamma_u:\mathbf{u}=\mathbf{u}^* \\ \Gamma_\sigma:\mathbf{t}^*=\boldsymbol{\sigma}\cdot\mathbf{n} \end{array}\right\} \quad \text{Boundary conditions in space}} \qquad (6.68)$$

where the inverse constitutive (6.24) (strains in terms of stresses) has been considered in (6.67).

The starting point of the stress formulation is the geometric equation of (6.67) from which, by means of successive differentiation, the displacements are eliminated and the *compatibility equations*[14] are obtained,

$$\varepsilon_{ij,kl}+\varepsilon_{kl,ij}-\varepsilon_{ik,jl}-\varepsilon_{jl,ik}=0 \qquad i,j,k,l\in\{1,2,3\} \; . \qquad (6.69)$$

Then, the equations of the problem are deduced in the following manner:

a) The constitutive equation of (6.67) is replaced in the compatibility equations (6.69).
b) The resulting expression is introduced in the equilibrium equation of (6.67).

This results in the equation

$$\boxed{\begin{array}{c} \text{Beltrami-Michell equation} \\ \nabla^2\sigma_{ij}+\dfrac{1}{1+\nu}\sigma_{ll,ij}=-\dfrac{\nu}{1-\nu}\delta_{ij}\left(\rho_0 b_l\right)_{,l}-\left(\rho_0 b_i\right)_{,j}-\left(\rho_0 b_j\right)_{,i} \\ \hfill i,j\in\{1,2,3\} \end{array}} \qquad (6.70)$$

which receives the name of *Beltrami-Michell equation* and constitutes a system of second-order PDEs in stresses $\boldsymbol{\sigma}(\mathbf{x})$ that must be solved in $\mathbb{R}^3$.

The *boundary conditions* of this system are the equilibrium equation of (6.67), which, being a system of first-order PDEs, acts as the boundary conditions of the second-order system in (6.70), and the boundary conditions in Γ_σ.

[14] The deduction of the compatibility equations has been studied in Chapter 3, Section 3.3.

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$$\boxed{\nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{b} = \mathbf{0} \qquad \text{Equilibrium equation}} \tag{6.71}$$

$$\boxed{\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t}^* \text{ in } \Gamma_\sigma \qquad \text{Boundary conditions in } \Gamma_\sigma} \tag{6.72}$$

The integration of the system in (6.70) yields the stress field $\boldsymbol{\sigma}(\mathbf{x})$. Substitution of the stresses in the inverse constitutive equation of (6.67) results in the strains $\boldsymbol{\varepsilon}(\mathbf{x})$. However, to obtain the displacement field $\mathbf{u}(\mathbf{x})$, the geometric equations must be integrated, taking into account the boundary conditions in Γ_u [15].

$$\begin{cases} \boldsymbol{\varepsilon}(\mathbf{x}) = \dfrac{1}{2}\big(\mathbf{u}(\mathbf{x}) \otimes \nabla + \nabla \otimes \mathbf{u}(\mathbf{x})\big) & \mathbf{x} \in V \\ \mathbf{u}(\mathbf{x}) = \mathbf{u}^*(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_u \end{cases} \tag{6.73}$$

Thus, the system of second-order PDEs must be integrated in $\mathbb{R}^3$.

Remark 6.10. The need to *integrate* the second system (6.73) (when the stress formulation is followed) is a disadvantage (with respect to the displacement formulation described in Section 6.7.1) when numerical methods are used to solve the linear elastic problem.

6.8 Unicity of the Solution to the Linear Elastic Problem

Theorem 6.1. *The solution*

$$\mathbb{R}(\mathbf{x},t) \stackrel{not}{\equiv} \begin{bmatrix} \mathbf{u}(\mathbf{x},t) \\ \boldsymbol{\varepsilon}(\mathbf{x},t) \\ \boldsymbol{\sigma}(\mathbf{x},t) \end{bmatrix}$$

to the linear elastic problem posed in (6.42) *to* (6.44) *is unique.*

Proof

Consider the actions defined by $\mathbb{A}(\mathbf{x},t) \stackrel{not}{\equiv} [\mathbf{b}(\mathbf{x},t),\ \mathbf{u}^*(\mathbf{x},t),\ \mathbf{t}^*(\mathbf{x},t),\ \mathbf{v}_0(\mathbf{x})]^T$, in the domains V, Γ_u, Γ_σ and V, respectively, (satisfying $\Gamma_\sigma \bigcup \Gamma_u = \partial V$ and

[15] An analytical procedure to integrate these geometric equations was provided in Chapter 3, Section 3.4.2

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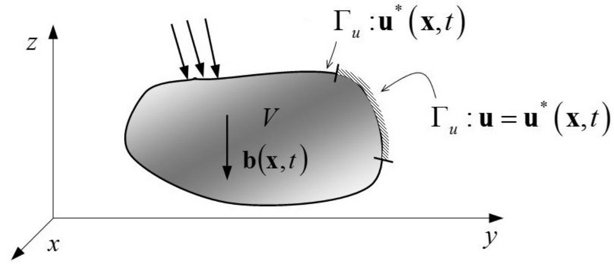

Figure 6.10: Linear elastic problem.

$\Gamma_\sigma \bigcap \Gamma_u = \emptyset$) act on the linear elastic problem schematically represented in Figure 6.10.

The possible solutions $\mathbb{R}(\mathbf{x},t) \stackrel{not}{\equiv} [\mathbf{u}(\mathbf{x},t)\,,\ \boldsymbol{\varepsilon}(\mathbf{x},t)\,,\ \boldsymbol{\sigma}(\mathbf{x},t)]^T$ to the linear elastic problem must satisfy the equations:

$$
\begin{array}{ll}
\nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{b} = \rho_0 \dfrac{\partial^2 \mathbf{u}}{\partial t^2} & \text{Cauchy's equation} \\
\boldsymbol{\sigma} = \lambda \operatorname{Tr}(\boldsymbol{\varepsilon})\,\mathbf{1} + 2\mu\boldsymbol{\varepsilon} & \text{Constitutive equation} \\
\boldsymbol{\varepsilon} = \nabla^S \mathbf{u} = \dfrac{1}{2}(\mathbf{u} \otimes \nabla + \nabla \otimes \mathbf{u}) & \text{Geometric equation}
\end{array}
\tag{6.74}
$$

$$
\left.\begin{array}{l}
\Gamma_u : \mathbf{u} = \mathbf{u}^* \\
\Gamma_\sigma : \mathbf{t}^* = \boldsymbol{\sigma} \cdot \mathbf{n}
\end{array}\right\} \quad \text{Boundary conditions in space}
\tag{6.75}
$$

$$
\left.\begin{array}{l}
\mathbf{u}(\mathbf{x},0) = \mathbf{0} \\
\dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{v}_0
\end{array}\right\} \quad \text{Initial conditions}
\tag{6.76}
$$

The unicity of the solution is proven as follows. Suppose the solution is not unique, that is, there exist two different solutions to the problem,

$$
\begin{gathered}
\mathbb{R}^{(1)}(\mathbf{x},t) \stackrel{not}{\equiv} \begin{bmatrix} \mathbf{u}^{(1)}(\mathbf{x},t) \\ \boldsymbol{\varepsilon}^{(1)}(\mathbf{x},t) \\ \boldsymbol{\sigma}^{(1)}(\mathbf{x},t) \end{bmatrix} \quad \text{and} \quad \mathbb{R}^{(2)}(\mathbf{x},t) \stackrel{not}{\equiv} \begin{bmatrix} \mathbf{u}^{(2)}(\mathbf{x},t) \\ \boldsymbol{\varepsilon}^{(2)}(\mathbf{x},t) \\ \boldsymbol{\sigma}^{(2)}(\mathbf{x},t) \end{bmatrix} \\
\text{such that} \quad \mathbb{R}^{(1)} \neq \mathbb{R}^{(2)}\,,
\end{gathered}
\tag{6.77}
$$

which, therefore, must satisfy equations (6.74) to (6.76) and are the elastic responses to the same action $\mathbb{A}(\mathbf{x},t) \stackrel{not}{\equiv} [\mathbf{b}(\mathbf{x},t)\,,\ \mathbf{u}^*(\mathbf{x},t)\,,\ \mathbf{t}^*(\mathbf{x},t)\,,\ \mathbf{v}_0(\mathbf{x})]^T$. Con-

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sider now a possible response constituted by the difference $\mathbb{R}^{(2)} - \mathbb{R}^{(1)}$,

$$\widetilde{\mathbb{R}}(\mathbf{x},t) \stackrel{def}{=} \mathbb{R}^{(2)} - \mathbb{R}^{(1)} \stackrel{not}{\equiv} \begin{bmatrix} \mathbf{u}^{(2)}(\mathbf{x},t) - \mathbf{u}^{(1)}(\mathbf{x},t) \\ \boldsymbol{\varepsilon}^{(2)}(\mathbf{x},t) - \boldsymbol{\varepsilon}^{(1)}(\mathbf{x},t) \\ \boldsymbol{\sigma}^{(2)}(\mathbf{x},t) - \boldsymbol{\sigma}^{(1)}(\mathbf{x},t) \end{bmatrix} \stackrel{def}{=} \begin{bmatrix} \widetilde{\mathbf{u}}(\mathbf{x},t) \\ \widetilde{\boldsymbol{\varepsilon}}(\mathbf{x},t) \\ \widetilde{\boldsymbol{\sigma}}(\mathbf{x},t) \end{bmatrix} . \tag{6.78}$$

Note how the answer $\widetilde{\mathbb{R}}$ satisfies the following equations:

- *Cauchy's equation with* $\mathbf{b} = \mathbf{0}$ [16]

$$\begin{aligned} \nabla \cdot \widetilde{\boldsymbol{\sigma}}(\mathbf{x},t) &= \nabla \cdot \left(\boldsymbol{\sigma}^{(2)}(\mathbf{x},t) - \boldsymbol{\sigma}^{(1)}(\mathbf{x},t) \right) = \nabla \cdot \boldsymbol{\sigma}^{(2)} - \nabla \cdot \boldsymbol{\sigma}^{(1)} = \\ &= \left(-\rho_0 \mathbf{b} + \rho_0 \frac{\partial^2 \mathbf{u}^{(2)}}{\partial t^2} \right) - \left(-\rho_0 \mathbf{b} + \rho_0 \frac{\partial^2 \mathbf{u}^{(1)}}{\partial t^2} \right) = \\ &= \rho_0 \frac{\partial^2 \mathbf{u}^{(2)}}{\partial t^2} - \rho_0 \frac{\partial^2 \mathbf{u}^{(1)}}{\partial t^2} = \rho_0 \frac{\partial^2 \widetilde{\mathbf{u}}}{\partial t^2} \end{aligned} \tag{6.79}$$

- *Constitutive equation*[17]

$$\begin{aligned} \widetilde{\boldsymbol{\sigma}}(\mathbf{x},t) &= \boldsymbol{\sigma}^{(2)}(\mathbf{x},t) - \boldsymbol{\sigma}^{(1)}(\mathbf{x},t) = \mathbb{C} : \boldsymbol{\varepsilon}^{(2)} - \mathbb{C} : \boldsymbol{\varepsilon}^{(1)} = \\ &= \mathbb{C} : \left(\boldsymbol{\varepsilon}^{(2)} - \boldsymbol{\varepsilon}^{(1)} \right) = \mathbb{C} : \widetilde{\boldsymbol{\varepsilon}} \end{aligned} \tag{6.80}$$

- *Geometric equation*

$$\begin{aligned} \widetilde{\boldsymbol{\varepsilon}}(\mathbf{x},t) &= \boldsymbol{\varepsilon}^{(2)}(\mathbf{x},t) - \boldsymbol{\varepsilon}^{(1)}(\mathbf{x},t) = \nabla^S \mathbf{u}^{(2)} - \nabla^S \mathbf{u}^{(1)} = \\ &= \nabla^S \left(\mathbf{u}^{(2)} - \mathbf{u}^{(1)} \right) = \nabla^S \widetilde{\mathbf{u}} \end{aligned} \tag{6.81}$$

- *Boundary conditions in* Γ_u *with* $\widetilde{\mathbf{u}}^* = \mathbf{0}$

$$\Gamma_u \rightarrow \begin{cases} \widetilde{\mathbf{u}}(\mathbf{x},t) = \mathbf{u}^{(2)}(\mathbf{x},t) - \mathbf{u}^{(1)}(\mathbf{x},t) = \mathbf{u}^* - \mathbf{u}^* = \mathbf{0} \quad \forall t \implies \\ \implies \dfrac{\partial \widetilde{\mathbf{u}}(\mathbf{x},t)}{\partial t} = \dot{\widetilde{\mathbf{u}}}(\mathbf{x},t) = \mathbf{0} \end{cases} \tag{6.82}$$

[16] The fact that the Nabla operator $(\nabla * (\bullet))$ is a linear operator is used advantageously here, that is, $\nabla * (\mathbf{a} + \mathbf{b}) = \nabla * \mathbf{a} + \nabla * \mathbf{b}$, where $*$ symbolizes any type of differential operation. Likewise, the operator $\partial^2 (\bullet, t)/\partial t^2$ is also a linear operator.

[17] The property that the operator $\mathbb{C}:$ is a linear operator is applied here, that is, $\mathbb{C} : (\mathbf{a} + \mathbf{b}) = \mathbb{C} : \mathbf{a} + \mathbb{C} : \mathbf{b}$.

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- *Boundary conditions in* Γ_σ *with* $\widetilde{\mathbf{t}}^* = \mathbf{0}$

$$\Gamma_\sigma \to \ \widetilde{\boldsymbol{\sigma}}(\mathbf{x},t)\cdot\mathbf{n} = \left(\boldsymbol{\sigma}^{(2)}(\mathbf{x},t) - \boldsymbol{\sigma}^{(1)}(\mathbf{x},t)\right)\cdot\mathbf{n} = \boldsymbol{\sigma}^{(2)}\cdot\mathbf{n} - \boldsymbol{\sigma}^{(1)}\cdot\mathbf{n} = \\ = \mathbf{t}^* - \mathbf{t}^* = \mathbf{0} \tag{6.83}$$

- *Initial conditions with* $\mathbf{v}_0 = \mathbf{0}$

$$\begin{cases} \widetilde{\mathbf{u}}(\mathbf{x},0) = \mathbf{u}^{(2)}(\mathbf{x},0) - \mathbf{u}^{(1)}(\mathbf{x},0) = \mathbf{0} - \mathbf{0} = \mathbf{0} \\ \dfrac{\partial \widetilde{\mathbf{u}}(\mathbf{x},0)}{\partial t} = \dot{\widetilde{\mathbf{u}}}(\mathbf{x},0) = \dot{\mathbf{u}}^{(2)}(\mathbf{x},0) - \dot{\mathbf{u}}^{(1)}(\mathbf{x},0) = \mathbf{v}_0 - \mathbf{v}_0 = \mathbf{0} \end{cases} \tag{6.84}$$

Consider now the calculation of the integral

$$\int_{\partial V} \mathbf{n}\cdot\left(\widetilde{\boldsymbol{\sigma}}\cdot\dot{\widetilde{\mathbf{u}}}\right) dS = \int_{\Gamma_u \cup \Gamma_\sigma} \overbrace{(\mathbf{n}\cdot\widetilde{\boldsymbol{\sigma}})}^{=0 \text{ in } \Gamma_\sigma} \cdot \underbrace{\dot{\widetilde{\mathbf{u}}}}_{=0 \text{ in } \Gamma_u} dS \overset{\substack{\text{Divergence} \\ \text{Theorem}}}{=} \int_V \nabla\cdot\left(\widetilde{\boldsymbol{\sigma}}\cdot\dot{\widetilde{\mathbf{u}}}\right) dV = 0\,, \tag{6.85}$$

where the conditions (6.82) and (6.83) have been applied. Operating on (6.85) results in

$$\begin{cases} \nabla\cdot\left(\widetilde{\boldsymbol{\sigma}}\cdot\dot{\widetilde{\mathbf{u}}}\right) = (\nabla\cdot\widetilde{\boldsymbol{\sigma}})\cdot\dot{\widetilde{\mathbf{u}}} + \widetilde{\boldsymbol{\sigma}} : \left(\nabla\dot{\widetilde{\mathbf{u}}}\right) = \rho_0 \dfrac{\partial^2 \widetilde{\mathbf{u}}}{\partial t^2}\cdot\dot{\widetilde{\mathbf{u}}} + \widetilde{\boldsymbol{\sigma}} : \left(\nabla\dot{\widetilde{\mathbf{u}}}\right)^T \\ \dfrac{\partial}{\partial x_i}\left(\widetilde{\sigma}_{ij}\dot{\widetilde{u}}_j\right) = \dfrac{\partial \widetilde{\sigma}_{ij}}{\partial x_i}\dot{\widetilde{u}}_j + \widetilde{\sigma}_{ij}\dfrac{\partial \dot{\widetilde{u}}_j}{\partial x_i} = \rho_0 \dfrac{\partial^2 \widetilde{u}_j}{\partial t^2}\dot{\widetilde{u}}_j + \widetilde{\sigma}_{ji}\dfrac{\partial \dot{\widetilde{u}}_j}{\partial x_i} \qquad i,j \in \{1,2,3\} \end{cases} \tag{6.86}$$

where the condition (6.79) has been considered. On the other hand[18],

$$\left(\nabla\dot{\widetilde{\mathbf{u}}}\right)^T = \dot{\widetilde{\mathbf{u}}}\otimes\nabla = \frac{1}{2}\underbrace{\left(\dot{\widetilde{\mathbf{u}}}\otimes\nabla + \nabla\otimes\dot{\widetilde{\mathbf{u}}}\right)}_{\dot{\widetilde{\boldsymbol{\varepsilon}}} = \nabla^S\dot{\widetilde{\mathbf{u}}}} + \frac{1}{2}\underbrace{\left(\dot{\widetilde{\mathbf{u}}}\otimes\nabla - \nabla\otimes\dot{\widetilde{\mathbf{u}}}\right)}_{\dot{\widetilde{\boldsymbol{\Omega}}} = \nabla^a\dot{\widetilde{\mathbf{u}}}} = \dot{\widetilde{\boldsymbol{\varepsilon}}} + \dot{\widetilde{\boldsymbol{\Omega}}} \implies \\ \widetilde{\boldsymbol{\sigma}} : \left(\nabla\dot{\widetilde{\mathbf{u}}}\right)^T = \widetilde{\boldsymbol{\sigma}} : \dot{\widetilde{\boldsymbol{\varepsilon}}} + \underbrace{\widetilde{\boldsymbol{\sigma}} : \dot{\widetilde{\boldsymbol{\Omega}}}}_{=0} \implies \boxed{\widetilde{\boldsymbol{\sigma}} : \left(\nabla\dot{\widetilde{\mathbf{u}}}\right)^T = \widetilde{\boldsymbol{\sigma}} : \dot{\widetilde{\boldsymbol{\varepsilon}}}}\,. \tag{6.87}$$

---

[18] The fact that $\widetilde{\boldsymbol{\sigma}}$ is a symmetric tensor and $\dot{\widetilde{\boldsymbol{\Omega}}}$ is an antisymmetric one is considered here, which leads to $\widetilde{\boldsymbol{\sigma}}:\dot{\widetilde{\boldsymbol{\Omega}}} = \widetilde{\sigma}_{ij}\dot{\widetilde{\Omega}}_{ij} = 0$.

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In addition[19],

$$
\begin{aligned}
&\rho_0 \frac{\partial^2 \tilde{\mathbf{u}}}{\partial t^2} \cdot \dot{\tilde{\mathbf{u}}} = \rho_0 \frac{\partial^2 \dot{\tilde{\mathbf{u}}}}{\partial t^2} \cdot \dot{\tilde{\mathbf{u}}} = \frac{1}{2}\rho_0 \frac{\partial \left(\dot{\tilde{\mathbf{u}}} \cdot \dot{\tilde{\mathbf{u}}}\right)}{\partial t} = \frac{1}{2}\rho_0 \frac{\partial \big(\overbrace{\tilde{\mathbf{v}} \cdot \tilde{\mathbf{v}}}^{\tilde{\mathrm{v}}^2}\big)}{\partial t} = \\
&= \rho_0 \frac{d}{dt}\left(\frac{1}{2}\tilde{\mathrm{v}}^2\right) \quad \Longrightarrow \quad \boxed{\rho_0 \frac{\partial^2 \tilde{\mathbf{u}}}{\partial t^2} \cdot \dot{\tilde{\mathbf{u}}} = \rho_0 \frac{d}{dt}\left(\frac{1}{2}\tilde{\mathrm{v}}^2\right)} \,.
\end{aligned}
\tag{6.88}
$$

Replacing (6.88) and (6.87) in (6.86), and the resulting expression in (6.85), and taking into account the definition of internal energy $\mathcal{U}$ given in (6.10) produces

$$
\begin{aligned}
&\int_V \nabla \cdot \left(\tilde{\boldsymbol{\sigma}} \cdot \dot{\tilde{\mathbf{u}}}\right) dV = \int_V \rho_0 \frac{d}{dt}\left(\frac{1}{2}\tilde{\mathrm{v}}^2\right) dV + \int_V \tilde{\boldsymbol{\sigma}} : \dot{\tilde{\boldsymbol{\varepsilon}}}\, dV = 0 \Longrightarrow \\
&\qquad \underbrace{\frac{d}{dt}\int_V \frac{1}{2}\rho_0\, \tilde{\mathrm{v}}^2 dV}_{d\tilde{\mathcal{K}}/dt} + \underbrace{\int_V \tilde{\boldsymbol{\sigma}} : \dot{\tilde{\boldsymbol{\varepsilon}}}\, dV}_{d\tilde{\mathcal{U}}/dt} = 0 \quad \Longrightarrow
\end{aligned}
\tag{6.89}
$$

$$
\boxed{\frac{d\tilde{\mathcal{K}}}{dt} + \frac{d\tilde{\mathcal{U}}}{dt} = \frac{d}{dt}\left(\tilde{\mathcal{K}} + \tilde{\mathcal{U}}\right) = 0 \quad \forall t \geq 0} \,. \tag{6.90}
$$

Note, though, that at the initial time $t = 0$ the following is satisfied (see (6.10), (6.13) and (6.84))

$$
\left.
\begin{aligned}
&\tilde{\mathcal{K}}\Big|_{t=0} = \int_V \frac{1}{2}\rho_0 \tilde{\mathrm{v}}^2\Big|_{t=0} dV = \int_V \frac{1}{2}\rho_0\, \underbrace{\tilde{\mathbf{v}}_0}_{\dot{\tilde{\mathbf{u}}}_0 = 0} \cdot\, \tilde{\mathbf{v}}_0 \, dV = 0 \\
&\tilde{\mathcal{U}}\Big|_{t=0} = \int_V \hat{u}(\mathbf{x},t)\,\Big|_{t=0} dV = \int_V \frac{1}{2}\, \underbrace{\tilde{\boldsymbol{\varepsilon}}\Big|_{t=0}}_{=\mathbf{0}} : \mathbb{C} : \tilde{\boldsymbol{\varepsilon}}\Big|_{t=0} dV = 0
\end{aligned}
\right\} \Rightarrow \left(\tilde{\mathcal{K}} + \tilde{\mathcal{U}}\right)\Big|_{t=0} = 0
\tag{6.91}
$$

and the integration of (6.90) with the initial condition (6.91) leads to

$$
\boxed{\tilde{\mathcal{K}} + \tilde{\mathcal{U}} = 0 \qquad \forall t \geq 0} \,, \tag{6.92}
$$

where

$$
\tilde{\mathcal{K}} = \int_V \frac{1}{2}\, \underbrace{\rho_0\, \tilde{\mathrm{v}}^2}_{\geq 0}\, dV \geq 0 \qquad \forall t \geq 0 \,. \tag{6.93}
$$

[19] Here, the definition $|\tilde{\mathbf{v}}| \stackrel{def}{=} \tilde{\mathrm{v}}$ is used.

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Comparing (6.92) and (6.93) necessarily leads to the conclusion

$$\left.\begin{array}{l}\widetilde{\mathcal{K}}+\widetilde{\mathcal{U}}=0\\ \widetilde{\mathcal{K}}\geq 0\end{array}\right\}\ \forall t\geq 0 \quad \Longrightarrow \quad \widetilde{\mathcal{U}}=\int_V \frac{1}{2}\widetilde{\boldsymbol{\varepsilon}}:\mathbb{C}:\widetilde{\boldsymbol{\varepsilon}}\,dV\leq 0 \quad \forall t\geq 0\,. \tag{6.94}$$

On the other hand, since the tensor of elastic constants $\mathbb{C}$ is positive-definite (see (6.32)) ,

$$\begin{array}{l}\widetilde{\boldsymbol{\varepsilon}}(\mathbf{x},t):\mathbb{C}:\widetilde{\boldsymbol{\varepsilon}}(\mathbf{x},t)\geq 0 \qquad \forall\mathbf{x}\in V \quad \forall t\geq 0 \quad \Longrightarrow\\ \qquad \widetilde{\mathcal{U}}=\displaystyle\int_V \frac{1}{2}\widetilde{\boldsymbol{\varepsilon}}:\mathbb{C}:\widetilde{\boldsymbol{\varepsilon}}\,dV\geq 0 \qquad \forall t\geq 0\,.\end{array} \tag{6.95}$$

Then, comparing (6.94) and (6.95) necessarily leads to

$$\left.\begin{array}{l}\widetilde{\mathcal{U}}(t)\leq 0\\ \widetilde{\mathcal{U}}(t)\geq 0\end{array}\right\}\ \forall t\geq 0 \quad \Longrightarrow \quad \widetilde{\mathcal{U}}(t)=\int_V \frac{1}{2}\widetilde{\boldsymbol{\varepsilon}}:\mathbb{C}:\widetilde{\boldsymbol{\varepsilon}}\,dV=0 \quad \forall t\geq 0\,. \tag{6.96}$$

Considering once more the positive-definite condition of tensor $\mathbb{C}$ [20],

$$\widetilde{\mathcal{U}}=\int_V \frac{1}{2}\underbrace{\widetilde{\boldsymbol{\varepsilon}}:\mathbb{C}:\widetilde{\boldsymbol{\varepsilon}}}_{\geq 0}\,dV=0 \quad \forall t\geq 0 \implies \widetilde{\boldsymbol{\varepsilon}}:\mathbb{C}:\widetilde{\boldsymbol{\varepsilon}}=0 \quad \forall\mathbf{x},\ \forall t\geq 0 \tag{6.97}$$

and, necessarily, from the positive-definite condition of $\mathbb{C}$ it is deduced that

$$\widetilde{\boldsymbol{\varepsilon}}:\mathbb{C}:\widetilde{\boldsymbol{\varepsilon}}=0 \iff \widetilde{\boldsymbol{\varepsilon}}(\mathbf{x},t)=\mathbf{0} \quad \forall\mathbf{x},\ \forall t\geq 0 \tag{6.98}$$

$$\widetilde{\boldsymbol{\varepsilon}}(\mathbf{x},t)=\boldsymbol{\varepsilon}^{(2)}-\boldsymbol{\varepsilon}^{(1)}=\mathbf{0} \qquad \Longrightarrow \qquad \boxed{\boldsymbol{\varepsilon}^{(2)}=\boldsymbol{\varepsilon}^{(1)}}\,. \tag{6.99}$$

In addition, replacing (6.99) in (6.81) results in

$$\widetilde{\boldsymbol{\varepsilon}}(\mathbf{x},t)=\nabla^S\cdot\widetilde{\mathbf{u}}=\mathbf{0} \quad \Longrightarrow \quad \frac{1}{2}\left(\frac{\partial\widetilde{u}_i}{\partial x_j}+\frac{\partial\widetilde{u}_j}{\partial x_i}\right)=0 \quad i,j\in\{1,2,3\}\ , \tag{6.100}$$

[20] The following theorem of integral calculus is applied here:

$$\text{If } \phi(\mathbf{x})\geq 0 \text{ and } \int_\Omega \phi(\mathbf{x})\,d\Omega=0 \quad \Longrightarrow \quad \phi(\mathbf{x})=0 \quad \forall\mathbf{x}\in\Omega\,.$$

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which is a system of six homogeneous and first-order PDEs. Its integration leads to the solution[21]

$$\widetilde{\mathbf{u}}(\mathbf{x},t) = \underbrace{\widetilde{\boldsymbol{\Omega}}\cdot\mathbf{x}}_{\text{rotation}} + \underbrace{\widetilde{\mathbf{c}}}_{\text{translation}} \qquad \text{with}$$

$$\widetilde{\boldsymbol{\Omega}} \stackrel{not}{\equiv} \begin{bmatrix} 0 & -\widetilde{\theta}_3 & \widetilde{\theta}_2 \\ \widetilde{\theta}_3 & 0 & -\widetilde{\theta}_1 \\ -\widetilde{\theta}_2 & \widetilde{\theta}_1 & 0 \end{bmatrix} \quad \text{and} \quad \widetilde{\mathbf{c}} \stackrel{not}{\equiv} \begin{bmatrix} \widetilde{c}_1 \\ \widetilde{c}_2 \\ \widetilde{c}_3 \end{bmatrix}, \tag{6.101}$$

where $\widetilde{\boldsymbol{\Omega}}$ is an antisymmetric tensor (rotation tensor dependent on three constants $\widetilde{\theta}_1, \widetilde{\theta}_2$ and $\widetilde{\theta}_3$) and $\widetilde{\mathbf{c}}$ is a constant vector equivalent to a translation. Ultimately, the solution (6.100) to the system (6.101) are the displacements $\widetilde{\mathbf{u}}(\mathbf{x},t)$ compatible with a null strain $\widetilde{\boldsymbol{\varepsilon}}(\mathbf{x},t) = \mathbf{0}$, which correspond to a rigid body motion. The integration constants in $\widetilde{\boldsymbol{\Omega}}$ and $\widetilde{\mathbf{c}}$ are determined by imposing the boundary conditions (6.82) ($\widetilde{\mathbf{u}}(\mathbf{x},t) = \mathbf{0} \ \forall \mathbf{x} \in \Gamma_u$), therefore, if the rigid body motion is impeded through the restrictions in Γ_u, one obtains $\widetilde{\boldsymbol{\Omega}} = \mathbf{0}$ and $\widetilde{\mathbf{c}} = \mathbf{0}$. In conclusion,

$$\left.\begin{array}{l} \widetilde{\mathbf{u}}(\mathbf{x},t) = \widetilde{\boldsymbol{\Omega}}\cdot\mathbf{x} + \widetilde{\mathbf{c}} \\ \widetilde{\boldsymbol{\Omega}} \equiv \mathbf{0} \ ; \quad \widetilde{\mathbf{c}} \equiv \mathbf{0} \end{array}\right\} \Longrightarrow \widetilde{\mathbf{u}}(\mathbf{x},t) = \mathbf{u}^{(2)} - \mathbf{u}^{(1)} = \mathbf{0} \Longrightarrow \boxed{\mathbf{u}^{(2)} = \mathbf{u}^{(1)}}. \tag{6.102}$$

Finally, replacing (6.99) in (6.80) yields

$$\widetilde{\boldsymbol{\sigma}}(\mathbf{x},t) = \mathbb{C} : \widetilde{\boldsymbol{\varepsilon}} = \mathbf{0} = \boldsymbol{\sigma}^{(2)} - \boldsymbol{\sigma}^{(1)} \quad \Longrightarrow \quad \boxed{\boldsymbol{\sigma}^{(2)} = \boldsymbol{\sigma}^{(1)}}. \tag{6.103}$$

Then, observing (6.99) , (6.102) and (6.103) leads to the conclusion

$$\left.\begin{array}{l} \mathbf{u}^{(2)} = \mathbf{u}^{(1)} \\ \boldsymbol{\varepsilon}^{(2)} = \boldsymbol{\varepsilon}^{(1)} \\ \boldsymbol{\sigma}^{(2)} = \boldsymbol{\sigma}^{(1)} \end{array}\right\} \quad \Longrightarrow \quad \mathbb{R}^{(2)} = \mathbb{R}^{(1)}. \tag{6.104}$$

Therefore, *the solution is unique (QED).*

6.9 Saint-Venant's Principle

Saint-Venant's principle is an empirical principle that does not have a rigorous proof. Consider a solid Ω that is subjected to a system of forces on its

[21] This solution can be obtained applying the methodology used in Chapter 3, Section 3.4.2 to integrate the strain field.

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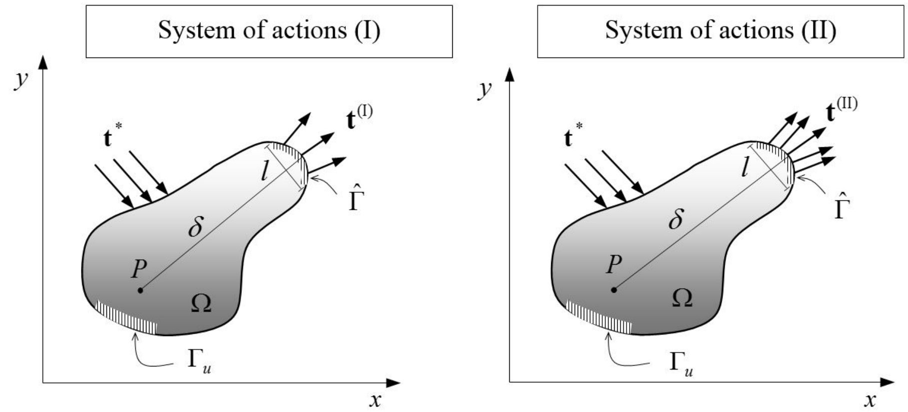


Figure 6.11: Saint-Venant's principle.

boundary characterized by the traction vector $\mathbf{t}^*$ (see Figure 6.11). These actions will lead to a solution or response in displacements, strains and stresses, $\mathbb{R}^{(I)}(\mathbf{x},t) \stackrel{not}{\equiv} \left[\mathbf{u}^{(I)}(\mathbf{x},t), \boldsymbol{\varepsilon}^{(I)}(\mathbf{x},t), \boldsymbol{\sigma}^{(I)}(\mathbf{x},t)\right]^T$. Consider now a part $\hat{\Gamma}$ of the boundary Γ_σ $\left(\hat{\Gamma} \subset \Gamma_\sigma\right)$ of said medium, whose typical dimension is ℓ, and replace the system of actions applied on the boundary, $\mathbf{t}^{(I)}$, by another system, $\mathbf{t}^{(II)}$, that is *statically equivalent* to $\mathbf{t}^{(I)}$ [22], without modifying the actions on the rest of Γ_σ. Modifying the actions in this way will presumably result in the new responses $\mathbb{R}^{(II)}(\mathbf{x},t) \stackrel{not}{\equiv} \left[\mathbf{u}^{(II)}(\mathbf{x},t), \boldsymbol{\varepsilon}^{(II)}(\mathbf{x},t), \boldsymbol{\sigma}^{(II)}(\mathbf{x},t)\right]^T$.

Saint-Venant's principle states that, for the points belonging to the domain Ω that are sufficiently far from the boundary $\hat{\Gamma}$, the solution in both cases is practically the same, that is, for a point P of the interior of Ω,

$$\boxed{\left.\begin{array}{l} \mathbf{u}^{(I)}(\mathbf{x}_p,t) \approx \mathbf{u}^{(II)}(\mathbf{x}_p,t) \\ \boldsymbol{\varepsilon}^{(I)}(\mathbf{x}_p,t) \approx \boldsymbol{\varepsilon}^{(II)}(\mathbf{x}_p,t) \\ \boldsymbol{\sigma}^{(I)}(\mathbf{x}_p,t) \approx \boldsymbol{\sigma}^{(II)}(\mathbf{x}_p,t) \end{array}\right\} \ \forall P \ \big| \ \delta \gg \ell} \ . \tag{6.105}$$

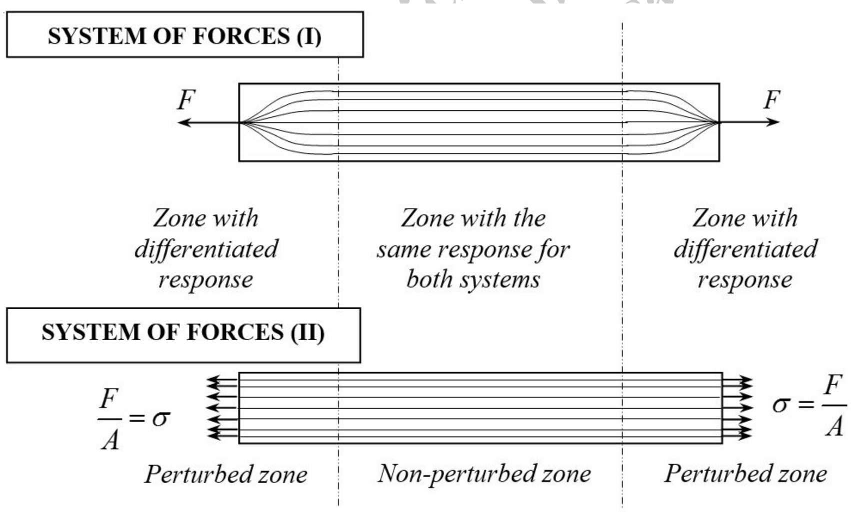
In other words, if the distance δ between the point being considered and the part of the boundary in which the actions have been modified is large in comparison with the dimension ℓ of the modified zone, the response in said point is equivalent in both cases.

[22] Two systems of forces $\mathbf{t}^{(I)}$ and $\mathbf{t}^{(II)}$ are said to be *statically equivalent* if the resultant (forces and moments) of both systems is the same.

Example 6.4 – *Description of Saint-Venant's principle in strength of materials and how it relates to the concept of* stress.

Solution

Consider a beam (or prismatic piece) with a cross-section A subjected to a tensile point force F in its ends, as shown in the figure below. The exact solution to the original elastic problem (system (I)) is extremely complicated, especially in the vicinity of the points of application of the point forces. If the forces F are now replaced by a statically equivalent system of uniformly distributed tensile loads in the end sections $\sigma = F/A$ (system (II)), the elastic solution to the corresponding problem is extremely simple and coincides (for a Poisson's ratio of $\nu = 0$) with the axial stress solution provided by strength of materials (uniformly distributed stresses in all the piece, $\sigma_x = F/A$). At a far enough distance from the beam's ends (once or twice the edge), Saint-Venant's principle allows approximating solution (I) with solution (II), and also allows dimensioning the strength characteristics of the piece for practical purposes.



6.10 Linear Thermoelasticity. Thermal Stresses and Strains

The main difference of linear thermoelasticity with respect to the linear elasticity studied up to this point is that the deformation process is no longer assumed to be isothermal (see Section 6.1). Now, the thermal effects are included and the temperature $\theta(\mathbf{x},t)$ is considered to evolve along time, that is,

$$
\begin{aligned}
&\theta(\mathbf{x},t) \neq \theta(\mathbf{x},0) \stackrel{not}{=} \theta_0 \,, \\
&\dot{\theta}(\mathbf{x},t) = \frac{\partial \theta(\mathbf{x},t)}{\partial t} \neq 0 \,.
\end{aligned}
\tag{6.106}
$$

Nevertheless, the hypothesis that the processes are adiabatic (slow) is maintained and, thus,

$$
\rho_0 \, r - \nabla \cdot \mathbf{q} \approx 0 \,. \tag{6.107}
$$

6.10.1 Linear Thermoelastic Constitutive Equation

Hooke's law (6.6) in this case is generalized to

$$
\boxed{
\begin{aligned}
&\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} - \boldsymbol{\beta}(\theta - \theta_0) \\
&\sigma_{ij} = \mathbb{C}_{ijkl}\, \varepsilon_{kl} - \beta_{ij}(\theta - \theta_0) \quad i,j \in \{1,2,3\}
\end{aligned}
} \,, \tag{6.108}
$$

Here, $\mathbb{C}$ is the tensor of elastic constants defined in (6.7), $\theta(\mathbf{x},t)$ is the temperature field, $\theta_0(\mathbf{x}) = \theta(\mathbf{x},0)$ is the distribution of temperatures in the neutral state (reference configuration) and $\boldsymbol{\beta}$ is the (symmetric) tensor of thermal properties.

$$
\boxed{
\begin{array}{c} \text{Tensor of thermal} \\ \text{properties} \end{array}
\begin{cases}
\boldsymbol{\beta} = \boldsymbol{\beta}^T \\
\beta_{ij} = \beta_{ji} \quad i,j \in \{1,2,3\}
\end{cases}
} \tag{6.109}
$$

In the case of an isotropic material, tensor $\mathbb{C}$ must be a fourth-order isotropic tensor and $\boldsymbol{\beta}$, a second-order isotropic one[23], that is,

$$
\begin{aligned}
&\begin{cases}
\mathbb{C} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I} \\
\mathbb{C}_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \quad i,j,k,l \in \{1,2,3\}
\end{cases} \\
&\begin{cases}
\boldsymbol{\beta} = \beta \mathbf{1} \\
\beta_{ij} = \beta \delta_{ij} \quad i,j \in \{1,2,3\}
\end{cases}
\end{aligned}
\tag{6.110}
$$

where now a single thermal property β appears in addition to the elastic constants λ and μ. Replacing (6.110) in the constitutive equation (6.108) and defining $(\theta - \theta_0) \stackrel{not}{=} \Delta\theta$, yields

[23] The most general expression of a second-order isotropic tensor is $\boldsymbol{\beta} = \beta \mathbf{1} \;\; \forall \beta$.

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Constitutive equation of an
isotropic linear thermoelastic material

$$\boldsymbol{\sigma} = \lambda \operatorname{Tr}(\boldsymbol{\varepsilon}) \mathbf{1} + 2\mu \boldsymbol{\varepsilon} - \beta \Delta\theta \mathbf{1}$$

$$\sigma_{ij} = \lambda \varepsilon_{ll} \delta_{ij} + 2\mu \varepsilon_{ij} - \beta \Delta\theta \delta_{ij} \quad i,j \in \{1,2,3\} \tag{6.111}$$

6.10.2 Inverse Constitutive Equation

Equation (6.111) can be inverted as follows.

$$\begin{cases} \boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} - \Delta\theta\, \boldsymbol{\beta} \;\; \Rightarrow \;\; \boldsymbol{\varepsilon} = \mathbb{C}^{-1} : \boldsymbol{\sigma} + \Delta\theta \underbrace{\mathbb{C}^{-1} : \boldsymbol{\beta}}_{\boldsymbol{\alpha}} = \mathbb{C}^{-1} : \boldsymbol{\sigma} + \Delta\theta\, \boldsymbol{\alpha} \\ \boldsymbol{\alpha} \overset{def}{=} \mathbb{C}^{-1} : \boldsymbol{\beta} \rightarrow \text{Tensor of thermal expansion coefficients} \end{cases} \tag{6.112}$$

where $\boldsymbol{\alpha}$ is a second-order (symmetric) tensor involving six thermal properties named *coefficients of thermal expansion*. For an isotropic case, in agreement with (6.111) and (6.24), and after certain algebraic manipulation, one obtains

Inverse constitutive equation of an
isotropic linear thermoelastic material

$$\boldsymbol{\varepsilon} = -\frac{\nu}{E} \operatorname{Tr}(\boldsymbol{\sigma}) \mathbf{1} + \frac{1+\nu}{E} \boldsymbol{\sigma} + \alpha \Delta\theta\, \mathbf{1}$$

$$\varepsilon_{ij} = -\frac{\nu}{E} \sigma_{ll}\, \delta_{ij} + \frac{1+\nu}{E} \sigma_{ij} + \alpha \Delta\theta\, \delta_{ij} \quad i,j \in \{1,2,3\} \tag{6.113}$$

Here, α is a scalar denoted as coefficient of thermal expansion, related to the thermal property β in (6.111) by means of

$$\boxed{\text{Thermal expansion coefficient} \;\; \rightarrow \;\; \alpha = \frac{1-2\nu}{E} \beta} \,. \tag{6.114}$$

6.10.3 Thermal Stresses and Strains

Comparing the linear elastic constitutive equation (6.20) and the linear thermoelastic one (6.111) suggests the following decomposition.

$$
\begin{aligned}
&\boldsymbol{\sigma} = \underbrace{\lambda \operatorname{Tr}(\boldsymbol{\varepsilon})\,\mathbf{1} + 2\mu\boldsymbol{\varepsilon}}_{\boldsymbol{\sigma}^{nt}} - \underbrace{\beta\,\Delta\theta\,\mathbf{1}}_{\boldsymbol{\sigma}^{t}} = \boldsymbol{\sigma}^{nt} - \boldsymbol{\sigma}^{t} \\
&\begin{cases} \text{Non-thermal stress} & \rightarrow \quad \boldsymbol{\sigma}^{nt} \overset{def}{=} \lambda \operatorname{Tr}(\boldsymbol{\varepsilon})\,\mathbf{1} + 2\mu\boldsymbol{\varepsilon} \\ \text{Thermal stress} & \rightarrow \quad \boldsymbol{\sigma}^{t} \overset{def}{=} \beta\,\Delta\theta\,\mathbf{1} \end{cases}
\end{aligned}
\qquad (6.115)
$$

Here, $\boldsymbol{\sigma}^{nt}$ represents the stress produced if there do not exist any thermal phenomena and $\boldsymbol{\sigma}^{t}$ is named *thermal stress* and acts as the "correcting" stress due to the thermal increment.

A similar operation can be performed on the inverse constitutive equations for the linear elastic and linear thermoelastic cases of (6.24) and (6.113), respectively, resulting in

$$
\begin{aligned}
&\boldsymbol{\varepsilon} = \underbrace{-\frac{\nu}{E}\operatorname{Tr}(\boldsymbol{\sigma})\,\mathbf{1} + \frac{1+\nu}{E}\boldsymbol{\sigma}}_{\boldsymbol{\varepsilon}^{nt}} + \underbrace{\alpha\,\Delta\theta\,\mathbf{1}}_{\boldsymbol{\varepsilon}^{t}} = \boldsymbol{\varepsilon}^{nt} + \boldsymbol{\varepsilon}^{t} \\
&\begin{cases} \text{Non-thermal strain} & \rightarrow \quad \boldsymbol{\varepsilon}^{nt} \overset{def}{=} -\dfrac{\nu}{E}\operatorname{Tr}(\boldsymbol{\sigma})\,\mathbf{1} + \dfrac{1+\nu}{E}\boldsymbol{\sigma} \\ \text{Thermal strain} & \rightarrow \quad \boldsymbol{\varepsilon}^{t} \overset{def}{=} \alpha\,\Delta\theta\,\mathbf{1} \end{cases}
\end{aligned}
\qquad (6.116)
$$

In conclusion, the stress and strain tensors in linear thermoelasticity can be decomposed into

Total	**Non-thermal component**	**Thermal component**	
$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{nt} - \boldsymbol{\sigma}^{t}$	$\boldsymbol{\sigma}^{nt} = \mathbb{C} : \boldsymbol{\varepsilon}$ *Isotropic material:* $\boldsymbol{\sigma}^{nt} = \lambda \operatorname{Tr}(\boldsymbol{\varepsilon})\,\mathbf{1} + 2\mu\boldsymbol{\varepsilon}$	$\boldsymbol{\sigma}^{t} = \Delta\theta\,\boldsymbol{\beta}$ *Isotropic material:* $\boldsymbol{\sigma}^{t} = \beta\,\Delta\theta\,\mathbf{1}$	(6.117)
$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{nt} + \boldsymbol{\varepsilon}^{t}$	$\boldsymbol{\varepsilon}^{nt} = \mathbb{C}^{-1} : \boldsymbol{\sigma}$ *Isotropic material:* $\boldsymbol{\varepsilon}^{nt} = -\frac{\nu}{E}\operatorname{Tr}(\boldsymbol{\sigma})\,\mathbf{1} + \frac{1+\nu}{E}\boldsymbol{\sigma}$	$\boldsymbol{\varepsilon}^{t} = \Delta\theta\,\boldsymbol{\alpha}$ *Isotropic material:* $\boldsymbol{\varepsilon}^{t} = \alpha\,\Delta\theta\,\mathbf{1}$	(6.118)

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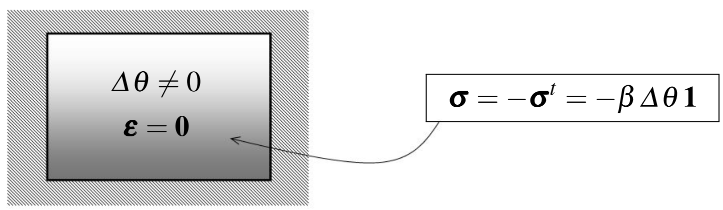
where the thermal components appear due to the thermal processes being taken into account. The following expressions result from (6.117) and (6.118).

$$\boxed{\boldsymbol{\varepsilon}^{nt} = \mathbb{C}^{-1} : \boldsymbol{\sigma} \quad \Longrightarrow \quad \boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon}^{nt} = \mathbb{C} : \left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{t}\right)} \tag{6.119}$$

$$\boxed{\boldsymbol{\sigma}^{nt} = \mathbb{C} : \boldsymbol{\varepsilon} \quad \Longrightarrow \quad \boldsymbol{\varepsilon} = \mathbb{C}^{-1} : \boldsymbol{\sigma}^{nt} = \mathbb{C}^{-1} : \left(\boldsymbol{\sigma} + \boldsymbol{\sigma}^{t}\right)} \tag{6.120}$$

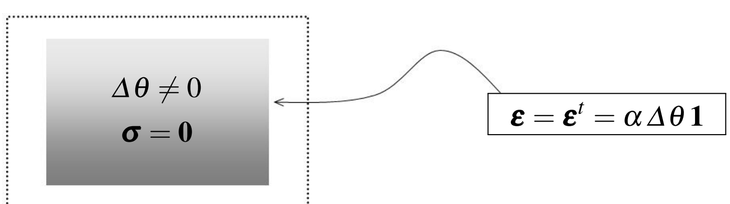
Remark 6.11. Unlike what occurs in elasticity, in the thermoelastic case *a state of null strain* in a point of a medium *does not imply a state of null stress* in said point. In effect, for $\boldsymbol{\varepsilon} = \mathbf{0}$ in (6.117),

$$\boldsymbol{\varepsilon} = \mathbf{0} \implies \boldsymbol{\sigma}^{nt} = \mathbf{0} \implies \boldsymbol{\sigma} = -\boldsymbol{\sigma}^{t} = -\beta\,\Delta\theta\,\mathbf{1} \neq \mathbf{0}\,.$$



Remark 6.12. Analogously, in thermoelasticity *a state of null stress* in a point of a medium *does not imply a state of null strain* in said point since (6.118) with $\boldsymbol{\sigma} = \mathbf{0}$ yields

$$\boldsymbol{\sigma} = \mathbf{0} \implies \boldsymbol{\varepsilon}^{nt} = \mathbf{0} \implies \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{t} = \alpha\,\Delta\theta\,\mathbf{1} \neq \mathbf{0}\,.$$



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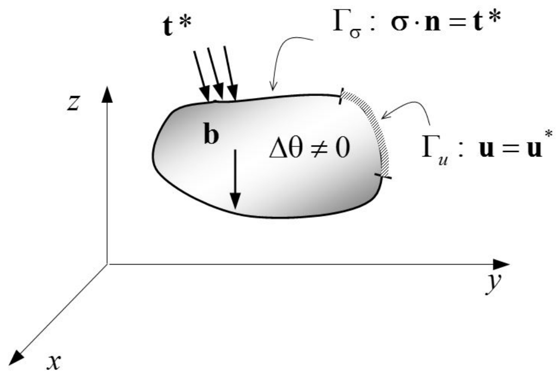

Figure 6.12: Actions on a continuous medium.

6.11 Thermal Analogies

The thermal analogies arise from the search of procedures to solve the linear *thermoelastic* problem using the strategies and methodologies developed in Section 6.7 for the linear *elastic* problem (without considering thermal effects).

Two analogies are presented in this section which, for the sake of simplicity, are restricted to the isotropic quasi-static problem, although they can be directly extrapolated to the general anisotropic dynamic problem.

6.11.1 First Thermal Analogy (Duhamel-Newman Analogy)

Consider the continuous medium in Figure 6.12 on which the body forces $\mathbf{b}(\mathbf{x},t)$ and an increment of temperature $\Delta\theta(\mathbf{x},t)$ are acting, and on whose boundaries Γ_u and Γ_σ act the prescribed displacements $\mathbf{u}^*(\mathbf{x},t)$ and a traction vector $\mathbf{t}^*(\mathbf{x},t)$, respectively.

The equations of the (isotropic quasi-static) linear thermoelastic problem are

$$
\begin{array}{|c|ll|}
\hline
\begin{array}{c}\text{Governing}\\\text{equations}\end{array} & \left\{\begin{array}{l}\nabla\cdot\boldsymbol{\sigma}+\rho_0\mathbf{b}=\mathbf{0}\\[1ex] \boldsymbol{\sigma}=\mathbb{C}:\boldsymbol{\varepsilon}-\beta\,\Delta\theta\,\mathbf{1}\\[1ex] \boldsymbol{\varepsilon}=\nabla^S\mathbf{u}\end{array}\right. & \begin{array}{l}\text{Equilibrium equation}\\[1ex]\text{Constitutive equation}\\[1ex]\text{Geometric equation}\end{array}\\
\hline
\begin{array}{c}\text{Boundary}\\\text{conditions}\end{array} & \left\{\begin{array}{l}\Gamma_u:\mathbf{u}=\mathbf{u}^*\\ \Gamma_\sigma:\boldsymbol{\sigma}\cdot\mathbf{n}=\mathbf{t}^*\end{array}\right. & \\
\hline
\end{array}
\qquad (6.121)
$$

which compose the actions (data) $\mathbb{A}(\mathbf{x},t)$ and responses (unknowns) $\mathbb{R}(\mathbf{x},t)$ of the problem[24].

$$\boxed{\underbrace{\left.\begin{array}{c}\mathbf{b}(\mathbf{x},t)\\ \mathbf{u}^{*}(\mathbf{x},t)\\ \mathbf{t}^{*}(\mathbf{x},t)\\ \Delta\theta(\mathbf{x},t)\end{array}\right\}}_{\text{Actions}\stackrel{not}{=}\mathbb{A}^{(I)}(\mathbf{x},t)}\Rightarrow\left\langle\begin{array}{c}MATHEMATICAL\\ MODEL:\\ PDEs+BCs\end{array}\right\rangle\Rightarrow\underbrace{\left\{\begin{array}{l}\mathbf{u}(\mathbf{x},t)\\ \boldsymbol{\varepsilon}(\mathbf{x},t)\\ \boldsymbol{\sigma}(\mathbf{x},t)\end{array}\right.}_{\text{Responses}\stackrel{not}{=}\mathbb{R}^{(I)}(\mathbf{x},t)}} \tag{6.122}$$

To be able to apply the resolution methods typical of the liner elastic problem developed in Section 6.7, the thermal term in the equations of the thermoelastic problem (6.121) must be eliminated (at least, in appearance). To this aim, the decomposition of the stress tensor $\boldsymbol{\sigma}=\boldsymbol{\sigma}^{nt}-\boldsymbol{\sigma}^{t}$ is replaced in (6.121) as follows.

a) Equilibrium equation

$$\begin{array}{c}\boldsymbol{\sigma}=\boldsymbol{\sigma}^{nt}-\boldsymbol{\sigma}^{t}\implies\\ \nabla\cdot\boldsymbol{\sigma}=\nabla\cdot\boldsymbol{\sigma}^{nt}-\nabla\cdot\underbrace{\boldsymbol{\sigma}^{t}}_{\beta\,\Delta\theta\mathbf{1}}=\nabla\cdot\boldsymbol{\sigma}^{nt}-\nabla(\beta\,\Delta\theta)\end{array} \tag{6.123}$$

$$\begin{array}{l}\nabla\cdot\boldsymbol{\sigma}+\rho_0\mathbf{b}=\mathbf{0}\quad\implies\quad\nabla\cdot\boldsymbol{\sigma}^{nt}+\rho_0\underbrace{\left(\mathbf{b}-\dfrac{1}{\rho_0}\nabla(\beta\,\Delta\theta)\right)}_{\stackrel{not}{=}\hat{\mathbf{b}}}=\mathbf{0}\\ \qquad\qquad\implies\quad\boxed{\nabla\cdot\boldsymbol{\sigma}^{nt}+\hat{\mathbf{b}}=\mathbf{0}}\end{array} \tag{6.124}$$

which constitutes the equilibrium equation of the medium subjected to the *pseudo-body forces* $\hat{\mathbf{b}}(\mathbf{x},t)$ defined by

$$\begin{cases}\hat{\mathbf{b}}(\mathbf{x},t)=\mathbf{b}(\mathbf{x},t)-\dfrac{1}{\rho_0}\nabla(\beta\,\Delta\theta)\\ \hat{b}_i(\mathbf{x},t)=b_i(\mathbf{x},t)-\dfrac{1}{\rho_0}\dfrac{\partial(\beta\,\Delta\theta)}{\partial x_i}\qquad i\in\{1,2,3\}\end{cases} \tag{6.125}$$

b) Constitutive equation

$$\boxed{\boldsymbol{\sigma}^{nt}=\mathbb{C}:\boldsymbol{\varepsilon}=\lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\mathbf{1}+2\mu\boldsymbol{\varepsilon}} \tag{6.126}$$

[24] The field of thermal increments $\Delta\theta(\mathbf{x},t)$ is assumed to be known *a priori* and, therefore, independent of the mechanical response of the problem. This situation is known as the *uncoupled* thermomechanical problem.

c) Geometric equation (remains unchanged)

$$\boxed{\boldsymbol{\varepsilon} = \nabla^2 \mathbf{u}} \tag{6.127}$$

d) Boundary condition in Γ_u

$$\boxed{\Gamma_u : \mathbf{u} = \mathbf{u}^*} \tag{6.128}$$

e) Boundary condition in Γ_σ

$$\begin{aligned}
&\left.\begin{array}{l} \boldsymbol{\sigma} = \boldsymbol{\sigma}^{nt} - \boldsymbol{\sigma}^{t} \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t}^* \end{array}\right\} \Longrightarrow \quad \boldsymbol{\sigma}^{nt} \cdot \mathbf{n} - \boldsymbol{\sigma}^{t} \cdot \mathbf{n} = \mathbf{t}^* \quad \Longrightarrow \\
&\boldsymbol{\sigma}^{nt} \cdot \mathbf{n} = \mathbf{t}^* + \underbrace{\boldsymbol{\sigma}^{t} \cdot \mathbf{n}}_{\beta\,\Delta\theta\,\mathbf{1}\cdot\mathbf{n}} = \underbrace{\mathbf{t}^* + \beta\,\Delta\theta\,\mathbf{n}}_{\hat{\mathbf{t}}^*} \quad \Longrightarrow \quad \boxed{\Gamma_\sigma : \boldsymbol{\sigma}^{nt} \cdot \mathbf{n} = \hat{\mathbf{t}}^*}
\end{aligned} \tag{6.129}$$

where $\hat{\mathbf{t}}^*(\mathbf{x},t)$ is a *pseudo-traction vector* defined by

$$\boxed{\hat{\mathbf{t}}^* = \mathbf{t}^* + \beta\,\Delta\theta\,\mathbf{n}}\ . \tag{6.130}$$

Equations (6.123) to (6.130) allow rewriting the original problem (6.121) as

$$\boxed{\begin{array}{ll|}
\begin{array}{c}\text{Governing}\\ \text{equations}\end{array} & \left\{\begin{array}{ll}
\nabla \cdot \boldsymbol{\sigma}^{nt} + \rho_0 \hat{\mathbf{b}} = \mathbf{0} & \\
\text{with } \hat{\mathbf{b}} = \mathbf{b} - \dfrac{1}{\rho_0}\nabla(\beta\,\Delta\theta) & \begin{array}{c}\text{Equilibrium}\\ \text{equation}\end{array} \\[2ex]
\boldsymbol{\sigma}^{nt} = \mathbb{C} : \boldsymbol{\varepsilon} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\,\mathbf{1} + 2\mu\boldsymbol{\varepsilon} & \begin{array}{c}\text{Constitutive}\\ \text{equation}\end{array} \\[2ex]
\boldsymbol{\varepsilon} = \nabla^S \mathbf{u} & \begin{array}{c}\text{Geometric}\\ \text{equation}\end{array}
\end{array}\right. \\ \hline
\begin{array}{c}\text{Boundary}\\ \text{conditions}\end{array} & \left\{\begin{array}{l}
\Gamma_u : \mathbf{u} = \mathbf{u}^* \\
\Gamma_\sigma : \boldsymbol{\sigma}^{nt} \cdot \mathbf{n} = \hat{\mathbf{t}}^* \text{ with } \hat{\mathbf{t}} = \mathbf{t} + \beta\,\Delta\theta\,\mathbf{n}
\end{array}\right.
\end{array}} \tag{6.131}$$

which constitutes the so-called *analogous problem*, a linear elastic problem that can be solved with the methodology indicated for this type of problems in Section 6.7 and characterized by the following actions and responses.

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$$\underbrace{\left.\begin{array}{c}\hat{\mathbf{b}}(\mathbf{x},t)\\ \mathbf{u}^*(\mathbf{x},t)\\ \hat{\mathbf{t}}^*(\mathbf{x},t)\end{array}\right\}}_{\text{Actions}\overset{not}{=}\mathbb{A}^{(II)}(\mathbf{x},t)} \Rightarrow \left\langle\begin{array}{c} MATHEMATICAL\\ MODEL:\\ PDEs+BCs\end{array}\right\rangle \Rightarrow \underbrace{\left\{\begin{array}{c}\mathbf{u}(\mathbf{x},t)\\ \boldsymbol{\varepsilon}(\mathbf{x},t)\\ \boldsymbol{\sigma}^{nt}(\mathbf{x},t)\end{array}\right.}_{\text{Responses}\overset{not}{=}\mathbb{R}^{(II)}(\mathbf{x},t)} \tag{6.132}$$

Comparing the actions and responses of the original problem (6.122) with those of the analogous problem (6.132), reveals the difference between them to be

$$\begin{aligned}
&\mathbb{A}^{(I)}-\mathbb{A}^{(II)} \overset{not}{\equiv} \begin{bmatrix}\mathbf{b}\\ \mathbf{u}^*\\ \mathbf{t}^*\\ \Delta\theta\end{bmatrix} - \begin{bmatrix}\hat{\mathbf{b}}\\ \mathbf{u}^*\\ \hat{\mathbf{t}}^*\\ 0\end{bmatrix} = \begin{bmatrix}\mathbf{b}-\hat{\mathbf{b}}\\ \mathbf{0}\\ \mathbf{t}^*-\hat{\mathbf{t}}^*\\ \Delta\theta\end{bmatrix} = \begin{bmatrix}\dfrac{1}{\rho_0}\nabla(\beta\,\Delta\theta)\\ \mathbf{0}\\ -\beta\,\Delta\theta\,\mathbf{n}\\ \Delta\theta\end{bmatrix} \overset{def}{=} \mathbb{A}^{(III)}(\mathbf{x},t)\\
&\mathbb{R}^{(I)}-\mathbb{R}^{(II)} \overset{not}{\equiv} \begin{bmatrix}\mathbf{u}\\ \boldsymbol{\varepsilon}\\ \boldsymbol{\sigma}\end{bmatrix} - \begin{bmatrix}\mathbf{u}\\ \boldsymbol{\varepsilon}\\ \boldsymbol{\sigma}^{nt}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ \underbrace{\boldsymbol{\sigma}-\boldsymbol{\sigma}^{nt}}_{-\boldsymbol{\sigma}^{t}}\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ -\beta\,\Delta\theta\,\mathbf{1}\end{bmatrix} \overset{def}{=} \mathbb{R}^{(III)}(\mathbf{x},t)
\end{aligned} \tag{6.133}$$

where (6.130) and (6.117) have been taken into account.

> *Remark 6.13.* It can be directly verified that, in (6.133), $\mathbb{R}^{(III)}$ is the response corresponding to the system of actions $\mathbb{A}^{(III)}$ in the thermoelastic problem (6.121).

Equation (6.133) suggests that the original problem (I) may be interpreted as the sum (superposition) of two problems or states:

STATE (II) (to be solved): *analogous elastic* state in which the temperature does not intervene and that can be solved by means of *elastic* procedures.

+

STATE (III) (trivial): trivial *thermoelastic* state in which the responses $\mathbb{R}^{(III)}(\mathbf{x})$ given in (6.133) are known without the need of any calculations.

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Once *STATE* (II) is computed, the solution to the original thermoelastic problem of *STATE* (I) is obtained as

$$\boxed{\text{Solution to the original thermoelastic problem}\quad \begin{cases} \mathbf{u}^{(I)} = \mathbf{u}^{(II)} \\ \boldsymbol{\varepsilon}^{(I)} = \boldsymbol{\varepsilon}^{(II)} \\ \boldsymbol{\sigma}^{(I)} = \boldsymbol{\sigma}^{(II)} - \beta\,\Delta\theta\,\mathbf{1} \end{cases}} \tag{6.134}$$

The procedure to solve the thermoelastic problem based on the first thermal analogy is summarized as a superposition of states in Figure 6.13.

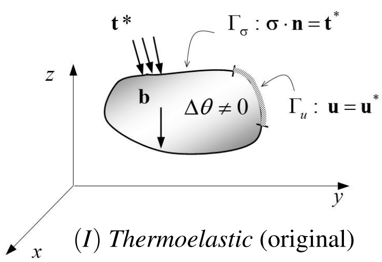
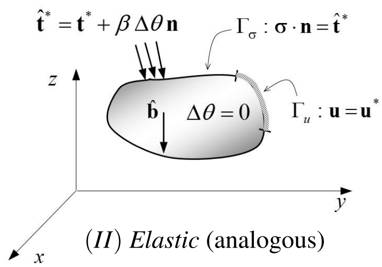
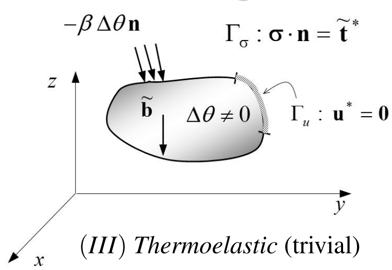
STATE	ACTION	RESPONSE
 (I) *Thermoelastic* (original)	$\begin{bmatrix} \mathbf{b}(\mathbf{x},t) \\ \mathbf{u}^*(\mathbf{x},t) \\ \mathbf{t}^*(\mathbf{x},t) \\ \Delta\theta(\mathbf{x},t) \end{bmatrix}$	$\begin{bmatrix} \mathbf{u}(\mathbf{x},t) \\ \boldsymbol{\varepsilon}(\mathbf{x},t) \\ \boldsymbol{\sigma}(\mathbf{x},t) \end{bmatrix}$
 (II) *Elastic* (analogous)	$\begin{bmatrix} \hat{\mathbf{b}} = \mathbf{b} - \dfrac{1}{\rho_0}\nabla(\beta\,\Delta\theta) \\ \mathbf{u}^*(\mathbf{x},t) \\ \hat{\mathbf{t}}^* = \mathbf{t}^* + \beta\,\Delta\theta\,\mathbf{n} \\ \Delta\theta = 0 \end{bmatrix}$	$\begin{bmatrix} \mathbf{u}(\mathbf{x},t) \\ \boldsymbol{\varepsilon}(\mathbf{x},t) \\ \boldsymbol{\sigma}^{nt}(\mathbf{x},t) \end{bmatrix}$
 (III) *Thermoelastic* (trivial)	$\begin{bmatrix} \tilde{\mathbf{b}} = \dfrac{1}{\rho_0}\nabla(\beta\,\Delta\theta) \\ \tilde{\mathbf{u}}^* = \mathbf{0} \\ \tilde{\mathbf{t}}^* = -\beta\,\Delta\theta\,\mathbf{n} \\ \Delta\theta(\mathbf{x},t) \end{bmatrix}$	$\begin{bmatrix} \mathbf{u} = \mathbf{0} \\ \boldsymbol{\varepsilon} = \mathbf{0} \\ \boldsymbol{\sigma} = -\beta\,\Delta\theta\,\mathbf{1} \end{bmatrix}$

Figure 6.13: First thermal analogy.

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6.11.2 Second Thermal Analogy

The second thermal analogy is based on expressing the equations that constitute the problem in terms of the thermal strains $\boldsymbol{\varepsilon}^t$ defined in (6.118). Consider the equations of the original thermoelastic problem, with the constitutive equation in its inverse form

$$
\begin{array}{|ll|}
\hline
\text{Governing equations} & \begin{cases} \nabla\cdot\boldsymbol{\sigma}+\rho_0\mathbf{b}=\mathbf{0} & \text{Equilibrium equation} \\ \boldsymbol{\varepsilon}=\mathbb{C}^{-1}:\boldsymbol{\sigma}+\alpha\,\Delta\theta\,\mathbf{1} & \text{Inverse constitutive equation} \\ \boldsymbol{\varepsilon}=\nabla^S\mathbf{u} & \text{Geometric equation} \end{cases} \\
\hline
\text{Boundary conditions} & \begin{cases} \Gamma_u:\mathbf{u}=\mathbf{u}^* \\ \Gamma_\sigma:\boldsymbol{\sigma}\cdot\mathbf{n}=\mathbf{t}^* \end{cases} \\
\hline
\end{array}
\tag{6.135}
$$

which constitute the actions (data) $\mathbb{A}(\mathbf{x},t)$ and responses (unknowns) $\mathbb{R}(\mathbf{x},t)$ of the problem.

$$
\underbrace{\left.\begin{array}{c}\mathbf{b}(\mathbf{x},t)\\ \mathbf{u}^*(\mathbf{x},t)\\ \mathbf{t}^*(\mathbf{x},t)\\ \Delta\theta(\mathbf{x},t)\end{array}\right\}}_{\text{Actions} \overset{not}{=} \mathbb{A}^{(I)}(\mathbf{x},t)} \Rightarrow \left\langle\begin{array}{c} MATHEMATICAL \\ MODEL: \\ PDEs+BCs\end{array}\right\rangle \Rightarrow \underbrace{\left\{\begin{array}{c}\mathbf{u}(\mathbf{x},t)\\ \boldsymbol{\varepsilon}(\mathbf{x},t)\\ \boldsymbol{\sigma}(\mathbf{x},t)\end{array}\right.}_{\text{Responses} \overset{not}{=} \mathbb{R}^{(I)}(\mathbf{x},t)}
\tag{6.136}
$$

Hypothesis 6.1. *Assume that the coefficient of thermal expansion $\alpha(\mathbf{x})$ and the thermal increment $\Delta\theta(\mathbf{x},t)$ are such that the thermal strain field*

$$\boldsymbol{\varepsilon}^t(\mathbf{x},t)=\alpha(\mathbf{x})\,\Delta\theta(\mathbf{x},t)\,\mathbf{1}$$

is integrable (satisfies the compatibility conditions).

Consequently, there exists a thermal displacement field $\mathbf{u}^t(\mathbf{x},t)$ that satisfies

$$
\begin{cases}
\boldsymbol{\varepsilon}^t(\mathbf{x},t)=\alpha\,\Delta\theta\,\mathbf{1}=\nabla^S\mathbf{u}^t=\dfrac{1}{2}\left(\mathbf{u}^t\otimes\nabla+\nabla\otimes\mathbf{u}^t\right) \\
\varepsilon^t_{ij}=\alpha\,\Delta\theta\,\delta_{ij}=\dfrac{1}{2}\left(\dfrac{\partial u^t_i}{\partial x_j}+\dfrac{\partial u^t_j}{\partial x_i}\right) \qquad i,j\in\{1,2,3\}
\end{cases}
\tag{6.137}
$$

doi:10.13140/RG.2.2.25821.20961

Remark 6.14. The solution $\mathbf{u}^t(\mathbf{x},t)$ to the system of differential equations (6.137) exists if and only if the strain field $\boldsymbol{\varepsilon}^t(\mathbf{x},t)$ satisfies the compatibility conditions (see Chapter 3). In addition, this solution is determined except for a rigid body motion characterized by a rotation tensor $\boldsymbol{\Omega}^*$ and a displacement vector $\mathbf{c}^*$ (both constant). That is, there exists a family of admissible solutions of the form

$$\mathbf{u}^t(\mathbf{x},t) = \widetilde{\mathbf{u}}(\mathbf{x},t) + \underbrace{\underbrace{\boldsymbol{\Omega}^* \cdot \mathbf{x}}_{\text{rotation}} + \underbrace{\mathbf{c}^*}_{\text{translation}}}_{\text{rigid body motion}} \ .$$

The rigid body motion may be chosen *arbitrarily* (in the form which is most convenient for the resolution process).

Once the thermal displacements have been defined, a decomposition of the total displacements into their thermal and non-thermal parts can be performed as follows.

$$\mathbf{u}^{nt}(\mathbf{x},t) \stackrel{def}{=} \mathbf{u}(\mathbf{x},t) - \mathbf{u}^t(\mathbf{x},t) \quad \Longrightarrow \quad \boxed{\mathbf{u} = \mathbf{u}^{nt} + \mathbf{u}^t} \tag{6.138}$$

To eliminate the thermal term in the equations that constitute the thermoelastic problem (6.135), the decompositions of the displacements and strains ($\mathbf{u} = \mathbf{u}^{nt} + \mathbf{u}^t$ and $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{nt} + \boldsymbol{\varepsilon}^t$) is introduced in the equations of (6.135), which result in

a) Equilibrium equation (remains unchanged)

$$\boxed{\nabla \cdot \boldsymbol{\sigma} + \rho_0 \mathbf{b} = \mathbf{0}} \tag{6.139}$$

b) Inverse constitutive equation

$$\boxed{\boldsymbol{\varepsilon}^{nt} = \mathbb{C}^{-1} : \boldsymbol{\sigma} = -\frac{\nu}{E}\,\mathrm{Tr}(\boldsymbol{\sigma})\,\mathbf{1} + \frac{1+\nu}{E}\boldsymbol{\sigma}} \tag{6.140}$$

c) Geometric equation

$$\left.\begin{array}{l} \boldsymbol{\varepsilon} = \nabla^S \mathbf{u} = \nabla^S(\mathbf{u}^{nt} + \mathbf{u}^t) = \nabla^S \mathbf{u}^{nt} + \underbrace{\nabla^S \mathbf{u}^t}_{\boldsymbol{\varepsilon}^t} = \nabla^S \mathbf{u}^{nt} + \boldsymbol{\varepsilon}^t \\ \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{nt} + \boldsymbol{\varepsilon}^t \end{array}\right\} \Longrightarrow \boxed{\boldsymbol{\varepsilon}^{nt} = \nabla^S \mathbf{u}^{nt}} \tag{6.141}$$

d) Boundary condition in Γ_u

$$\left.\begin{array}{l}\mathbf{u}=\mathbf{u}^* \\ \mathbf{u}=\mathbf{u}^{nt}+\mathbf{u}^t\end{array}\right\} \implies \boxed{\Gamma_u : \mathbf{u}^{nt}=\mathbf{u}^*-\mathbf{u}^t} \tag{6.142}$$

e) Boundary condition in Γ_σ (remains unchanged)

$$\boxed{\Gamma_\sigma : \boldsymbol{\sigma}\cdot\mathbf{n}=\mathbf{t}^*} \tag{6.143}$$

Equations (6.139) to (6.143) allow rewriting the original problem (6.135) as

$$\boxed{\begin{array}{l|}
\text{Governing equations} \begin{cases} \nabla\cdot\boldsymbol{\sigma}+\rho_0\mathbf{b}=\mathbf{0} & \text{Equilibrium equation} \\ \boldsymbol{\varepsilon}^{nt}=\mathbb{C}^{-1}:\boldsymbol{\sigma}= \\ \quad =-\dfrac{\nu}{E}\operatorname{Tr}(\boldsymbol{\sigma})\,\mathbf{1}+\dfrac{1+\nu}{E}\boldsymbol{\sigma} & \text{Constitutive equation} \\ \boldsymbol{\varepsilon}^{nt}=\nabla^S\mathbf{u}^{nt} & \text{Geometric equation} \end{cases} \\ \hline
\text{Boundary conditions} \begin{cases} \Gamma_u : \mathbf{u}=\mathbf{u}^*-\mathbf{u}^t \\ \Gamma_\sigma : \boldsymbol{\sigma}\cdot\mathbf{n}=\mathbf{t}^* \end{cases}
\end{array}} \tag{6.144}$$

which constitutes the so-called *analogous problem*, a linear elastic problem characterized by the following actions and responses

$$\boxed{\underbrace{\left.\begin{array}{l}\hat{\mathbf{b}}(\mathbf{x},t) \\ \mathbf{u}^*(\mathbf{x},t)-\mathbf{u}^t(\mathbf{x},t) \\ \hat{\mathbf{t}}^*(\mathbf{x},t)\end{array}\right\}}_{\text{Actions} \stackrel{not}{=} \mathbb{A}^{(II)}(\mathbf{x},t)} \Rightarrow \left\langle\begin{array}{c} MATHEM. \\ MODEL: \\ PDEs+BCs \end{array}\right\rangle \Rightarrow \underbrace{\left\{\begin{array}{l}\mathbf{u}^{nt}(\mathbf{x},t) \\ \boldsymbol{\varepsilon}^{nt}(\mathbf{x},t) \\ \boldsymbol{\sigma}(\mathbf{x},t)\end{array}\right.}_{\text{Responses} \stackrel{not}{=} \mathbb{R}^{(II)}(\mathbf{x},t)}} \tag{6.145}$$

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Comparing the actions and responses of the original problem (6.136) and the analogous problem (6.145), reveals the difference between them to be

$$\begin{array}{l}
\mathbb{A}^{(I)} - \mathbb{A}^{(II)} \stackrel{not}{\equiv} \begin{bmatrix} \mathbf{b} \\ \mathbf{u}^* \\ \mathbf{t}^* \\ \Delta\theta \end{bmatrix} - \begin{bmatrix} \mathbf{b} \\ \mathbf{u}^* - \mathbf{u}^t \\ \mathbf{t}^* \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{u}^t \\ \mathbf{0} \\ \Delta\theta \end{bmatrix} \stackrel{def}{=} \mathbb{A}^{(III)}(\mathbf{x},t) \\
\mathbb{R}^{(I)} - \mathbb{R}^{(II)} \stackrel{not}{\equiv} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\varepsilon} \\ \boldsymbol{\sigma} \end{bmatrix} - \begin{bmatrix} \mathbf{u}^{nt} \\ \boldsymbol{\varepsilon}^{nt} \\ \boldsymbol{\sigma} \end{bmatrix} = \begin{bmatrix} \mathbf{u}^t \\ \boldsymbol{\varepsilon}^t \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{u}^t \\ \alpha\Delta\theta\mathbf{1} \\ \mathbf{0} \end{bmatrix} \stackrel{def}{=} \mathbb{R}^{(III)}(\mathbf{x},t)
\end{array} \qquad (6.146)$$

where equations (6.138) and (6.118) have been taken into account.

Remark 6.15. It can be directly verified that, in (6.146), $\mathbb{R}^{(III)}$ is the response corresponding to the system of actions $\mathbb{A}^{(III)}$ in the thermoelastic problem (6.135).

Therefore, the original problem (I) can be interpreted as the sum (superposition) of two problems or states:

STATE (II) (to be solved): *analogous elastic* state in which the temperature does not intervene and that can be solved by means of *elastic* procedures.

+

STATE (III) (trivial): trivial *thermoelastic* state in which the responses $\mathbb{R}^{(III)}(\mathbf{x})$ given in (6.146) are known without the need of any calculations.

Once *STATE* (II) is computed, the solution to the original thermoelastic problem of *STATE* (I) is obtained as

$$\text{Solution to the original thermoelastic problem} \quad \begin{cases} \mathbf{u}^{(I)} = \mathbf{u}^{(II)} + \mathbf{u}^t \\ \boldsymbol{\varepsilon}^{(I)} = \boldsymbol{\varepsilon}^{(II)} + \alpha\Delta\theta\mathbf{1} \\ \boldsymbol{\sigma}^{(I)} = \boldsymbol{\sigma}^{(II)} \end{cases} \qquad (6.147)$$

where $\mathbf{u}^t$ is known from the integration process of the thermal strain field in (6.137). The procedure to solve the thermoelastic problem based on the second thermal analogy is summarized as a superposition of states in Figure 6.14.

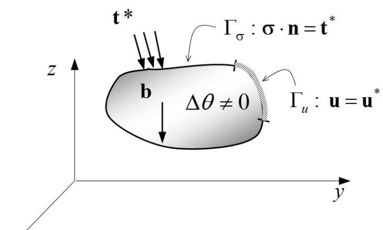
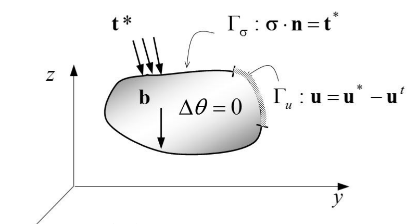
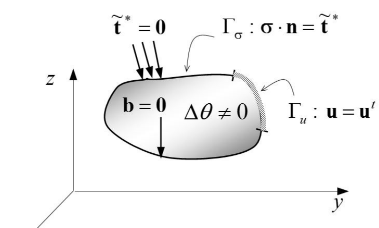
STATE	ACTION	RESPONSE
 (I) *Thermoelastic* (original)	$\begin{bmatrix}\mathbf{b}(\mathbf{x},t)\\ \mathbf{u}^*(\mathbf{x},t)\\ \mathbf{t}^*(\mathbf{x},t)\\ \Delta\theta(\mathbf{x},t)\end{bmatrix}$	$\begin{bmatrix}\mathbf{u}(\mathbf{x},t)\\ \boldsymbol{\varepsilon}(\mathbf{x},t)\\ \boldsymbol{\sigma}(\mathbf{x},t)\end{bmatrix}$
 (II) *Elastic* (analogous)	$\begin{bmatrix}\mathbf{b}(\mathbf{x},t)\\ \mathbf{u}^*-\mathbf{u}^t\\ \mathbf{t}^*(\mathbf{x},t)\\ \Delta\theta=0\end{bmatrix}$	$\begin{bmatrix}\mathbf{u}^{nt}(\mathbf{x},t)\\ \boldsymbol{\varepsilon}^{nt}(\mathbf{x},t)\\ \boldsymbol{\sigma}(\mathbf{x},t)\end{bmatrix}$
 (III) *Thermoelastic* (trivial)	$\begin{bmatrix}\mathbf{b}=\mathbf{0}\\ \tilde{\mathbf{u}}^*=\mathbf{u}^t\\ \tilde{\mathbf{t}}^*=\mathbf{0}\\ \Delta\theta(\mathbf{x},t)\end{bmatrix}$	$\begin{bmatrix}\mathbf{u}=\mathbf{u}^t(\mathbf{x},t)\\ \boldsymbol{\varepsilon}=\alpha\Delta\theta\mathbf{1}\\ \boldsymbol{\sigma}=\mathbf{0}\end{bmatrix}$

Figure 6.14: Second thermal analogy.

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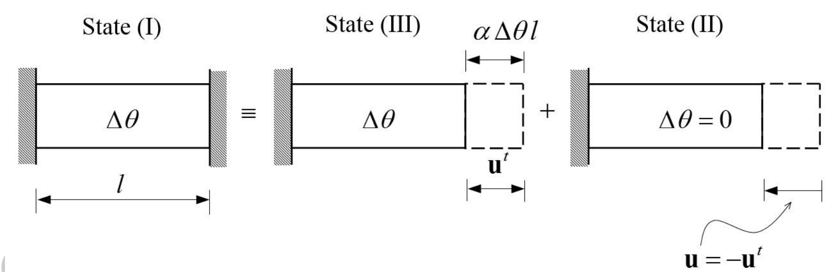
Example 6.5 – *Solve the problem of a beam fully-fixed at its ends and subjected to a constant thermal increment $\Delta\theta$ using the second thermal analogy.*

Solution

The classic procedure followed in strength of materials to solve this problem consists in the superposition (sum) of the following situations: 1) The structure is initially considered to be hyperstatic; 2) the right end is *freed* to allow for thermal expansion, which takes place with null stresses (since it is an isostatic structure); and 3) the displacement of the beam's right end is *recovered* until it is brought again to zero.
This procedure coincides exactly with the application of the second thermal analogy in which the thermal displacement field $\mathbf{u}^t$ is defined by the thermal expansion of the piece with its right end freed (state III). Said expansion produces a displacement in the right end of value $u|_{x=\ell} = \alpha\,\Delta\theta\ell$ and, when *recovering* the displacement at this end, the boundary condition

$$\Gamma_u : \mathbf{u} = \underbrace{\mathbf{u}^*}_{\mathbf{0}} - \mathbf{u}^t = -\mathbf{u}^t ,$$

which corresponds exactly to state II of Figure 6.14, is being implicitly applied.



Remark 6.16. The application of the second thermal analogy essentially resides in the integration of the thermal strain field $\boldsymbol{\varepsilon}^t(\mathbf{x},t)$ to obtain the thermal displacement field $\mathbf{u}^t(\mathbf{x},t)$ (see Remark 6.14). If the thermal strains are not integrable, the analogy cannot be applied. Comparing its advantages and disadvantages with respect to the first thermal analogy, it is also recommended that the integration of the thermal strains be, in addition to possible, *simple* to perform.

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Remark 6.17. The case involving

- a *homogeneous material* $(\boldsymbol{\alpha}(\mathbf{x}) = const. = \boldsymbol{\alpha})$
- a *linear thermal increment* $(\Delta\theta = ax + by + cz + d)$

is of particular interest. In this case, the product $\Delta\theta\boldsymbol{\alpha}$ is a linear polynomial and the thermal strains $\boldsymbol{\varepsilon}^t = \Delta\theta\boldsymbol{\alpha}$ automatically satisfy the compatibility conditions (6.69) (which are equations that only contain second-order derivatives) and, therefore, the *thermal strain field is guaranteed to be integrable*.

Remark 6.18. In the case involving

- a *homogeneous material* $(\boldsymbol{\alpha}(\mathbf{x}) = const. = \alpha)$
- a *constant thermal increment* $(\Delta\theta = const.)$

the integration of the thermal strain field $\boldsymbol{\varepsilon}^t = \Delta\theta\alpha\mathbf{1} = const.$ is trivial, resulting in

$$\mathbf{u}^t(\mathbf{x},t) = \alpha\Delta\theta\,\mathbf{x} + \underbrace{\boldsymbol{\Omega}^* \cdot \mathbf{x} + \mathbf{c}^*}_{\text{rigid body motion}} ,$$

where the rigid body motion can be chosen arbitrarily (see Remark 6.14). If this motion is considered to be null, the solution to the thermal displacement field is

$$\mathbf{u}^t(\mathbf{x},t) = \alpha\Delta\theta\,\mathbf{x} \quad \Longrightarrow \quad \mathbf{x} + \mathbf{u}^t = \mathbf{x} + \alpha\Delta\theta\,\mathbf{x} = (1 + \alpha\Delta\theta)\,\mathbf{x} ,$$

which means that *STATE* (III) in the second thermal analogy (see Figure 6.15) *is an homothecy, with respect to the origin of coordinates, of value* $(1 + \alpha\Delta\theta)$. This homothecy is known as *free thermal expansion* (see Figure 6.15).
The value of the thermal displacement (associated with the free thermal expansion) in the boundary Γ_u can be trivially determined in this case without need of *formally* integrating the thermal strains.

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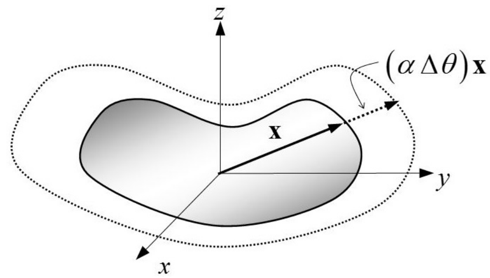

Figure 6.15: Free thermal expansion in a homogeneous material subjected to a constant thermal increment.

6.12 Superposition Principle in Linear Thermoelasticity

Consider the linear thermoelastic problem in Figure 6.16 and its corresponding governing equations

$$
\begin{array}{ll}
\nabla\cdot\boldsymbol{\sigma}+\rho_0\mathbf{b}=\rho_0\dfrac{\partial^2\mathbf{u}}{\partial t^2} & \text{Cauchy's equation} \\[2ex]
\boldsymbol{\sigma}=\underbrace{\lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\,\mathbf{1}+2\mu\boldsymbol{\varepsilon}}_{\mathbb{C}:\boldsymbol{\varepsilon}}-\beta\,\Delta\theta\,\mathbf{1} & \text{Constitutive equation} \\[2ex]
\boldsymbol{\varepsilon}=\nabla^S\mathbf{u}=\dfrac{1}{2}(\mathbf{u}\otimes\nabla+\nabla\otimes\mathbf{u}) & \text{Geometric equation}
\end{array}
\qquad (6.148)
$$

$$
\left.\begin{array}{l}
\Gamma_u:\mathbf{u}=\mathbf{u}^* \\
\Gamma_\sigma:\mathbf{t}^*=\boldsymbol{\sigma}\cdot\mathbf{n}
\end{array}\right\}\quad \text{Boundary conditions in space}
\qquad (6.149)
$$

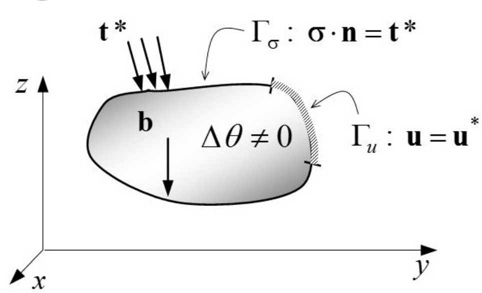

Figure 6.16: Linear thermoelastic problem.

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$$\boxed{\left.\begin{array}{l}\mathbf{u}(\mathbf{x},0)=\mathbf{0}\\ \dot{\mathbf{u}}(\mathbf{x},0)=\mathbf{v}_0\end{array}\right\}\qquad \text{Initial conditions}} \qquad (6.150)$$

which define the generic set of actions and responses

$$\boxed{\underbrace{\left.\begin{array}{l}\hat{\mathbf{b}}(\mathbf{x},t)\\ \mathbf{u}^*(\mathbf{x},t)\\ \mathbf{t}^*(\mathbf{x},t)\\ \Delta\theta(\mathbf{x},t)\\ \mathbf{v}_0(\mathbf{x})\end{array}\right\}}_{\text{Actions}\overset{not}{=}\mathbb{A}(\mathbf{x},t)} \Rightarrow \left\langle\begin{array}{c}MATHEMATICAL\\ MODEL:\\ PDEs+BCs\end{array}\right\rangle \Rightarrow \underbrace{\left\{\begin{array}{l}\mathbf{u}(\mathbf{x},t)\\ \boldsymbol{\varepsilon}(\mathbf{x},t)\\ \boldsymbol{\sigma}(\mathbf{x},t)\end{array}\right.}_{\text{Responses}\overset{not}{=}\mathbb{R}(\mathbf{x},t)}} \qquad (6.151)$$

Remark 6.19. The different (scalar, vector, tensor and differential) operators that intervene in the governing equations of the problem (6.148) to (6.150) are linear, that is, given any two scalars a and b,

$$\begin{aligned}
&\nabla\cdot(\bullet) \ \rightarrow\ \text{linear} \Longrightarrow \nabla\cdot(a\,\mathbf{x}+b\,\mathbf{y}) = a\,\nabla\cdot\mathbf{x}+b\,\nabla\cdot\mathbf{y}\ ,\\
&\mathbb{C}:(\bullet) \ \rightarrow\ \text{linear} \Longrightarrow \mathbb{C}:(a\,\mathbf{x}+b\,\mathbf{y}) = a\,\mathbb{C}:\mathbf{x}+b\,\mathbb{C}:\mathbf{y}\ ,\\
&\nabla^S(\bullet) \ \rightarrow\ \text{linear} \Longrightarrow \nabla^S(a\,\mathbf{x}+b\,\mathbf{y}) = a\,\nabla^S\mathbf{x}+b\,\nabla^S\mathbf{y}\ ,\\
&\frac{\partial^2}{\partial t^2}(\bullet) \rightarrow\ \text{linear} \Longrightarrow \frac{\partial^2(a\,\mathbf{x}+b\,\mathbf{y})}{\partial t^2} = a\frac{\partial^2\mathbf{x}}{\partial t^2}+b\frac{\partial^2\mathbf{y}}{\partial t^2}\ .
\end{aligned}$$

Consider now two possible systems of actions $\mathbb{A}^{(1)}$ and $\mathbb{A}^{(2)}$,

$$\mathbb{A}^{(1)}(\mathbf{x},t)\overset{not}{\equiv}\begin{bmatrix}\mathbf{b}^{(1)}(\mathbf{x},t)\\ \mathbf{u}^{*(1)}(\mathbf{x},t)\\ \mathbf{t}^{*(1)}(\mathbf{x},t)\\ \Delta\theta^{(1)}(\mathbf{x},t)\\ \mathbf{v}_0^{(1)}(\mathbf{x})\end{bmatrix} \quad\text{and}\quad \mathbb{A}^{(2)}(\mathbf{x},t)\overset{not}{\equiv}\begin{bmatrix}\mathbf{b}^{(2)}(\mathbf{x},t)\\ \mathbf{u}^{*(2)}(\mathbf{x},t)\\ \mathbf{t}^{*(2)}(\mathbf{x},t)\\ \Delta\theta^{(2)}(\mathbf{x},t)\\ \mathbf{v}_0^{(2)}(\mathbf{x})\end{bmatrix}\ , \qquad (6.152)$$

and their corresponding responses, $\mathbb{R}^{(1)}$ and $\mathbb{R}^{(2)}$,

$$\mathbb{R}^{(1)}(\mathbf{x},t) \stackrel{not}{\equiv} \begin{bmatrix} \mathbf{u}^{(1)}(\mathbf{x},t) \\ \boldsymbol{\varepsilon}^{(1)}(\mathbf{x},t) \\ \boldsymbol{\sigma}^{(1)}(\mathbf{x},t) \end{bmatrix} \quad \text{and} \quad \mathbb{R}^{(2)}(\mathbf{x},t) \stackrel{not}{\equiv} \begin{bmatrix} \mathbf{u}^{(2)}(\mathbf{x},t) \\ \boldsymbol{\varepsilon}^{(2)}(\mathbf{x},t) \\ \boldsymbol{\sigma}^{(2)}(\mathbf{x},t) \end{bmatrix} . \tag{6.153}$$

Theorem 6.2. Superposition principle.

The solution (response) to the system of actions

$$\mathbb{A}^{(3)} = \lambda^{(1)}\,\mathbb{A}^{(1)} + \lambda^{(2)}\,\mathbb{A}^{(2)}$$

(where $\lambda^{(1)}$ and $\lambda^{(2)}$ are any two scalars) is

$$\mathbb{R}^{(3)} = \lambda^{(1)}\,\mathbb{R}^{(1)} + \lambda^{(2)}\,\mathbb{R}^{(2)}$$

In other words, the solution to the linear thermoelastic problem when considering a linear combination of different systems of actions is the same linear combination of the individual solutions to each of these systems of actions.

Proof

Replacing the actions $\mathbb{A}^{(3)} = \lambda^{(1)}\,\mathbb{A}^{(1)} + \lambda^{(2)}\,\mathbb{A}^{(2)}$ and the responses $\mathbb{R}^{(3)} = \lambda^{(1)}\,\mathbb{R}^{(1)} + \lambda^{(2)}\,\mathbb{R}^{(2)}$ in the equations of the problem, and taking into account the linearity of the different operators (see Remark 6.19) yields

a) Cauchy's equation

$$\nabla\cdot\boldsymbol{\sigma}^{(3)} + \rho_0\,\mathbf{b}^{(3)} = \lambda^{(1)} \underbrace{\left(\nabla\cdot\boldsymbol{\sigma}^{(1)} + \rho_0\mathbf{b}^{(1)}\right)}_{\rho_0 \dfrac{\partial^2\mathbf{u}^{(1)}}{\partial t^2}} + \lambda^{(2)} \underbrace{\left(\nabla\cdot\boldsymbol{\sigma}^{(2)} + \rho_0\mathbf{b}^{(2)}\right)}_{\rho_0 \dfrac{\partial^2\mathbf{u}^{(2)}}{\partial t^2}} =$$

$$= \rho_0 \frac{\partial^2\left(\lambda^{(1)}\,\mathbf{u}^{(1)} + \lambda^{(2)}\,\mathbf{u}^{(2)}\right)}{\partial t^2} = \rho_0 \frac{\partial^2\mathbf{u}^{(3)}}{\partial t^2}$$

$$\boxed{\nabla\cdot\boldsymbol{\sigma}^{(3)} + \rho_0\,\mathbf{b}^{(3)} = \rho_0 \frac{\partial^2\mathbf{u}^{(3)}}{\partial t^2}} \tag{6.154}$$

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b) Constitutive equation

$$\boldsymbol{\sigma}^{(3)} - \left(\mathbb{C}:\boldsymbol{\varepsilon}^{(3)} - \beta\,\Delta\theta^{(3)}\mathbf{1}\right) = \lambda^{(1)}\underbrace{\left(\boldsymbol{\sigma}^{(1)} - \left(\mathbb{C}:\boldsymbol{\varepsilon}^{(1)} - \beta\,\Delta\theta^{(1)}\mathbf{1}\right)\right)}_{=\mathbf{0}} + \lambda^{(2)}\underbrace{\left(\boldsymbol{\sigma}^{(2)} - \left(\mathbb{C}:\boldsymbol{\varepsilon}^{(2)} - \beta\,\Delta\theta^{(2)}\mathbf{1}\right)\right)}_{=\mathbf{0}} = \mathbf{0}$$

$$\boxed{\boldsymbol{\sigma}^{(3)} = \mathbb{C}:\boldsymbol{\varepsilon}^{(3)} - \beta\,\Delta\theta^{(3)}\mathbf{1}} \tag{6.155}$$

c) Geometric equation

$$\boldsymbol{\varepsilon}^{(3)} - \nabla^S\mathbf{u}^{(3)} = \lambda^{(1)}\underbrace{\left(\boldsymbol{\varepsilon}^{(1)} - \nabla^S\mathbf{u}^{(1)}\right)}_{=\mathbf{0}} + \lambda^{(2)}\underbrace{\left(\boldsymbol{\varepsilon}^{(2)} - \nabla^S\mathbf{u}^{(2)}\right)}_{=\mathbf{0}} = \mathbf{0} \qquad \boxed{\boldsymbol{\varepsilon}^{(3)} = \nabla^S\mathbf{u}^{(3)}} \tag{6.156}$$

d) Boundary condition in Γ_u

$$\mathbf{u}^{(3)} - \mathbf{u}^{*(3)} = \lambda^{(1)}\underbrace{\left(\mathbf{u}^{(1)} - \mathbf{u}^{*(1)}\right)}_{=\mathbf{0}} + \lambda^{(2)}\underbrace{\left(\mathbf{u}^{(2)} - \mathbf{u}^{*(2)}\right)}_{=\mathbf{0}} = \mathbf{0} \qquad \boxed{\Gamma_u : \mathbf{u}^{(3)} = \mathbf{u}^{*(3)}} \tag{6.157}$$

e) Boundary condition in Γ_σ

$$\boldsymbol{\sigma}^{(3)}\cdot\mathbf{n} - \mathbf{t}^{*(3)} = \lambda^{(1)}\underbrace{\left(\boldsymbol{\sigma}^{(1)}\cdot\mathbf{n} - \mathbf{t}^{*(1)}\right)}_{=\mathbf{0}} + \lambda^{(2)}\underbrace{\left(\boldsymbol{\sigma}^{(2)}\cdot\mathbf{n} - \mathbf{t}^{*(2)}\right)}_{=\mathbf{0}} = \mathbf{0} \qquad \boxed{\Gamma_\sigma : \boldsymbol{\sigma}^{(3)}\cdot\mathbf{n} = \mathbf{t}^{*(3)}} \tag{6.158}$$

f) Initial conditions

$$\dot{\mathbf{u}}^{(3)}(\mathbf{x},0) - \mathbf{v}_0^{(3)} = \lambda^{(1)}\underbrace{\left(\dot{\mathbf{u}}^{(1)}(\mathbf{x},0) - \mathbf{v}_0^{(1)}\right)}_{=\mathbf{0}} + \lambda^{(2)}\underbrace{\left(\dot{\mathbf{u}}^{(2)}(\mathbf{x},0) - \mathbf{v}_0^{(2)}\right)}_{=\mathbf{0}} = \mathbf{0}$$

$$\boxed{\dot{\mathbf{u}}^{(3)}(\mathbf{x},0) = \mathbf{v}_0^{(3)}} \tag{6.159}$$

Consequently, $\mathbb{R}^{(3)} = \lambda^{(1)}\,\mathbb{R}^{(1)} + \lambda^{(2)}\,\mathbb{R}^{(2)} \overset{not}{\equiv} \left[\mathbf{u}^{(3)}, \boldsymbol{\varepsilon}^{(3)}, \boldsymbol{\sigma}^{(3)}\right]^T$ is the solution to the thermoelastic problem subjected to the actions: $\mathbb{A}^{(3)} = \lambda^{(1)}\,\mathbb{A}^{(1)} + \lambda^{(2)}\,\mathbb{A}^{(2)}$ *(QED)*.

6.13 Hooke's Law in terms of the Stress and Strain "Vectors"

The symmetry of the stress and the strain tensors, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, means that only six of its nine components in a certain Cartesian system are different. Therefore, and to "economize" in writing, only these six different components are used in engineering, and they are expressed in the form of the stress and strain "vectors". These are constructed in $\mathbb{R}^6$, systematically arranging the elements of the *upper triangle* of the matrix of components of the corresponding tensor in the following manner[25].

$$\boldsymbol{\sigma} \overset{not}{\equiv} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \rightarrow \quad \{\boldsymbol{\sigma}\} \overset{def}{=} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} \tag{6.160}$$

The same arrangement is followed in the case of the strains, with the particularity that the strain vector $\{\boldsymbol{\varepsilon}\}$ is constructed using the angular strains $\gamma_{xy} = 2\,\varepsilon_{xy}$, $\gamma_{xz} = 2\,\varepsilon_{xz}$ and $\gamma_{yz} = 2\,\varepsilon_{yz}$ (see Chapter 2, Section 2.11.4).

$$\boldsymbol{\varepsilon} \overset{not}{\equiv} \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z \end{bmatrix} \overset{not}{=} \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix} \quad \rightarrow \quad \{\boldsymbol{\varepsilon}\} \overset{def}{=} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \tag{6.161}$$

[25] The notation $\{\mathbf{x}\}$ is used to denote the vector in $\mathbb{R}^6$ constructed from the symmetric tensor $\mathbf{x}$.

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Remark 6.20. An interesting property of this construction is that the double contraction of the stress and strain tensors is transformed into the dot product (in $\mathbb{R}^6$) of the stress and strain vectors,

$$\underbrace{\boldsymbol{\sigma}:\boldsymbol{\varepsilon}}_{\substack{\text{second-order}\\ \text{tensors}}} = \underbrace{\{\boldsymbol{\sigma}\}\cdot\{\boldsymbol{\varepsilon}\}}_{\text{vectors}} \qquad \Longleftrightarrow \qquad \sigma_{ij}\,\varepsilon_{ij} = \sigma_i\,\varepsilon_i$$

which can be verified by performing said operations, using the definitions in (6.160) and (6.161).

The inverse constitutive equation (6.113),

$$\boldsymbol{\varepsilon} = -\frac{\nu}{E}\,\mathrm{Tr}(\boldsymbol{\sigma})\,\mathbf{1} + \frac{1+\nu}{E}\boldsymbol{\sigma} + \alpha\Delta\theta\,\mathbf{1}\,, \tag{6.162}$$

can now be rewritten in terms of the stress and strain vectors as

$$\{\boldsymbol{\varepsilon}\} = \hat{\mathbb{C}}^{-1}\cdot\{\boldsymbol{\sigma}\} + \{\boldsymbol{\varepsilon}\}^t\ , \tag{6.163}$$

where $\hat{\mathbb{C}}^{-1}$ is the inverse matrix of elastic constants,

$$\hat{\mathbb{C}}^{-1} \stackrel{not}{\equiv} \left[\begin{array}{ccc|ccc} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{array}\right] \tag{6.164}$$

and $\{\boldsymbol{\varepsilon}\}^t$ is a thermal strain vector defined by means of an adequate *translation* of the thermal strain tensor $\boldsymbol{\varepsilon}^t = \alpha\,\Delta\theta\,\mathbf{1}$,

$$\boldsymbol{\varepsilon}^t \stackrel{not}{\equiv} \begin{bmatrix} \alpha\Delta\theta & 0 & 0 \\ 0 & \alpha\Delta\theta & 0 \\ 0 & 0 & \alpha\Delta\theta \end{bmatrix} \quad \rightarrow \quad \{\boldsymbol{\varepsilon}\}^t \stackrel{def}{=} \begin{bmatrix} \alpha\Delta\theta \\ \alpha\Delta\theta \\ \alpha\Delta\theta \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{6.165}$$

Finally, the inversion of equation (6.163) provides *Hooke's law* in terms of the stress and strain vectors,

$$\boxed{\begin{array}{c}\text{Hooke's law}\\ \text{in terms of the}\\ \text{stress and strain vectors}\end{array} \qquad \{\boldsymbol{\sigma}\} = \hat{\mathbb{C}} \cdot \left(\{\boldsymbol{\varepsilon}\} - \{\boldsymbol{\varepsilon}\}^t\right)} \tag{6.166}$$

where $\hat{\mathbb{C}}$ is the matrix of elastic constants.

$$\hat{\mathbb{C}} \stackrel{not}{\equiv} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \left[\begin{array}{ccc|ccc} 1 & \dfrac{\nu}{1-\nu} & \dfrac{\nu}{1-\nu} & 0 & 0 & 0 \\ \dfrac{\nu}{1-\nu} & 1 & \dfrac{\nu}{1-\nu} & 0 & 0 & 0 \\ \dfrac{\nu}{1-\nu} & \dfrac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1-2\nu}{2(1-\nu)} \end{array}\right] \tag{6.167}$$

PROBLEMS

Problem 6.1 – *Justify whether the following statements are true or false.*

a) The terms isentropic and adiabatic are equivalent when dealing with a thermoelastic material.

b) The second thermal analogy is always applicable to linear thermoelastic materials.

Solution

a) According to the second law of thermodynamics (5.114),

$$\rho_0 \theta \dot{s}^i_{loc} = \rho_0 \theta \dot{s} - (\rho_0 r - \nabla \cdot \mathbf{q}) \geq 0 \; .$$

All processes are reversible in the case of a thermoelastic material and, thus, the inequality becomes an equality,

$$\rho_0 \theta \dot{s}^i_{loc} = \rho_0 \theta \dot{s} - (\rho_0 r - \nabla \cdot \mathbf{q}) = 0 \; . \qquad [1]$$

An isentropic process (entropy remains constant) is characterized by $\dot{s} = 0$. On the other hand, an adiabatoc process (variation of heat is null) satisfies

$$\rho_0 r - \nabla \cdot \mathbf{q} = 0 \; .$$

Therefore, if an isentropic process is assumed, and its mathematical expression is introduced in [1], the definition of an adiabatic process is obtained,

$$\underbrace{\rho_0 \theta \dot{s}}_{=0} - (\rho_0 r - \nabla \cdot \mathbf{q}) = 0 \quad \Longrightarrow \quad \rho_0 r - \nabla \cdot \mathbf{q} = 0 \; .$$

Conversely, if an adiabatic process is assumed, and its mathematical expression is introduced in [1], the definition of an isentropic process is obtained,

$$\rho_0 \theta \dot{s} - \underbrace{(\rho_0 r - \nabla \cdot \mathbf{q})}_{=0} = 0 \quad \Longrightarrow \quad \dot{s} = 0 \; .$$

In conclusion, *the statement is true*.

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b) The second thermal analogy is not always applicable. The condition that the thermal strain field be integrable must be verified, that is, the thermal strain field $\boldsymbol{\varepsilon}^t(\mathbf{x},t)$ must satisfy the compatibility conditions (3.19),

$$\varepsilon_{ij,kl}+\varepsilon_{kl,ij}-\varepsilon_{ik,jl}-\varepsilon_{jl,ik}=0 \qquad i,j,k,l\in\{1,2,3\}\ .$$

Given that these involve second-order derivatives of the components of the strain tensor with respect to x, y and z, they will be automatically satisfied if $\alpha = const.$ and $\Delta\theta = const.$, or if $\alpha\Delta\theta$ is linear in x, y and z (which is the definition of a linear thermoelastic material). Therefore, *the statement is true*.

Problem 6.2 – *An isotropic linear elastic solid is subjected to a constant pressure of value p on all of its external boundary, in addition to a thermal increment of $\Delta\theta=\theta(x,y,z)$ in its interior. Both actions cancel each other out such that no displacements are observed in the solid. Obtain the value of $\Delta\theta$ in each point of the solid.*

Solution

The first thermal analogy described in Section 6.11.1 will be applied. To this aim, the original problem I is decomposed into the sum of problems II and III as described in Figure 6.13.

PROBLEM I

$$\text{Actions:}\begin{cases}\mathbf{b}=\mathbf{0} & \\ \mathbf{t}^*=-p\,\mathbf{n} & \text{in } \Gamma_\sigma \\ \mathbf{u}^*=\mathbf{0} & \text{in } \Gamma_u \\ \Delta\theta=\Delta\theta & \end{cases} \qquad \text{Responses:}\begin{cases}\mathbf{u}\\ \boldsymbol{\varepsilon}\\ \boldsymbol{\sigma}\end{cases}$$

PROBLEM III
This problem is solved first since its solution is trivial.

$$\text{Actions:}\begin{cases}\mathbf{b}_{III}=\dfrac{1}{\rho}\beta\nabla(\Delta\theta) & \\ \mathbf{t}^*_{III}=-\beta\,\Delta\theta\,\mathbf{n} & \text{in } \Gamma_\sigma \\ \mathbf{u}^*_{III}=\mathbf{0} & \text{in } \Gamma_u \\ \Delta\theta_{III}=\Delta\theta & \end{cases} \qquad \text{Responses:}\begin{cases}\mathbf{u}_{III}=\mathbf{0}\\ \boldsymbol{\varepsilon}_{III}=\mathbf{0}\\ \boldsymbol{\sigma}_{III}=-\beta\,\Delta\theta\mathbf{1}\end{cases}$$

PROBLEM II

$$\text{Actions:}\begin{cases} \mathbf{b}_{II}=\dfrac{1}{\rho}\beta\nabla(\Delta\theta) \\ \mathbf{t}^*_{II}=(-p+\beta\,\Delta\theta)\,\mathbf{n} & \text{in } \Gamma_\sigma \\ \mathbf{u}^*_{II}=\mathbf{u}^*=\mathbf{0} & \text{in } \Gamma_u \\ \Delta\theta_{II}=0 \end{cases}$$

To solve problem II, Navier's equation (6.62) is taken into account, together with the fact that $\mathbf{u}_{II}=\mathbf{0}$.

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}_{II})+\mu\nabla^2\mathbf{u}_{II}+\rho\mathbf{b}_{II}=\mathbf{0} \implies$$

$$\mathbf{b}_{II}=\mathbf{0} \implies \beta\nabla(\Delta\theta)=0 \implies \Delta\theta \text{ is uniform}$$

In addition, $\mathbf{u}_{II}=\mathbf{0}$ also results in

$$\boldsymbol{\varepsilon}_{II}=\frac{1}{2}(\mathbf{u}_{II}\otimes\nabla+\nabla\otimes\mathbf{u}_{II})=\mathbf{0}\,,$$

$$\boldsymbol{\sigma}_{II}=\lambda(\nabla\cdot\mathbf{u}_{II})\mathbf{1}+\mu(\mathbf{u}_{II}\otimes\nabla+\nabla\otimes\mathbf{u}_{II})=\mathbf{0}\,.$$

Since the traction vector $\mathbf{t}^*_{II}$ is defined in terms of the stress tensor $\boldsymbol{\sigma}_{II}$,

$$\boldsymbol{\sigma}_{II}\cdot\mathbf{n}=\mathbf{t}^*_{II}=(-p+\beta\,\Delta\theta)\,\mathbf{n}=\mathbf{0}\quad\forall\mathbf{n} \implies -p+\beta\,\Delta\theta=0\,,$$

and the value of the thermal increment is finally obtained,

$$\boxed{\Delta\theta=\frac{p}{\beta}}\,.$$

Problem 6.3 – *A cylindrical shell of height* h*, internal radius* R *and external radius* 2R *is placed inside an infinitely rigid cylindrical cavity of height* h *and radius* 2R + a*, with* a << R*. Assume the cylindrical shell is subjected to a uniform temperature field* $\Delta\theta$*.*

a) Determine the value of $\Delta\theta^*$ *required for the external lateral walls of the cylindrical shell and the rigid walls of the cavity to come into contact.*

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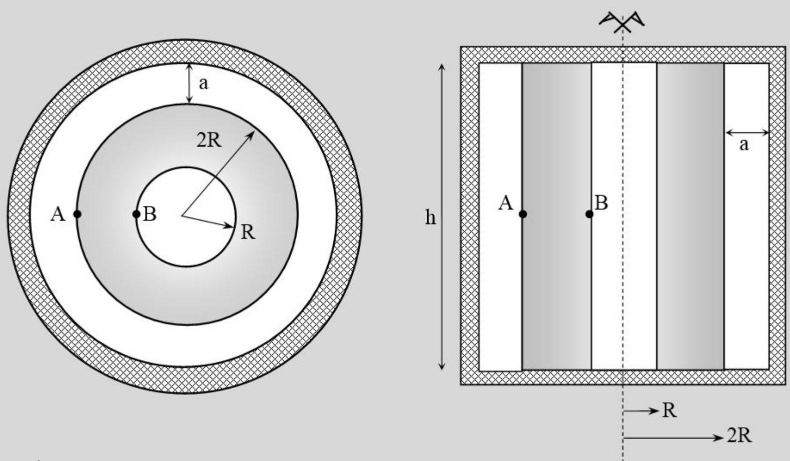
b) Plot, indicating the most significant values, the curve $\delta - \Delta\theta$, where δ is the lengthening of the internal radius of the cylindrical shell. Determine the value of $\Delta\theta$ such that this radius recovers its initial value.

c) Plot, indicating the most significant values, the curves $\sigma_{rr} - \Delta\theta$, $\sigma_{\theta\theta} - \Delta\theta$ and $\sigma_{zz} - \Delta\theta$, in points A *and* B.



Hypotheses:

1) *Young's modulus: E*
2) *Poisson's coefficient: $\nu = 0$*
3) *Thermal expansion coefficient: α*
4) *Isotropic linear elastic material*
5) *Weights can be neglected*
6) *The friction between the walls is negligible*

Solution

a) Two distinct phases can be identified in this problem:

First phase

The cylindrical shell has not come into contact with the rigid walls of the cavity. The boundary condition on the lateral walls, both internal and external, will be null radial stress. The two cylinders will come into contact when

$$u_r(r = 2\mathrm{R}) = \mathrm{a}\ .$$

Second phase

The cylindrical shell and the rigid walls of the cavity are in contact and, therefore, the boundary condition on the external lateral wall is different than that of the first phase. In this case, a null radial displacement will be imposed. Nonetheless, the internal wall will retain the same boundary condition as in the previous phase.
A positive $\Delta\theta$ will reduce the internal radius since the external radius cannot increase because it is limited by the infinitely rigid walls of the cavity. Then, the only possibility is that the cylindrical shell continues expanding inwards. There will be a point in which the internal radius, which had increased in the first phase, will recover its initial value.

The first thermal analogy (see Section 6.11.1) and the superposition principle (see Section 6.12) will be applied. To this aim, the original problem (problem I) is decomposed into the sum of problems II and II as described in Figure 6.13.

PROBLEM III
The actions in problem III, the trivial problem, are

$$\mathbf{b}_{III} = \frac{1}{\rho}\nabla\cdot(\boldsymbol{\beta}\,\Delta\theta) \ .$$

In this case, however, $\Delta\theta$ is uniform and $\boldsymbol{\beta}$ is a spherical and constant tensor ($\boldsymbol{\beta} = \beta\mathbf{1}$). Therefore,

$$\mathbf{b}_{III} = \mathbf{0} \ .$$

The boundary conditions are

1) Prescribed displacements in $\Gamma_u : \mathbf{u}_{III} = \mathbf{0}$.
2) Prescribed stresses in $\Gamma_\sigma : \mathbf{t} = -\boldsymbol{\beta}\,\Delta\theta\mathbf{n} = -\beta\,\Delta\theta\mathbf{n}$.

The solution to this problem is known to be

$$\left.\begin{aligned} &\mathbf{u}_{III} = \mathbf{0} \\ &\boldsymbol{\varepsilon}_{III} = \mathbf{0} \\ &\boldsymbol{\sigma}_{III} = -\beta\,\Delta\theta\mathbf{1} \end{aligned}\right. \qquad [1]$$

PROBLEM II
The actions in problem II, the analogous problem, are

$$\mathbf{b}_{II} = \mathbf{b} - \frac{1}{\rho}\nabla\cdot(\boldsymbol{\beta}\,\Delta\theta) \ .$$

Here, $\mathbf{b} = \mathbf{0}$ because the weight of the cylinder is assumed to be negligible and the second term is zero, as seen in problem III. Therefore,

$$\mathbf{b}_{II} = \mathbf{0} \ .$$

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The boundary conditions are

1) Prescribed displacements in $\Gamma_u : \mathbf{u}_{II} = \mathbf{u}^*$, where $\mathbf{u}^*$ is the displacement imposed in problem I.
2) Prescribed stresses in $\Gamma_\sigma : \mathbf{t}_{II} = \boldsymbol{\sigma}_{II} \cdot \mathbf{n} = \mathbf{t}^* + \boldsymbol{\beta}\, \Delta\theta\, \mathbf{n} = \mathbf{t}^* + \beta\, \Delta\theta\, \mathbf{n}$, where $\mathbf{t}^*$ is the traction vector imposed in problem I.

The analogous problem will now be solved assuming an infinitesimal strains hypothesis, since a << R and the strains are due to $\Delta\theta$, which are generally infinitesimal. Due to cylindrical symmetry, the displacement vector $\mathbf{u}$ is known to be of the form

$$\mathbf{u}_{II}(r,z) \stackrel{not}{\equiv} \left[u_r(r) \ , \ 0 \ , \ u_z(z) \right]^T \ .$$

In addition, $u_z(z) = 0$ will be imposed in all points since no information on the top and bottom surfaces of the cylindrical shell is given. Boundary conditions in displacements cannot be imposed for these surfaces because there is no way to determine the integration constants of u_z that would appear if $u_z \neq 0$ were considered. Therefore, the displacement vector

$$\mathbf{u}_{II}(r,z) \stackrel{not}{\equiv} \left[u_r(r) \ , \ 0 \ , \ 0 \ \right]^T$$

is adopted. Navier's equation (6.62) will be used to solve this problem,

$$(\lambda + \mu)\, \nabla (\nabla \cdot \mathbf{u}_{II}) + \mu \nabla^2 \mathbf{u}_{II} + \rho_0 \mathbf{b}_{II} = \rho_0 \frac{\partial^2 \mathbf{u}_{II}}{\partial t^2} = \mathbf{0} \ .$$

Note that the problem requires working in cylindrical coordinates and, thus, the equation must be adapted to this system of coordinates. Given the simplifications introduced into the problem, only the radial component of the equation will result in a non-trivial solution,

$$(\lambda + 2G) \frac{\partial e}{\partial r} - \frac{2G}{r} \frac{\partial \omega_z}{\partial \theta} + 2G \frac{\partial \omega_\theta}{\partial z} + \rho b_r = \rho \frac{\partial^2 u_r}{\partial t^2} \ , \qquad [2]$$

where b_r is the radial component of $\mathbf{b}_{II}$ and with

$$\omega_\theta = \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) = 0 \ ,$$

$$\omega_z = \frac{1}{2} \left(\frac{1}{r} \frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right) = 0 \ ,$$

$$e = \frac{1}{r} \frac{\partial (r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = \frac{1}{r} \frac{\partial (r u_r)}{\partial r} \ .$$

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The values of the parameters λ, G and β that intervene in Navier's equation must also be determined from the known parameters $(E\,,\ \alpha\,,\ \nu=0)$ as follows.

$$\begin{aligned}
\nu &= \frac{\lambda}{2(\lambda+\mu)} = 0 && \Longrightarrow && \lambda = 0 \\
\mu &= \frac{E}{2(1+\nu)} && \Longrightarrow && \mu = G = \frac{E}{2} \\
\beta &= \frac{E}{1-2\nu}\alpha && \Longrightarrow && \beta = E\alpha
\end{aligned} \qquad [3]$$

The problem can be considered to be a quasi-static and, taking into account $\mathbf{b}_{II}=\mathbf{0}$ and the relations derived in [3], the Navier's stokes equation [2] is reduced to

$$(\lambda+2G)\frac{\partial e}{\partial r} = 0 \quad \Longrightarrow \quad E\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\,u_r)\right) = 0\,.$$

Integrating this last expression leads to

$$\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}(r\,u_r) = 2\mathrm{A} \quad \Longrightarrow \quad \frac{\partial}{\partial r}(r\,u_r) = 2\mathrm{A}r \quad \Longrightarrow \quad r\,u_r = \mathrm{A}r^2 + \mathrm{B} \\
&\Longrightarrow \quad u_r = \mathrm{A}r + \frac{\mathrm{B}}{r} \quad \Longrightarrow \quad \mathbf{u}_{II}(r) \overset{not}{\equiv} \left[\mathrm{A}r + \frac{\mathrm{B}}{r},\ 0,\ 0\right]^T ,
\end{aligned} \qquad [4]$$

where A and B are the integration constants. The strain tensor corresponding to this displacement vector is easily obtained by means of the geometric equation (6.3),

$$\boldsymbol{\varepsilon}_{II}(r) \overset{not}{\equiv} \begin{bmatrix} \mathrm{A}-\dfrac{\mathrm{B}}{r^2} & 0 & 0 \\ 0 & \mathrm{A}+\dfrac{\mathrm{B}}{r^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} . \qquad [5]$$

Finally, the stress tensor is obtained through the constitutive equation of an isotropic linear elastic material (6.20), particularized with the expressions in [3],

$$\boldsymbol{\sigma}_{II} = \lambda\,(\mathrm{Tr}(\boldsymbol{\varepsilon}_{II}))\,\mathbf{1} + 2\mu\boldsymbol{\varepsilon}_{II} \quad \Longrightarrow \quad \boldsymbol{\sigma}_{II} = E\boldsymbol{\varepsilon}_{II}\ . \qquad [6]$$

First phase

The integration constants A and B must be determined by means of the boundary conditions. Stresses can be imposed in both lateral walls of the cylindrical shell as follows.

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BOUNDARY CONDITION AT $r = 2\mathrm{R}$

If $r = 2\mathrm{R}$ and according to the boundary conditions in Γ_σ of the analogous problem,

$$\mathbf{t}_{II} = \boldsymbol{\sigma}_{II} \cdot \mathbf{n} = \mathbf{t}^* + \boldsymbol{\beta}\,\Delta\theta\,\mathbf{n} = \mathbf{t}^* + \beta\,\Delta\theta\,\mathbf{n} \ .$$

Here, the following is known:

$\mathbf{n} = [1\,,\ 0\,,\ 0]^T$: outward unit normal vector.
$\mathbf{t}^* = \mathbf{0}$, since, for this phase, problem I has no loading on the lateral walls.
$\boldsymbol{\sigma}_{II}$ is given by [5] and [6].

Therefore, the boundary condition is reduced to

$$\sigma_{rr}\,(r = 2\mathrm{R}) = \beta\,\Delta\theta \ ,$$

which, replacing the value of the radial stress from [6] and, considering [3], results in

$$\mathrm{A} - \frac{\mathrm{B}}{4\mathrm{R}^2} = \alpha\,\Delta\theta \ . \qquad [7]$$

BOUNDARY CONDITION AT $r = \mathrm{R}$

If $r = \mathrm{R}$ and according to the boundary conditions in Γ_σ of the analogous problem,

$$\mathbf{t}_{II} = \boldsymbol{\sigma}_{II} \cdot \mathbf{n} = \mathbf{t}^* + \boldsymbol{\beta}\,\Delta\theta\,\mathbf{n} = \mathbf{t}^* + \beta\,\Delta\theta\,\mathbf{n} \ .$$

Here, the following is known:

$\mathbf{n} = [-1\,,\ 0\,,\ 0]^T$: outward unit normal vector.
$\mathbf{t}^* = \mathbf{0}$, since, for this phase, problem I has no loading on the lateral walls.
$\boldsymbol{\sigma}_{II}$ is given by [5] and [6].

Therefore, the boundary condition is reduced to

$$\sigma_{rr}\,(r = \mathrm{R}) = \beta\,\Delta\theta \ ,$$

which, replacing the value of the radial stress from [6] and, considering [3], results in

$$\mathrm{A} - \frac{\mathrm{B}}{\mathrm{R}^2} = \alpha\,\Delta\theta \ . \qquad [8]$$

From [7] and [8], the values

$$\mathrm{A} = \alpha\,\Delta\theta \quad \text{and} \quad \mathrm{B} = 0 \qquad [9]$$

are obtained. Now, replacing [9] in [4], [5] and [6] results in the displacements, strains and stresses of the analogous problem.

$$\mathbf{u}_{II} \stackrel{not}{\equiv} \left[\alpha\Delta\theta r,\, 0,\, 0\right]^T$$

$$\boldsymbol{\varepsilon}_{II} \stackrel{not}{\equiv} \begin{bmatrix} \alpha\Delta\theta & 0 & 0 \\ 0 & \alpha\Delta\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [10]$$

$$\boldsymbol{\sigma}_{II} \stackrel{not}{\equiv} \begin{bmatrix} E\alpha\Delta\theta & 0 & 0 \\ 0 & E\alpha\Delta\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Taking into account the superposition principle (see Section 6.12), and expressions [1], [3] and [10], the original problem is solved for the first phase.

$$\mathbf{u} \stackrel{not}{\equiv} \left[\alpha\Delta\theta r,\, 0,\, 0\right]^T$$

$$\boldsymbol{\varepsilon} \stackrel{not}{\equiv} \begin{bmatrix} \alpha\Delta\theta & 0 & 0 \\ 0 & \alpha\Delta\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [11]$$

$$\boldsymbol{\sigma} \stackrel{not}{\equiv} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -E\alpha\Delta\theta \end{bmatrix}$$

To obtain the value of $\Delta\theta^*$ for which the external lateral walls of the cylindrical shell and the rigid walls of the cavity come into contact, it is enough to impose that

$$u_r\left(r = 2\mathrm{R}\right) = \mathrm{a} \quad \Longrightarrow \quad \alpha\Delta\theta^* 2\mathrm{R} = \mathrm{a}\,.$$

Then, the temperature field required for the external lateral walls of the cylindrical shell and the rigid walls of the cavity to come into contact is

$$\boxed{\Delta\theta^* = \frac{\mathrm{a}}{2\alpha\mathrm{R}}}\,. \qquad [12]$$

b) First, the value $\Delta\theta^{**}$ for which the internal radius recovers its initial position will be determined. To this aim, the same geometry as in the initial problem will be used, but now there will exist contact between the cylindrical shell and the rigid walls of the cavity, which corresponds to the second phase defined in the previous section. So, a new problem must be solved, with the same geometry as before but considering different boundary conditions.

Second phase

The first phase will be obviated in this section, but one must bear in mind that the solid now starts from a state that results from the previous phase, that is, it has already suffered certain displacements, strains, stresses and thermal increments. The variable $\overline{\Delta\theta}$ will be used.
As before, the first thermal analogy will be applied. Problem III remains unchanged and, thus, so does its result [1]. Therefore, problem II must be solved with the same expressions [4], [5] and [6]. The integration constants A and B must be determined by means of the boundary conditions. Stresses can be imposed on the internal lateral wall of the cylindrical shell and displacements, on its external lateral wall.

BOUNDARY CONDITION AT $r = 2\mathrm{R}$

If $r = 2\mathrm{R}$ and according to the boundary conditions in Γ_u of the analogous problem,

$$u_r(r = 2\mathrm{R}) = 0 .$$

Therefore, the following condition is obtained,

$$\mathrm{A}\,2\mathrm{R} + \frac{\mathrm{B}}{2\mathrm{R}} = 0 . \qquad [13]$$

BOUNDARY CONDITION AT $r = \mathrm{R}$

If $r = \mathrm{R}$ and according to the boundary conditions in Γ_σ of the analogous problem,

$$\mathbf{t}_{II} = \boldsymbol{\sigma}_{II} \cdot \mathbf{n} = \mathbf{t}^* + \boldsymbol{\beta}\,\overline{\Delta\theta}\,\mathbf{n} = \mathbf{t}^* + \beta\,\overline{\Delta\theta}\,\mathbf{n} .$$

Here, the following is known:

$\mathbf{n} \overset{not}{\equiv} [-1\,,\ 0\,,\ 0]^T$: outward unit normal vector.
$\mathbf{t}^* = \mathbf{0}$, since, for this phase, problem I has no loading on the lateral walls.
$\boldsymbol{\sigma}_{II}$ is given by [5] and [6].

Therefore, the boundary condition is reduced to

$$\sigma_{rr}(r = \mathrm{R}) = \beta\,\overline{\Delta\theta} ,$$

which, replacing the value of the radial stress from [6], and considering [3], results in

$$\mathrm{A} - \frac{\mathrm{B}}{\mathrm{R}^2} = \alpha\,\overline{\Delta\theta} \ . \qquad [14]$$

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From [13] and [14], the values

$$\mathrm{A} = \frac{1}{5}\alpha\,\overline{\Delta\theta} \quad \text{and} \quad \mathrm{B} = -\frac{4}{5}\alpha\,\overline{\Delta\theta}\mathrm{R}^2 \qquad [15]$$

are obtained. Introducing now [15] in [4], [5] and [6] results in the displacements, strains and stresses of the analogous problem.

$$\mathbf{u}_{II} \stackrel{not}{\equiv} \left[\frac{1}{5}\alpha\,\overline{\Delta\theta}\left(r - \frac{4\mathrm{R}^2}{r}\right),\ 0,\ 0 \right]^T$$

$$\boldsymbol{\varepsilon}_{II} \stackrel{not}{\equiv} \begin{bmatrix} \frac{1}{5}\alpha\,\overline{\Delta\theta}\left(1 + \frac{4\mathrm{R}^2}{r^2}\right) & 0 & 0 \\ 0 & \frac{1}{5}\alpha\,\overline{\Delta\theta}\left(1 - \frac{4\mathrm{R}^2}{r^2}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [16]$$

$$\boldsymbol{\sigma}_{II} \stackrel{not}{\equiv} \begin{bmatrix} \frac{1}{5}E\alpha\,\overline{\Delta\theta}\left(1 + \frac{4\mathrm{R}^2}{r^2}\right) & 0 & 0 \\ 0 & \frac{1}{5}E\alpha\,\overline{\Delta\theta}\left(1 - \frac{4\mathrm{R}^2}{r^2}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Taking into account the superposition principle (see Section 6.12), and expressions [1], [3] and [16], the original problem is solved for the second phase.

$$\mathbf{u} \stackrel{not}{\equiv} \left[\frac{1}{5}\alpha\,\overline{\Delta\theta}\left(r - \frac{4\mathrm{R}^2}{r}\right),\ 0,\ 0 \right]^T$$

$$\boldsymbol{\varepsilon} \stackrel{not}{\equiv} \begin{bmatrix} \frac{1}{5}\alpha\,\overline{\Delta\theta}\left(1 + \frac{4\mathrm{R}^2}{r^2}\right) & 0 & 0 \\ 0 & \frac{1}{5}\alpha\,\overline{\Delta\theta}\left(1 - \frac{4\mathrm{R}^2}{r^2}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [17a]$$

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$$\boldsymbol{\sigma} \stackrel{not}{\equiv} \begin{bmatrix} \dfrac{4}{5}E\alpha\overline{\Delta\theta}\left(-1+\dfrac{\mathrm{R}^2}{r^2}\right) & 0 & 0 \\ 0 & \dfrac{4}{5}E\alpha\overline{\Delta\theta}\left(-1-\dfrac{\mathrm{R}^2}{r^2}\right) & 0 \\ 0 & 0 & -E\alpha\overline{\Delta\theta} \end{bmatrix} \qquad [17b]$$

Note that, up to this point, the second phase has been solved assuming an initial neutral state. In reality, this phase starts from the final state of the first phase, which has the displacements, strains, stresses and thermal increments corresponding to $\Delta\theta = \Delta\theta^*$,

$$\begin{aligned} \mathbf{u}_{initial} &= \mathbf{u}_{first\,phase}\left(\Delta\theta = \Delta\theta^*\right) \ , \\ \boldsymbol{\varepsilon}_{initial} &= \boldsymbol{\varepsilon}_{first\,phase}\left(\Delta\theta = \Delta\theta^*\right) \ , \\ \boldsymbol{\sigma}_{initial} &= \boldsymbol{\sigma}_{first\,phase}\left(\Delta\theta = \Delta\theta^*\right) \ . \end{aligned} \qquad [18]$$

In fact, the variable $\overline{\Delta\theta}$ in [17] is not a total thermal increment but the difference in temperature at the moment corresponding to $\Delta\theta^*$, that is,

$$\overline{\Delta\theta} = \Delta\theta - \Delta\theta^* \ . \qquad [19]$$

Then, considering [17], [18] and [19], the actual displacements, strains and stresses during the second phase are obtained,

$$\begin{aligned} \mathbf{u}_{second\,phase} &= \mathbf{u}_{initial} + \mathbf{u}\left(\overline{\Delta\theta}\right) \ , \\ \boldsymbol{\varepsilon}_{second\,phase} &= \boldsymbol{\varepsilon}_{initial} + \boldsymbol{\varepsilon}\left(\overline{\Delta\theta}\right) \ , \\ \boldsymbol{\sigma}_{second\,phase} &= \boldsymbol{\sigma}_{initial} + \boldsymbol{\sigma}\left(\overline{\Delta\theta}\right) \ . \end{aligned} \qquad [20]$$

Therefore, to determine $\Delta\theta^{**}$, it is enough to impose that the displacement, according to the first phase, of the internal radius be equal but of opposite sign to that of the second phase. In this way, the total displacement will be null.

First phase: displacement for $r = \mathrm{R}$ and $\Delta\theta = \Delta\theta^*$. From [11] and [12],

$$\delta_1 = u_r\left(r = \mathrm{R}, \Delta\theta = \Delta\theta^*\right) = \alpha\,\Delta\theta^*\,\mathrm{R} = \frac{\mathrm{a}}{2} \ . \qquad [21]$$

Second phase: displacement for $r = \mathrm{R}$ and $\overline{\Delta\theta} = \overline{\Delta\theta^{**}}$. From [17],

$$\delta_2 = u_r\left(r = \mathrm{R}, \overline{\Delta\theta} = \overline{\Delta\theta^{**}}\right) = -\frac{3}{5}\alpha\,\overline{\Delta\theta^{**}}\,\mathrm{R} \ . \qquad [22]$$

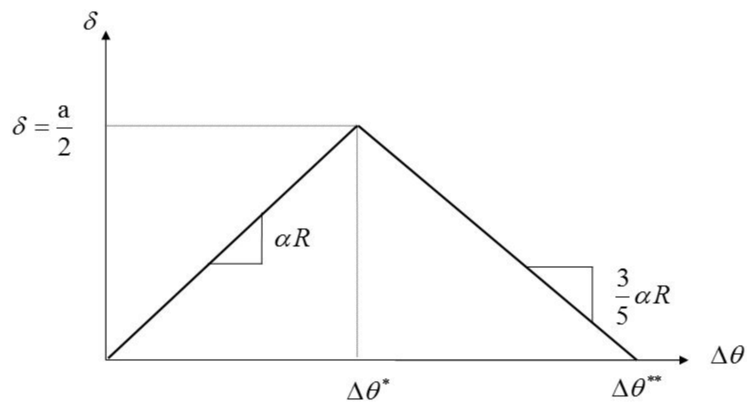
Then,

$$\delta_1 = -\delta_2 \qquad \Longrightarrow \qquad \overline{\Delta\theta^{**}} = \frac{5\mathrm{a}}{6\alpha\mathrm{R}} \ . \qquad [23]$$

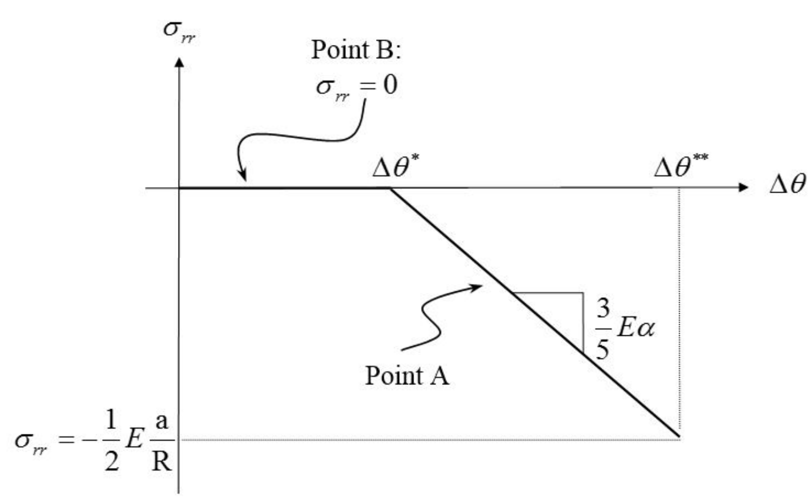
Finally, from [19] the total thermal increment is obtained,

$$\Delta\theta^{**} = \overline{\Delta\theta^{**}} + \Delta\theta^{*} = \frac{5\mathrm{a}}{6\alpha\mathrm{R}} + \frac{\mathrm{a}}{2\alpha\mathrm{R}} \quad \Longrightarrow \quad \boxed{\Delta\theta^{**} = \frac{4\mathrm{a}}{3\alpha\mathrm{R}}}$$

Now, the curve $\delta - \Delta\theta$ can be plotted, where δ is the displacement of the internal radius of the cylindrical shell.



c) Expressions [11] and [17] must be used to plot the curves $\sigma_{rr} - \Delta\theta$, $\sigma_{\theta\theta} - \Delta\theta$ and $\sigma_{zz} - \Delta\theta$ for points B $(r = \mathrm{R})$ and A $(r = 2\mathrm{R})$.



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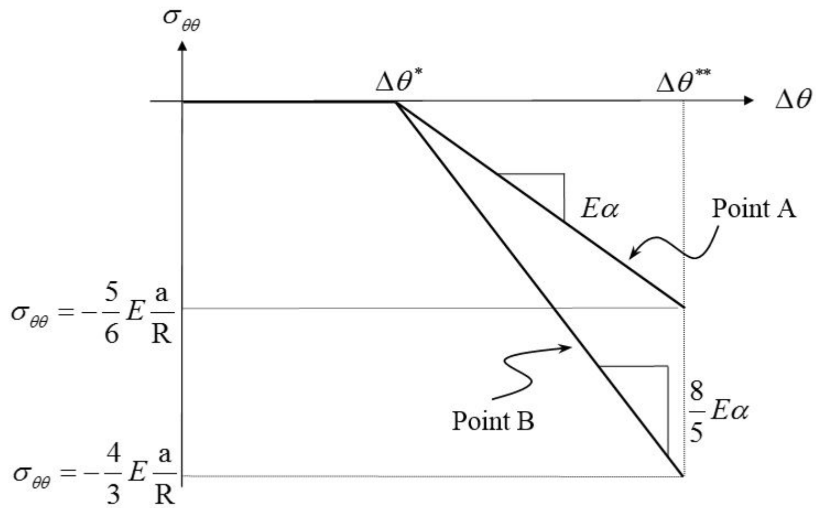
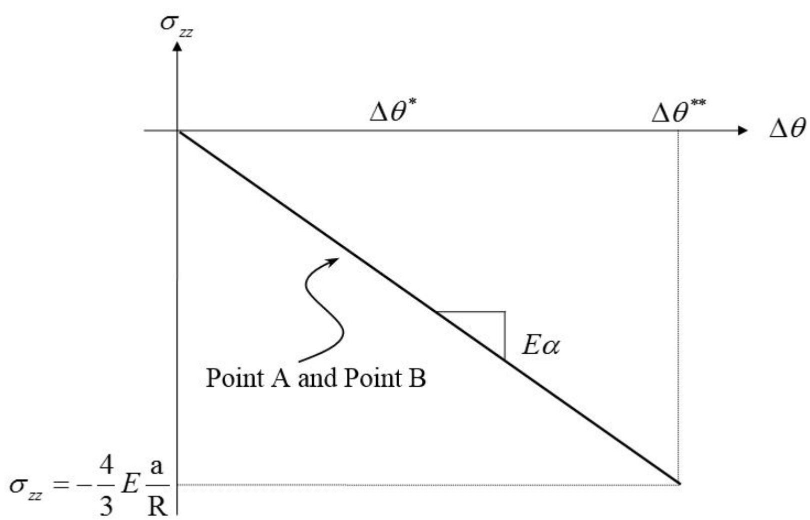


$\sigma_{\theta\theta}$
$\Delta\theta^*$
$\Delta\theta^{**}$
$\Delta\theta$
$E\alpha$
Point A
$\sigma_{\theta\theta} = -\frac{5}{6} E \frac{a}{R}$
Point B
$\frac{8}{5} E\alpha$
$\sigma_{\theta\theta} = -\frac{4}{3} E \frac{a}{R}$



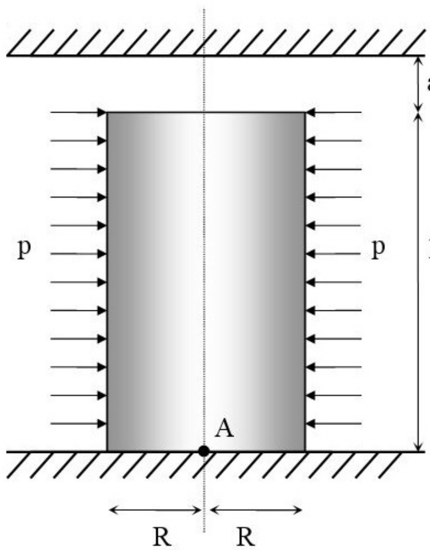
σ_{zz}
$\Delta\theta^*$
$\Delta\theta^{**}$
$\Delta\theta$
$E\alpha$
Point A and Point B
$\sigma_{zz} = -\frac{4}{3} E \frac{a}{R}$

EXERCISES

6.1 – *A cylinder composed of an isotropic linear elastic material stands on a rigid base. At a very small distance "*a*"* $(a \ll h)$ *of its top face there is another rigid surface. A uniform pressure* p *acts on all the lateral surface of the cylinder. Plot, indicating the most significant values, the following curves:*

a) Curve $p-\delta$*, where* δ *is the shortening of the radius of the cylinder,* R.

b) Curve $p-\sigma_A$*, where* σ_A *is the stress normal to the bottom contact surface at point* A.



Additional hypotheses:

1) Weights can be neglected.

2) Lamé's constants: $\lambda=\mu$

3) The problem is assumed to be quasi-static.

6.2 – *The solid sphere* A *with external radius* R_1 *and the solid spherical* B*, with external radius* R_2 *are composed of the same material. The external surface of* A *and the internal surface of* B *are separated by a very small distance "*a*"* $(a \ll R_1$ *and* $a \ll R_2)$.

a) Determine what value of the uniform normal pressure p *shown in the figure is required for the two surfaces to be in contact.*

b) Plot, indicating the most significant values, the curve $p-\delta$*, where* δ *is the shortening of* R_2.

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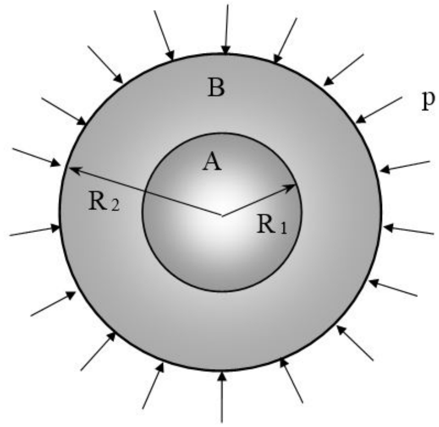
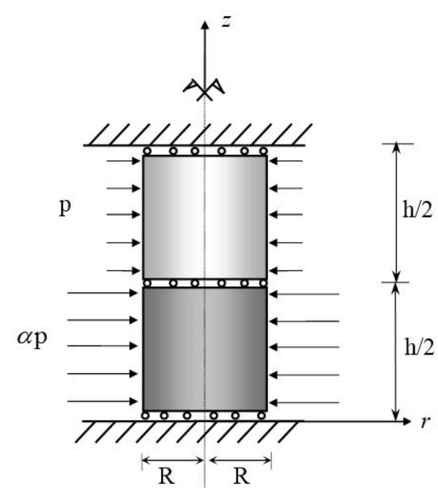
Additional hypotheses:

1) *Young's modulus: E*
2) *Lamé's constants:* $\lambda = \mu$
3) $R_1 = R$
4) $R_2 = 2R$



6.3 – *Two solid cylinders composed of different elastic materials are vertically superimposed and confined between two infinitely rigid walls. The cylinders are subjected to the external pressures* p *and* αp ($p > 0$, $\alpha > 0$) *as shown in the figure.*

a) *Determine the displacement field of the two cylinders in terms of the integration constants (justify the assumptions used).*
b) *Indicate the boundary conditions that need to be applied for the different boundaries of the problem.*
c) *Assuming a constant value* α *such that the contact surface between the two cylinders does not have a vertical displacement, calculate the integration constants and the value of* α.



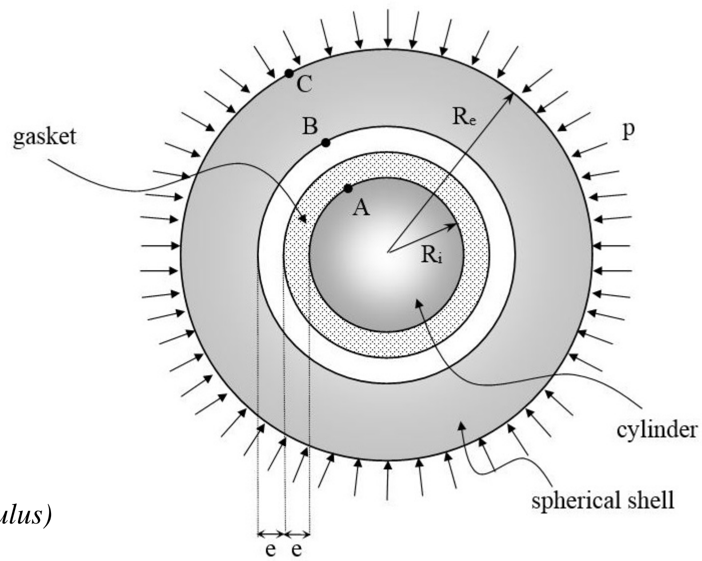
Additional hypotheses:

1) *Top cylinder:* $\lambda_1 = \mu_1$
2) *Bottom cylinder:* $\lambda_2 = \mu_2$
3) *The friction between the cylinders and between the cylinders and the walls is assumed to be null.*
4) *Weights can be neglected.*

6.4 – *A cylinder with radius* R_i *is placed in the interior of a cylindrical shell with internal radius* $\mathrm{R_i}+2\mathrm{e}$ *and external radius* $\mathrm{R_e}$*. There is an elastic gasket between the cylinder and the cylindrical shell which has an internal radius* $\mathrm{R_i}$ *and a thickness "*e*". The cylindrical shell is subjected to an external pressure* p*.*

a) Determine the displacement, strain and stress fields of the cylinder and the cylindrical shell.

b) Plot the curves $U_r - p$*, where* U_r *is the radial displacement, and* $\sigma_{rr} - p$*, where* σ_{rr} *is the radial stress at points* A, B *and* C *of the figure.*



Data:
$\mathrm{R_i} = 1$
$\mathrm{R_e} = 2$
$\nu = 0$
E (Young's modulus)

Additional hypotheses:

1) The constitutive law of the elastic gasket is $\mathrm{p}^* = K\,\delta^*$*, where* p^* *is the pressure acting on the gasket,* δ^* *is the shortening of its thickness and K is its elastic modulus.*

2) $\mathrm{e} \ll \mathrm{R_i}$

3) A plane strain behavior in an infinitesimal strain framework may be assumed.

6.5 – *The figure below schematizes the layout of a railway rail composed of straight rails of length "*L*", separated by an elastic gasket with elastic modulus K. Due to symmetry and construction considerations, it can be assumed that the section* $x=0$ *suffers no longitudinal displacements and the inferior part of the rail suffers no vertical displacements. A constant thermal increment* $\Delta\theta$ *is imposed in all points of the rail.*

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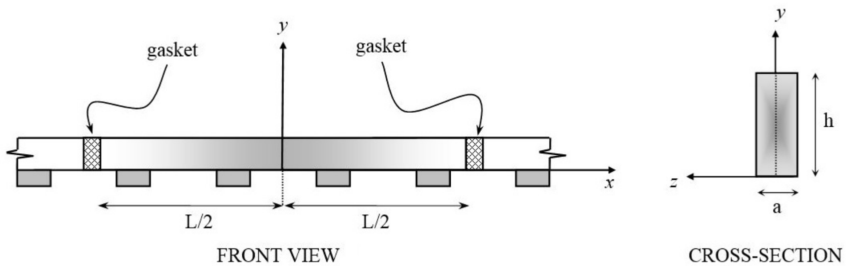
a) Obtain the displacement, strain and stress fields in terms of the corresponding integration constants.

b) Indicate the boundary conditions that must be applied to determine the integration constants.

c) Determine the integration constants and obtain the corresponding displacement, strain and stress fields.

d) Particularize these results for the cases $K = 0$ (open junction) and $K \to \infty$ (continuous rail).



Additional hypotheses:

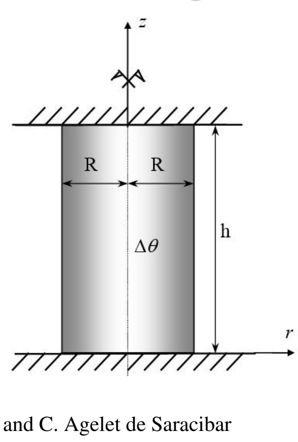
1) *Assume the displacements are of the form* $\mathbf{u} = [u(x)\ ,\ v(y)\ ,\ w(z)]^T$.
2) *Linear elastic material*
3) $\lambda = \mu$
4) *The weight of the rail can be neglected.*

6.6 – *A solid cylinder with radius* R *and height* h *is placed between two infinitely rigid walls, fitting perfectly between them without producing any stress. A thermal increment $\Delta\theta > 0$ is applied on the cylinder. Determine:*

a) The displacement field in terms of the corresponding integration constants.

b) The integration constants.

c) The stress state. Plot its variation along the radius.



Additional hypotheses:

1) *Material properties: $\lambda = \mu$ and $\alpha = \alpha(r) = \alpha_0 + \alpha_1 r$*
2) *The friction between the cylinder and the walls is negligible.*
3) *Weights can be neglected.*

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Chapter 7
Plane Linear Elasticity

7.1 Introduction

As seen in Chapter 6, from a mathematical point of view, the elastic problem consists in a system of PDEs that must be solved in the three dimensions of space and in the dimension associated with time $\left(\mathbb{R}^3 \times \mathbb{R}^+\right)$. However, in certain situations, the problem can be simplified so that it is reduced to two dimensions in space in addition to, obviously, the temporal dimension $\left(\mathbb{R}^2 \times \mathbb{R}^+\right)$. This simplification is possible because, in certain cases, the geometry and boundary conditions of the problem allow identifying an *irrelevant direction* (associated with a direction of the problem) such that solutions *independent of this dimension* can be posed *a priori* for this elastic problem.

Consider a local coordinate system $\{x, y, z\}$ in which the aforementioned irrelevant direction (assumed constant) coincides with the z-direction. Then, the analysis is reduced to the x-y plane and, hence, the name *plane elasticity* used to denote such problems. In turn, these are typically divided into two large groups associated with two families of simplifying hypotheses, *plane stress* problems and *plane strain* problems.

For the sake of simplicity, the *isothermal* case will be considered here, even though there is no intrinsic limitation to generalizing the results that will be obtained to the thermoelastic case.

7.2 Plane Stress State

The plane stress state is characterized by the following simplifying hypotheses:

1) *The stress state is of the type*

$$[\boldsymbol{\sigma}]_{xyz} \stackrel{not}{\equiv} \left[\begin{array}{cc|c} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ \hline 0 & 0 & 0 \end{array}\right] . \tag{7.1}$$

2) *The non-zero stresses (that is, those associated with the x-y plane) do not depend on the z-variable,*

$$\sigma_x = \sigma_x(x,y,t) \quad , \quad \sigma_y = \sigma_y(x,y,t) \quad \text{and} \quad \tau_{xy} = \tau_{xy}(x,y,t) \ . \tag{7.2}$$

To analyze under which conditions these hypotheses are reasonable, consider a plane elastic medium whose dimensions and form associated with the *x-y* plane (denoted as *plane of analysis*) are arbitrary and such that the third dimension (denoted as the *thickness* of the piece) is associated with the *z*-axis (see Figure 7.1). Assume the following circumstances hold for this elastic medium:

a) The thickness e is much smaller than the typical dimension associated with the plane of analysis *x-y*,

$$e \ll L \ . \tag{7.3}$$

b) The actions (body forces $\mathbf{b}(\mathbf{x},t)$, prescribed displacements $\mathbf{u}^*(\mathbf{x},t)$ and traction vector $\mathbf{t}^*(\mathbf{x},t)$) are contained within the plane of analysis *x-y* (its *z*-component is null) and, in addition, do not depend on the third dimension,

$$\begin{aligned} &\mathbf{b} \stackrel{not}{\equiv} \begin{bmatrix} b_x(x,y,t) \\ b_y(x,y,t) \\ 0 \end{bmatrix} , \qquad \Gamma_u \ : \ \mathbf{u}^* \stackrel{not}{\equiv} \begin{bmatrix} u_x^*(x,y,t) \\ u_y^*(x,y,t) \\ - \end{bmatrix} , \\ &\Gamma_\sigma = \Gamma_\sigma^+ \bigcup \Gamma_\sigma^- \bigcup \Gamma_\sigma^e \ : \ \mathbf{t}^* \stackrel{not}{\equiv} \begin{bmatrix} t_x^*(x,y,t) \\ t_y^*(x,y,t) \\ - \end{bmatrix} . \end{aligned} \tag{7.4}$$

c) The traction vector $\mathbf{t}^*(\mathbf{x},t)$ *is only non-zero on the boundary of the piece's thickness* (boundary Γ_σ^e), whilst on the lateral surfaces Γ_σ^+ and Γ_σ^- it is null (see Figure 7.1).

$$\Gamma_\sigma^+ \bigcup \Gamma_\sigma^- : \mathbf{t}^* \stackrel{not}{\equiv} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} . \tag{7.5}$$

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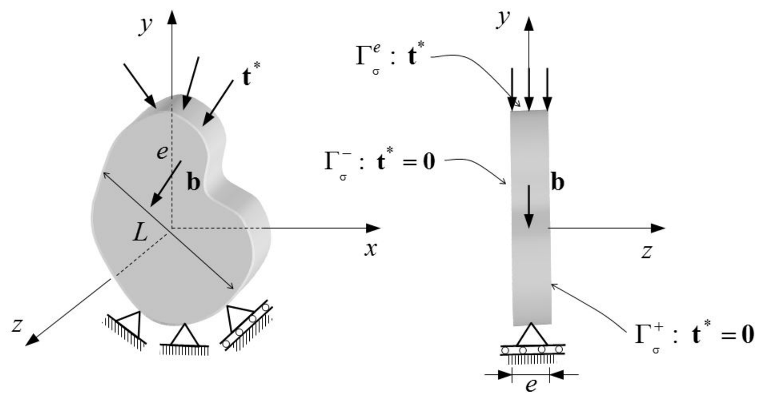

Figure 7.1: Example of a plane stress state.

Remark 7.1. The piece with the actions defined by (7.4) and (7.5) is compatible with the plane stress state given by (7.1) and (7.2), and schematized in Figure 7.2[1]. In effect, applying the boundary conditions Γ_σ on the piece yields:

- *Lateral surfaces Γ_σ^+ and Γ_σ^-*

$$\mathbf{n} \overset{not}{\equiv} \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix}, \qquad \boldsymbol{\sigma} \cdot \mathbf{n} \overset{not}{\equiv} \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

- *Edge Γ_σ^e*

$$\mathbf{n} \overset{not}{\equiv} \begin{bmatrix} n_x \\ n_y \\ 0 \end{bmatrix}, \qquad \boldsymbol{\sigma}(x,y,t) \cdot \mathbf{n} \overset{not}{\equiv} \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ 0 \end{bmatrix} = \begin{bmatrix} t_x(x,y,t) \\ t_y(x,y,t) \\ 0 \end{bmatrix},$$

which is compatible with the assumptions (7.4) and (7.5) .

[1] The fact that all the non-null *stresses* are contained in the *x*-*y* plane is what gives rise to the name *plane stress*.

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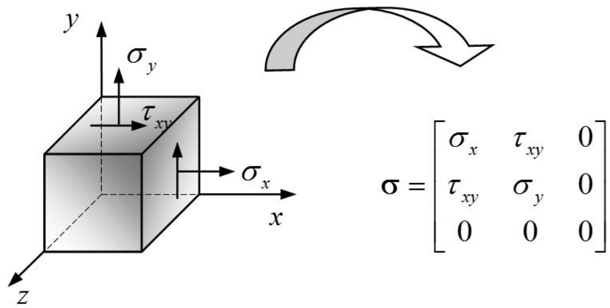

Figure 7.2: Plane stress state.

7.2.1 Strain Field. Constitutive Equation

Consider now the linear elastic constitutive equation (6.24),

$$\boldsymbol{\varepsilon} = -\frac{\nu}{E}\,\mathrm{Tr}(\boldsymbol{\sigma})\,\mathbf{1} + \frac{1+\nu}{E}\boldsymbol{\sigma} = -\frac{\nu}{E}\,\mathrm{Tr}(\boldsymbol{\sigma})\,\mathbf{1} + \frac{1}{2G}\boldsymbol{\sigma}\ , \tag{7.6}$$

which, applied on the stress state in (7.1) and in engineering notation, provides the strains (6.25)[2]

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right) = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy}\ , \\ \varepsilon_y &= \frac{1}{E}\left(\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right) = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right) \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz} = 0\ , \\ \varepsilon_z &= \frac{1}{E}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right) = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right) \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz} = 0\ , \end{aligned} \tag{7.7}$$

where the conditions $\sigma_z = \tau_{xz} = \tau_{yz} = 0$ have been taken into account. From (7.2) and (7.7) it is concluded that the strains do not depend on the z-coordinate either ($\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(x,y,t)$). In addition, the strain ε_z in (7.7) can be solved as

$$\varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right)\ . \tag{7.8}$$

[2] The engineering angular strains are defined as $\gamma_{xy} = 2\,\varepsilon_{xy}$, $\gamma_{xz} = 2\,\varepsilon_{xz}$ and $\gamma_{yz} = 2\,\varepsilon_{yz}$.

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In short, the strain tensor for the plane stress case results in

$$\boxed{\boldsymbol{\varepsilon}(x,y,t) \stackrel{not}{\equiv} \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \quad \text{with} \quad \varepsilon_z = -\frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y)} \tag{7.9}$$

and replacing (7.8) in (7.7) leads, after certain algebraic operations, to

$$\begin{aligned} &\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) \ , \quad \sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) \ , \\ &\text{and} \quad \tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \ , \end{aligned} \tag{7.10}$$

which can be rewritten as

$$\underbrace{\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}}_{\{\boldsymbol{\sigma}\}} = \underbrace{\frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}}_{\mathbb{C}^{plane}_{stress}} \underbrace{\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}}_{\{\boldsymbol{\varepsilon}\}} \Longrightarrow \boxed{\{\boldsymbol{\sigma}\} = \mathbb{C}^{plane}_{stress} \cdot \{\boldsymbol{\varepsilon}\}} \ . \tag{7.11}$$

7.2.2 Displacement Field

The components of the geometric equation of the problem (6.3),

$$\boldsymbol{\varepsilon}(\mathbf{x},t) = \nabla^S \mathbf{u}(\mathbf{x},t) = \frac{1}{2}(\mathbf{u} \otimes \nabla + \nabla \otimes \mathbf{u}) \ , \tag{7.12}$$

can be decomposed into two groups:

1) Those that do not affect the displacement u_z (and are hypothetically integrable in $\mathbb{R}^2$ for the x-y domain),

$$\left.\begin{aligned} &\varepsilon_x(x,y,t) = \frac{\partial u_x}{\partial x} \\ &\varepsilon_y(x,y,t) = \frac{\partial u_y}{\partial y} \\ &\gamma_{xy}(x,y,t) = 2\varepsilon_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \end{aligned}\right\} \begin{array}{c} \text{integration} \\ \text{in } \mathbb{R}^2 \\ \Longrightarrow \end{array} \left\{\begin{aligned} u_x &= u_x(x,y,t) \\ u_y &= u_y(x,y,t) \end{aligned}\right. \ . \tag{7.13}$$

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2) Those in which the displacement u_z intervenes,

$$
\begin{aligned}
\varepsilon_z(x,y,t) &= \frac{\partial u_z}{\partial z} = -\frac{\nu}{1-\nu}\left(\varepsilon_x+\varepsilon_y\right) \ , \\
\gamma_{xz}(x,y,t) &= 2\varepsilon_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0 \ , \\
\gamma_{yz}(x,y,t) &= 2\varepsilon_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 0 \ .
\end{aligned}
\tag{7.14}
$$

Observation of (7.1) to (7.14) suggests considering an *ideal elastic plane stress problem* reduced to the two dimensions of the plane of analysis and characterized by the unknowns

$$
\boxed{\mathbf{u}(x,y,t) \stackrel{not}{\equiv} \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \ \{\boldsymbol{\varepsilon}(x,y,t)\} \stackrel{not}{\equiv} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \text{ and } \{\boldsymbol{\sigma}(x,y,t)\} \stackrel{not}{\equiv} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix},}
\tag{7.15}
$$

in which the additional unknowns with respect to the general problem are either null, or can be calculated in terms of those in (7.15), or do not intervene in the reduced problem,

$$
\begin{aligned}
& \sigma_x = \tau_{xz} = \tau_{yz} = \gamma_{xz} = \gamma_{yz} = 0 \ , \quad \varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x+\varepsilon_y\right) \ , \\
& \text{and} \quad u_z(x,y,z,t) \ \text{ does not intervene in the problem.}
\end{aligned}
\tag{7.16}
$$

Remark 7.2. The plane stress problem is an *ideal* elastic problem since it cannot be exactly reproduced as a particular case of a three-dimensional elastic problem. In effect, there is no guarantee that the solution of the reduced plane stress $u_x(x,y,t)$ and $u_y(x,y,t)$ will allow obtaining a solution $u_z(x,y,z,t)$ for the rest of components of the geometric equation (7.14).

7.3 Plane Strain

The strain state is characterized by the simplifying hypotheses

$$
\mathbf{u} \stackrel{not}{\equiv} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_x(x,y,t) \\ u_y(x,y,t) \\ 0 \end{bmatrix} \ .
\tag{7.17}
$$

Again, it is illustrative to analyze in which situations these hypotheses are plausible. Consider, for example, an elastic medium whose geometry and actions can be generated from a bidimensional section (associated with the x-y plane and with the actions $\mathbf{b}(\mathbf{x},t)$, $\mathbf{u}^*(\mathbf{x},t)$ and $\mathbf{t}^*(\mathbf{x},t)$ contained in this plane) that is translated along a straight generatrix perpendicular to said section and, thus, associated with the z-axis (see Figure 7.3).

The actions of the problem can then be characterized by

$$\mathbf{b} \overset{not}{\equiv} \begin{bmatrix} b_x(x,y,t) \\ b_y(x,y,t) \\ 0 \end{bmatrix}, \ \Gamma_u : \ \mathbf{u}^* \overset{not}{\equiv} \begin{bmatrix} u_x^*(x,y,t) \\ u_y^*(x,y,t) \\ 0 \end{bmatrix} \ \text{and} \ \Gamma_\sigma : \ \mathbf{t}^* \overset{not}{\equiv} \begin{bmatrix} t_x^*(x,y,t) \\ t_y^*(x,y,t) \\ 0 \end{bmatrix}. \quad (7.18)$$

In the central section (which is a plane of symmetry with respect to the z-axis) the conditions

$$u_z = 0 \ , \qquad \frac{\partial u_x}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial u_y}{\partial z} = 0 \quad (7.19)$$

are satisfied and, thus, the displacement field in this central section is of the form

$$\mathbf{u}(x,y,t) \overset{not}{\equiv} \begin{bmatrix} u_x(x,y,t) \\ u_y(x,y,t) \\ 0 \end{bmatrix}. \quad (7.20)$$

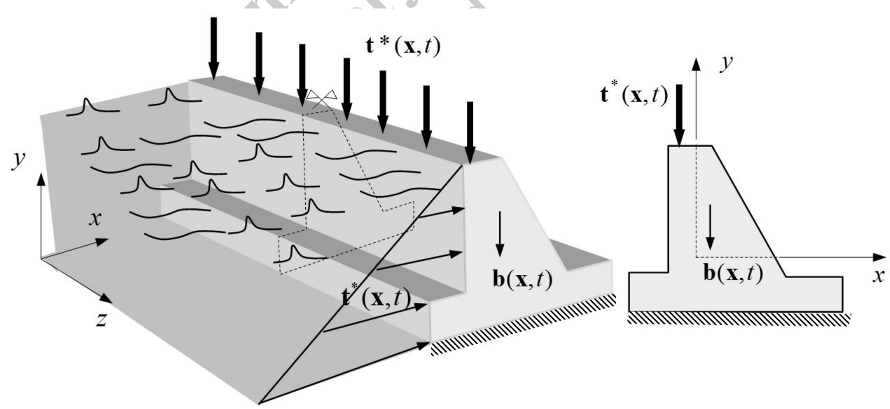

Figure 7.3: Example of a plane strain state.

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7.3.1 Strain and Stress Fields

The strain field corresponding with the displacement field characteristic of a plane strain state (7.20) is

$$\begin{aligned} &\varepsilon_x(x,y,t) = \frac{\partial u_x}{\partial x}\,, && \varepsilon_z(x,y,t) = \frac{\partial u_z}{\partial z} = 0\,, \\ &\varepsilon_y(x,y,t) = \frac{\partial u_y}{\partial y}\,, && \gamma_{xz}(x,y,t) = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0\,, \\ &\gamma_{xy}(x,y,t) = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\,, && \gamma_{yz}(x,y,t) = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} = 0\,. \end{aligned} \tag{7.21}$$

Therefore, the structure of the strain tensor is[3]

$$\boxed{\boldsymbol{\varepsilon}(x,y,t) \stackrel{not}{\equiv} \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & 0 \\ 0 & 0 & 0 \end{bmatrix}\,.} \tag{7.22}$$

Consider now the lineal elastic constitutive equation (6.20)

$$\boldsymbol{\sigma} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\,\mathbf{1} + 2\mu\boldsymbol{\varepsilon} = \lambda\,\mathrm{Tr}(\boldsymbol{\varepsilon})\,\mathbf{1} + 2G\boldsymbol{\varepsilon}\,, \tag{7.23}$$

which, applied to the strain field (7.21), produces the stresses

$$\begin{aligned} &\sigma_x = \lambda(\varepsilon_x + \varepsilon_y) + 2\mu\varepsilon_x = (\lambda + 2G)\,\varepsilon_x + \lambda\varepsilon_y\,, && \tau_{xy} = G\,\gamma_{xy}\,, \\ &\sigma_y = \lambda(\varepsilon_x + \varepsilon_y) + 2\mu\varepsilon_y = (\lambda + 2G)\,\varepsilon_y + \lambda\varepsilon_x\,, && \tau_{xz} = G\,\gamma_{xz} = 0\,, \\ &\sigma_z = \lambda(\varepsilon_x + \varepsilon_y)\,, && \tau_{yz} = G\,\gamma_{yz} = 0\,. \end{aligned} \tag{7.24}$$

Considering (7.21) and (7.24), one concludes that *stresses do not depend on the z-coordinate either* ($\boldsymbol{\sigma} = \boldsymbol{\sigma}(x,y,t)$). On the other hand, the stress σ_z in (7.24) can be solved as

$$\sigma_z = \frac{\lambda}{2(\lambda + \mu)}(\sigma_x + \sigma_y) = \nu(\sigma_x + \sigma_y) \tag{7.25}$$

[3] By analogy with the plane stress case, the fact that all non-null *strains* are contained in the *x*-*y* plane gives rise to the name *plane strain*.

and the stress tensor for the plane strain case results in

$$\boxed{\boldsymbol{\sigma}(x,y,t) \overset{not}{\equiv} \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} \quad \text{with} \quad \sigma_z = -\nu\,(\sigma_x + \sigma_y)} \ , \tag{7.26}$$

where the non-null components of the stress tensor (7.26) are

$$\begin{aligned} \sigma_x &= (\lambda + 2G)\,\varepsilon_x + \lambda\,\varepsilon_y = \frac{E\,(1-\nu)}{(1+\nu)\,(1-2\nu)}\left(\varepsilon_x + \frac{\nu}{1-\nu}\varepsilon_y\right) \ , \\ \sigma_y &= (\lambda + 2G)\,\varepsilon_y + \lambda\,\varepsilon_x = \frac{E\,(1-\nu)}{(1+\nu)\,(1-2\nu)}\left(\varepsilon_y + \frac{\nu}{1-\nu}\varepsilon_x\right) \ , \\ \text{and} \quad \tau_{xy} &= G\,\gamma_{xy} = \frac{E}{2\,(1+\nu)}\gamma_{xy} \ . \end{aligned} \tag{7.27}$$

Equation (7.27) can be rewritten in matrix form as

$$\underbrace{\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}}_{\{\boldsymbol{\sigma}\}} = \underbrace{\frac{E\,(1-\nu)}{(1+\nu)\,(1-2\nu)} \begin{bmatrix} 1 & \dfrac{\nu}{1-\nu} & 0 \\ \dfrac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2\,(1-\nu)} \end{bmatrix}}_{\mathbb{C}^{plane}_{strain}} \underbrace{\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}}_{\{\boldsymbol{\varepsilon}\}} \Rightarrow \tag{7.28}$$

$$\boxed{\{\boldsymbol{\sigma}\} = \mathbb{C}^{plane}_{strain} \cdot \{\boldsymbol{\varepsilon}\}} \ .$$

Similarly to the plane stress problem, (7.20), (7.21) and (7.26) suggest considering an *elastic plane strain problem* reduced to the two dimensions of the plane of analysis x-y and characterized by the unknowns

$$\boxed{\mathbf{u}(x,y,t) \overset{not}{\equiv} \begin{bmatrix} u_x \\ u_y \end{bmatrix} , \ \{\boldsymbol{\varepsilon}(x,y,t)\} \overset{not}{\equiv} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \text{ and } \{\boldsymbol{\sigma}(x,y,t)\} \overset{not}{\equiv} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \ ,} \tag{7.29}$$

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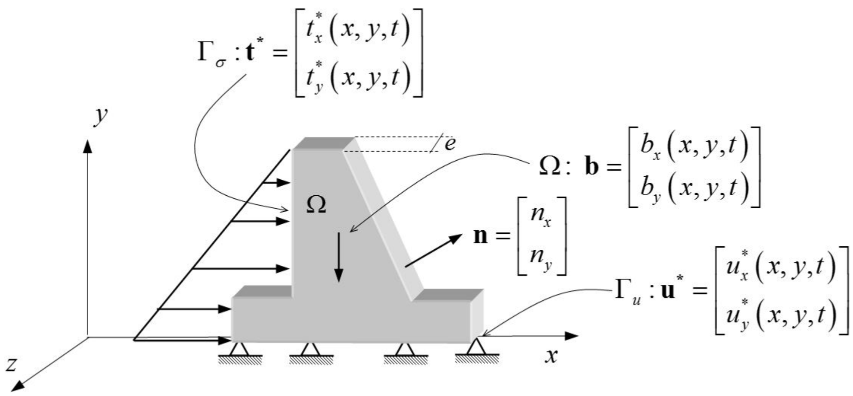

Figure 7.4: The plane linear elastic problem.

in which the additional unknowns with respect to the general problem are either null or can be calculated in terms of those in (7.29),

$$u_z = 0 \ , \quad \varepsilon_z = \gamma_{xz} = \gamma_{yz} = \tau_{xz} = \tau_{yz} = 0 \quad \text{and} \quad \sigma_z = \nu\left(\sigma_x + \sigma_y\right) . \tag{7.30}$$

7.4 The Plane Linear Elastic Problem

In view of the equations in Sections 7.2 and 7.3, the linear elastic problem for the plane stress and plane strain problems is characterized as follows (see Figure 7.4).

Equations [4]

a) Cauchy's equation

$$\boxed{\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho b_x = \rho \frac{\partial^2 u_x}{\partial t^2} \\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho b_y = \rho \frac{\partial^2 u_y}{\partial t^2} \end{aligned}} \tag{7.31}$$

[4] The equation corresponding to the z-component either does not intervene (plane stress), or is identically null (plane strain).

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b) Constitutive equation

$$\boxed{\{\boldsymbol{\sigma}\} \stackrel{not}{\equiv} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} , \quad \{\boldsymbol{\varepsilon}\} \stackrel{not}{\equiv} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} ; \qquad \{\boldsymbol{\sigma}\} = \mathbb{C} \cdot \{\boldsymbol{\varepsilon}\} ,} \tag{7.32}$$

where the constitutive matrix $\mathbb{C}$ can be written in a general form, from (7.11) and (7.28), as

$$\mathbb{C} \stackrel{not}{\equiv} \frac{\bar{E}}{1-\bar{\nu}^2} \begin{bmatrix} 1 & \bar{\nu} & 0 \\ \bar{\nu} & 1 & 0 \\ 0 & 0 & \dfrac{1-\bar{\nu}}{2} \end{bmatrix} \qquad \begin{array}{l} \text{Plane stress} \begin{cases} \bar{E} = E \\ \bar{\nu} = \nu \end{cases} \\ \text{Plane strain} \begin{cases} \bar{E} = \dfrac{E}{1-\nu^2} \\ \bar{\nu} = \dfrac{\nu}{1-\nu} \end{cases} \end{array} \tag{7.33}$$

c) Geometric equation

$$\boxed{\varepsilon_x = \frac{\partial u_x}{\partial x} , \quad \varepsilon_y = \frac{\partial u_y}{\partial y} , \quad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}} \tag{7.34}$$

d) Boundary conditions in space

$$\begin{gathered} \boxed{\Gamma_u : \ \mathbf{u}^* \stackrel{not}{\equiv} \begin{bmatrix} u_x^*(x,y,t) \\ u_y^*(x,y,t) \end{bmatrix} , \qquad \Gamma_\sigma : \ \mathbf{t}^* \stackrel{not}{\equiv} \begin{bmatrix} t_x^*(x,y,t) \\ t_y^*(x,y,t) \end{bmatrix}} \\ \mathbf{t}^* = \boldsymbol{\sigma} \cdot \mathbf{n} , \qquad \boldsymbol{\sigma} \stackrel{not}{\equiv} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} , \quad \mathbf{n} \stackrel{not}{\equiv} \begin{bmatrix} n_x \\ n_y \end{bmatrix} \end{gathered} \tag{7.35}$$

e) Initial conditions

$$\boxed{\mathbf{u}(x,y,t)\Big|_{t=0} = \mathbf{0} , \qquad \dot{\mathbf{u}}(x,y,t)\Big|_{t=0} = \mathbf{v}_0(x,y)} \tag{7.36}$$

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Unknowns

$$\boxed{\mathbf{u}(x,y,t) \stackrel{not}{\equiv} \begin{bmatrix} u_x \\ u_y \end{bmatrix},\ \boldsymbol{\varepsilon}(x,y,t) \stackrel{not}{\equiv} \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y \end{bmatrix},\ \boldsymbol{\sigma}(x,y,t) \stackrel{not}{\equiv} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}} \tag{7.37}$$

Equations (7.31) to (7.37) define a system of 8 PDEs with 8 unknowns that must be solved in the *reduced* space-time domain $\mathbb{R}^2 \times \mathbb{R}^+$. Once the problem is solved, the following can be explicitly calculated:

$$\begin{aligned} &\text{Plane stress} \ \rightarrow \ \varepsilon_z = \frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y) \\ &\text{Plane strain} \ \rightarrow \ \sigma_z = \nu(\sigma_x + \sigma_y) \end{aligned} \tag{7.38}$$

7.5 Problems Typically Assimilated to Plane Elasticity

7.5.1 Plane Stress

The stress and strain states produced in solids that have *a dimension considerably inferior to the other two* (which constitute the plane of analysis *x-y*) *and whose actions are contained in said plane* are typically assimilated to a plane stress state. The slab loaded on its mean plane and the deep beam of Figure 7.5 are classic examples of structures that can be analyzed as being in a plane stress state. As a particular case, the problems of *simple and complex bending in beams* considered in strength of materials can also be assimilated to plane stress problems.

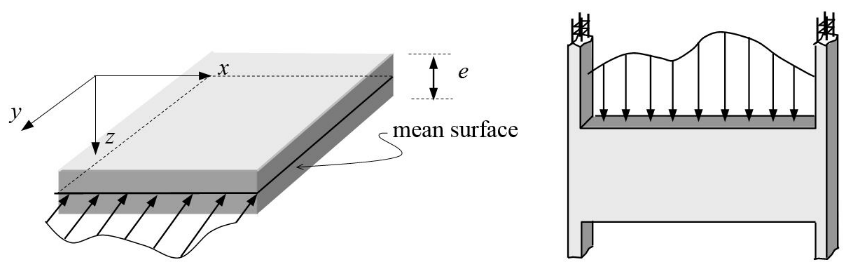

Figure 7.5: Slab loaded on its mean plane (left) and deep beam (right).

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7.5.2 Plane Strain

The solids whose geometry can be obtained by translation of a *plane section with actions contained in its plane* (plane of analysis x-y) along a generatrix line perpendicular to said section are typically assimilated to plane strain states. In addition, the plane strain hypothesis $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$ must be justifiable. In general, this situation occurs in two circumstances:

1) *The dimension of the piece in the z-direction is very large* (for the purposes of analysis, it is assumed to be infinite). In this case, any central transversal section (not close to the extremes) can be considered a symmetry plane and, thus, satisfies the conditions

$$u_z = 0 \ , \qquad \frac{\partial u_x}{\partial z} = 0 \quad \text{and} \qquad \frac{\partial u_y}{\partial z} = 0 \ , \tag{7.39}$$

which result in the initial condition of the plane strain state (7.17),

$$\mathbf{u} \stackrel{not}{\equiv} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} u_x\,(x,y,t) \\ u_y\,(x,y,t) \\ 0 \end{bmatrix} \ . \tag{7.40}$$

Examples of this case are a pipe under internal (and/or external) pressure (see Figure 7.6), a tunnel (see Figure 7.7) and a strip foundation (see Figure 7.8).

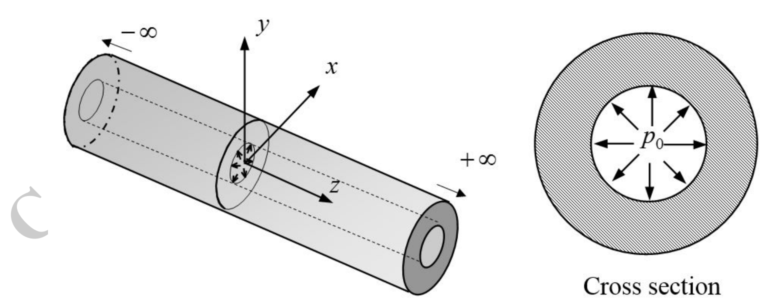

Figure 7.6: Pressure tube.

2) *The length of the piece in the longitudinal direction is reduced, but the displacements in the z-direction are impeded by the boundary conditions at the end sections* (see Figure 7.9).

   In this case, the plane strain hypothesis (7.17) can be assumed for all the transversal sections of the piece.

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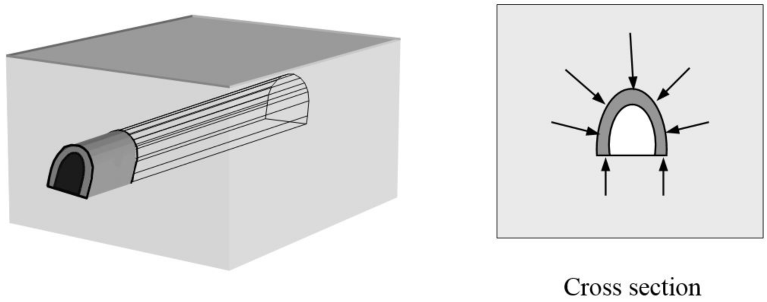

Figure 7.7: Tunnel.

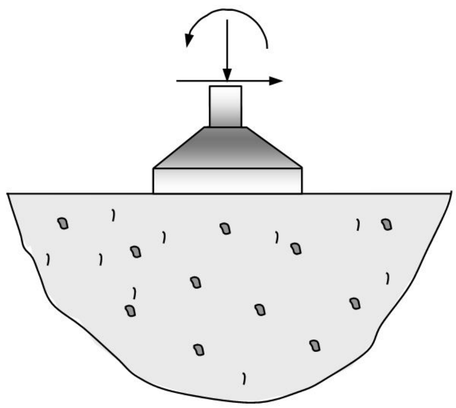

Figure 7.8: Strip foundation.

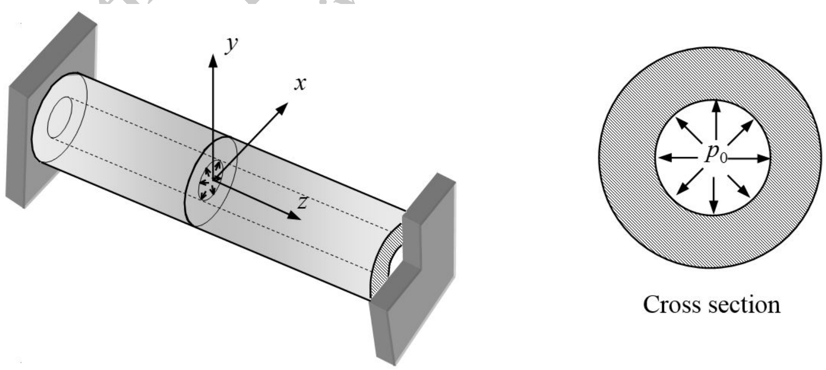

Figure 7.9: Solid with impeded z-displacements.

7.6 Representative Curves of Plane Elasticity

There is an important tradition in engineering of graphically representing the distribution of plane elasticity. To this aim, certain families of curves are used, whose plotting on the plane of analysis provides useful information of said stress state.

7.6.1 Isostatics or stress trajectories

Definition 7.1. The *isostatics* or *stress trajectories* are the *envelopes* of the vector field determined by the principal stresses.

Considering the definition of the envelope of a vector field, isostatics are, at each point, tangent to the two principal directions and, thus, there exist two families of isostatics:

– *Isostatics* σ_1, tangent to the direction of the largest principal stress.
– *Isostatics* σ_2, tangent to the direction of the smallest principal stress.

In addition, since the principal stress directions are orthogonal to each other, *both families of curves are also be orthogonal*. The isostatic lines provide information on the mode in which the flux of principal stresses occurs on the plane of analysis.

As an example, Figure 7.10 shows the distribution of isostatics on a supported beam with uniformly distributed loading.

Definition 7.2. A *singular point* is a point characterized by the stress state

$$\sigma_x = \sigma_y \quad \text{and} \quad \tau_{xy} = 0$$

and its Mohr's circle is *a point* on the axis σ (see Figure 7.11).

A *neutral point* is a singular point characterized by the stress state

$$\sigma_x = \sigma_y = \tau_{xy} = 0$$

and its Mohr's circle is the *origin* of the $\sigma - \tau$ space (see Figure 7.11).

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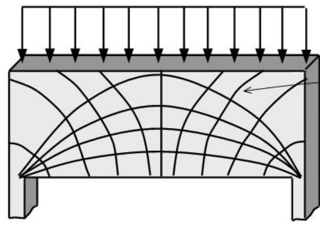

Figure 7.10: Isostatics or stress trajectories on a beam.

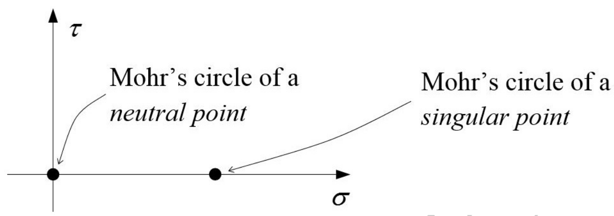

Figure 7.11: Singular and neutral points.

> *Remark 7.3.* All directions in a singular point are principal stress directions (the pole is the Mohr's circle itself, see Figure 7.11). Consequently, the isostatics tend to loose their regularity in singular points and *can brusquely change their direction*.

7.6.1.1 Differential Equation of the Isostatics

Consider the general equation of an isostatic line $y = f(x)$ and the value of the angle formed by the principal stress direction σ_1 with respect to the horizontal direction (see Figure 7.12),

$$\left.\begin{aligned} &\tan(2\alpha) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2\tan\alpha}{1-\tan^2\alpha} \\ &\tan\alpha = \frac{dy}{dx} \overset{not}{=} y' \end{aligned}\right\} \Rightarrow \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2y'}{1-(y')^2} \Rightarrow$$

$$(y')^2 + \frac{\sigma_x - \sigma_y}{\tau_{xy}}\, y' - 1 = 0 \tag{7.41}$$

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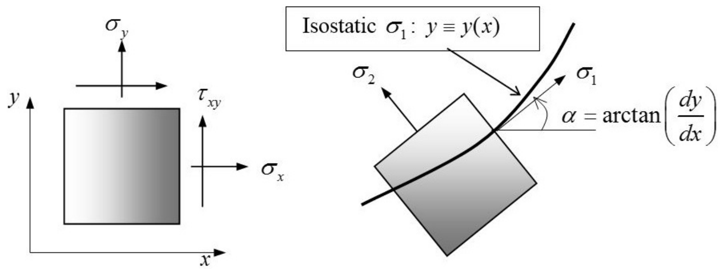


Figure 7.12: Determination of the differential equation of the isostatics.

and solving the second-order equation (7.41) for y', the differential equation of the isostatics is obtained.

$$\boxed{\begin{array}{c}\text{Differential}\\ \text{equation}\\ \text{of the isostatics}\end{array} \rightarrow \; y' = \underbrace{-\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right)^2 + 1}}_{\varphi(x,y)}} \tag{7.42}$$

If the function $\varphi(x,y)$ in (7.42) is known, this equation can be integrated to obtain the algebraic equation of the family of isostatics,

$$y = f(x) + C\,. \tag{7.43}$$

The double sign in (7.42) leads to two differential equations corresponding to the two families of isostatics.

Example 7.1 – *A rectangular plate is subjected to the following stress states.*

$$\sigma_x = -x^3\;; \quad \sigma_y = 2x^3 - 3xy^2\;; \quad \tau_{xy} = 3x^2 y\;; \quad \tau_{xz} = \tau_{yz} = \sigma_z = 0$$

Obtain and plot the singular points and distribution of isostatics.

Solution

The *singular points* are defined by $\sigma_x = \sigma_y$ and $\tau_{xy} = 0$. Then,

$$\tau_{xy} = 3x^2 y = 0 \implies \begin{cases} x = 0 \implies \begin{cases} \sigma_x = -x^3 = 0 \\ \sigma_y = 2x^3 - 3xy^2 = 0 \end{cases} \quad \forall y \\ \\ y = 0 \implies \begin{cases} \sigma_x = -x^3 \\ \sigma_y = 2x^3 - 3xy^2 = 2x^3 \end{cases} \implies x = 0 \end{cases}$$

Therefore, the locus of singular points is the straight line $x = 0$. These singular points are, in addition, neutral points ($\sigma_x = \sigma_y = 0$).

The *isostatics* are obtained from (7.42),

$$y' = \frac{dy}{dx} = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right)^2 + 1} \,,$$

which, for the given data of this problem, results in

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{x}{y} \\[2mm] \dfrac{dy}{dx} = \dfrac{-y}{x} \end{cases} \implies \text{integrating} \implies \boxed{\begin{array}{l} x^2 - y^2 = C_1 \\ xy = C_2 \end{array}} \,.$$

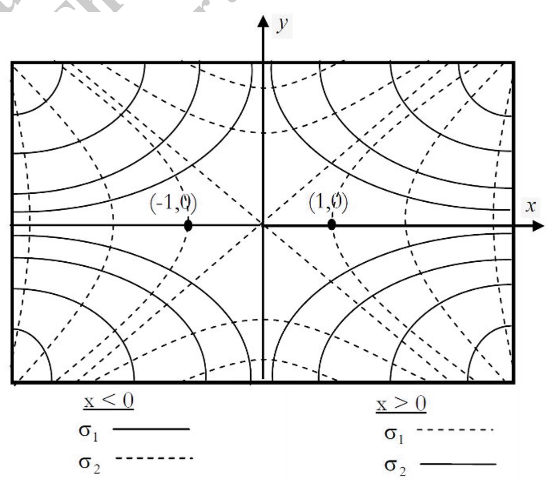
Therefore, *the isostatics are two families of equilateral hyperboles orthogonal to each other*.

On the line of singular points $x = 0$ (which divides the plate in two regions) the isostatics will brusquely change their slope. To identify the family of isostatics σ_1, consider a point in each region:

– Point $(1,0)$: $\sigma_x = \sigma_2 = -1; \quad \sigma_y = \sigma_1 = +2; \quad \tau_{xy} = 0$
(isostatic σ_1 in the y-direction)

– Point $(-1,0)$: $\sigma_x = \sigma_1 = +1; \quad \sigma_y = \sigma_2 = -2; \quad \tau_{xy} = 0$
(isostatic σ_1 in the x-direction)

Finally, the distribution of isostatics is as follows:



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7.6.2 Isoclines

Definition 7.3. *Isoclines* are the locus of the points in the plane of analysis along which the principal stress directions form a certain angle with the x-axis.

It follows from its definition that in all the points of a same isocline the principal stress directions are parallel to each other, forming a constant angle θ (which characterizes the isocline) with the x-axis (see Figure 7.13).

7.6.2.1 Equation of the Isoclines

The equation $y = f(x)$ of the isocline with an angle θ is obtained by establishing that the principal stress direction σ_1 forms an angle $\alpha = \theta$ with the horizontal direction, that is,

$$\boxed{\text{Algebraic equation of the isoclines} \quad \tan(2\theta) = \underbrace{\frac{2\tau_{xy}}{\sigma_x - \sigma_y}}_{\varphi(x,y)}} \tag{7.44}$$

This algebraic equation allows isolating, for each value of θ,

$$y = f(x, \theta) \ , \tag{7.45}$$

which constitutes the equation of the family of isoclines parametrized in terms of the angle θ.

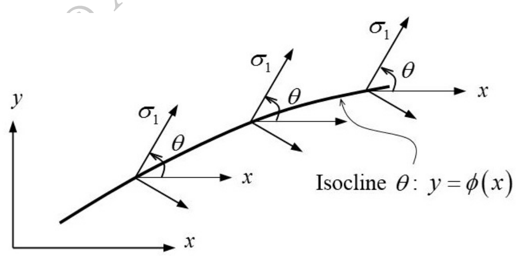

Figure 7.13: Isocline.

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Remark 7.4. Determining the family of isoclines allows knowing, at each point in the medium, the direction of the principal stresses and, thus, the obtainment of the isostatics can be sought. Given that *isoclines* can be determined by means of experimental methods (methods based on photoelasticity) they *provide, indirectly, a method for the experimental determination of the isostatics*.

7.6.3 Isobars

Definition 7.4. *Isobars* are the locus of points in the plane of analysis with the same value of principal stress σ_1 (or σ_2).

Two families of isobars will cross at each point of the plane of analysis: one corresponding to σ_1 and another to σ_2. Note that the isobars depend on the value of σ_1, but not on its direction (see Figure 7.14).

7.6.3.1 Equation of the Isobars

The equation that provides the value of the principal stresses (see Chapter 4) implicitly defines the algebraic equation of the two families of isobars $y = f_1(x, c_1)$ and $y = f_2(x, c_2)$,

$$\boxed{\begin{array}{c}\text{Algebraic}\\ \text{equation}\\ \text{of the}\\ \text{isobars}\end{array} \left\{ \begin{array}{l} \sigma_1 = \underbrace{\dfrac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}_{\varphi_1(x,y)} = const. = c_1 \\[2ex] \sigma_2 = \underbrace{\dfrac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}_{\varphi_2(x,y)} = const. = c_2 \end{array} \right.} \tag{7.46}$$

which leads to

$$\begin{cases} y = f_1(x, c_1) \\ y = f_2(x, c_2) \end{cases} \tag{7.47}$$

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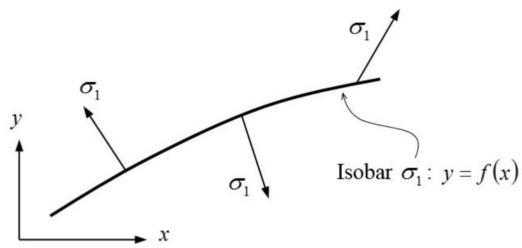

Figure 7.14: Isobars.

7.6.4 *Maximum Shear Stress or Slip Lines*

Definition 7.5. *Maximum shear stress lines* or *slip lines* are the envelopes of the directions that, at each point, correspond with the maximum value (in modulus) of the shear (or tangent) stress.

Remark 7.5. At each point of the plane of analysis there are two planes on which the shear stresses reach the same maximum value (in module) but that have opposite directions, τ_{max} and τ_{min}. These planes can be determined by means of the Mohr's circle and *form a* 45° *angle with the principal stress directions* (see Figure 7.15). Therefore, their envelopes (maximum shear stress lines) are two families of curves *orthogonal to each other* that form a 45° angle with the isostatics.

7.6.4.1 Differential Equation of the Maximum Shear Lines

Consider β is the angle formed by the direction of τ_{max} with the horizontal direction (see Figure 7.16). In accordance with Remark 7.5[5],

$$\beta = \alpha - \frac{\pi}{4} \quad \Longrightarrow \quad \tan(2\beta) = \tan\left(2\alpha - \frac{\pi}{2}\right) = -\frac{1}{\tan(2\alpha)} \,, \tag{7.48}$$

[5] Here, the trigonometric expression $\tan(\theta - \pi/2) = -\cot\theta = -1/\tan\theta$ is used.

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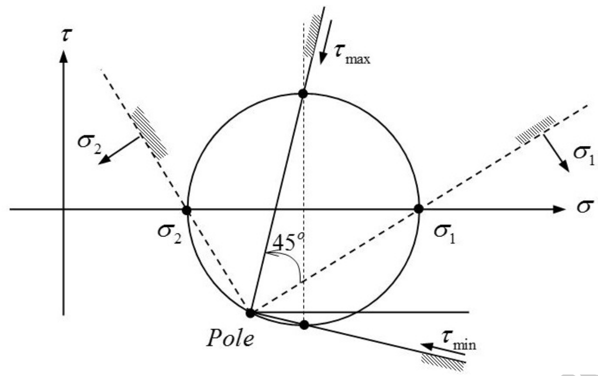

Figure 7.15: Maximum shear stress planes.

where α is the angle formed by the principal stress direction σ_1 with the horizontal direction. Consequently, considering the general equation of a slip line, $y = f(x)$, the expression (7.48) and the relation $\tan(2\alpha) = 2\tau_{xy}/(\sigma_x - \sigma_y)$ yields

$$\left.\begin{array}{l} \tan(2\beta) = -\dfrac{1}{\tan(2\alpha)} = \dfrac{\sigma_x - \sigma_y}{2\tau_{xy}} = \dfrac{2\tan\beta}{1-\tan^2\beta} \\ \tan\beta = \dfrac{dy}{dx} \overset{not}{=} y' \end{array}\right\} \Longrightarrow$$

$$-\frac{\sigma_x - \sigma_y}{2\tau_{xy}} = \frac{2y'}{1-(y')^2} \Longrightarrow \quad (y')^2 - \frac{4\tau_{xy}}{\sigma_x - \sigma_y}\, y' - 1 = 0. \tag{7.49}$$

Solving the second-order equation in (7.49) for y' provides the differential equation of the maximum shear stress lines.

Differential equation of the max. shear stress or slip lines	$y' = \underbrace{-\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y} \pm \sqrt{\left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)^2 + 1}}_{\varphi(x,y)}$	(7.50)

If the function $\varphi(x,y)$ in (7.50) is known, this differential equation can be integrated and the algebraic equation of the two families of orthogonal curves (corresponding to the double sign in (7.50)) is obtained.

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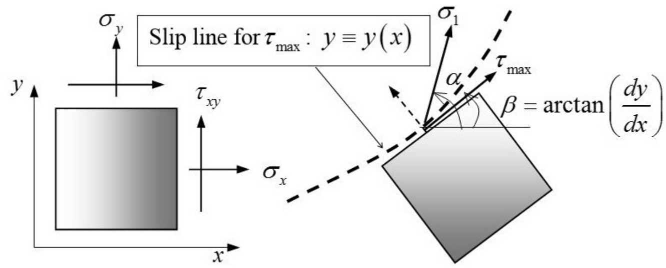

Figure 7.16: Maximum shear stress or slip lines.

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PROBLEMS

Problem 7.1 – *Justify whether the following statements are true or false.*

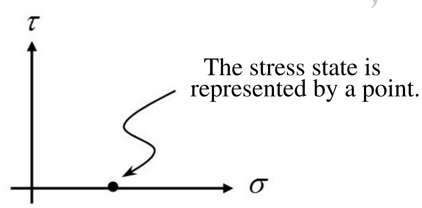
a) If a plane stress state has a singular point, all the isoclines cross this point.

b) If a plane stress state is uniform, all the slip lines are parallel to each other.

Solution

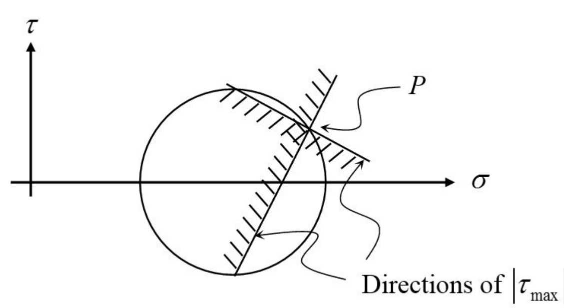
a) A singular point is defined as:

$$\begin{cases} \sigma_1 = \sigma_2 \\ \tau = 0 \end{cases}$$



Therefore, all directions are principal stress directions and, given an angle θ which can take any value, the principal stress direction will form an angle θ with the x-axis. Then, an isocline of angle θ will cross said point and, since this holds true for any value of θ, all the isoclines will cross this point. Therefore, *the statement is true*.

b) A uniform stress state implies that the Mohr's circle is equal in all points of the medium, therefore, the planes of maximum shear stress will be the same in all points. Then, the maximum shear stress lines (or slip lines) will be parallel to each other. In conclusion, *the statement is true*.



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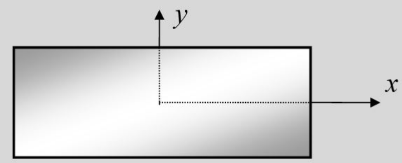
Problem 7.2 – *A rectangular plate is subjected to the following plane stress states.*



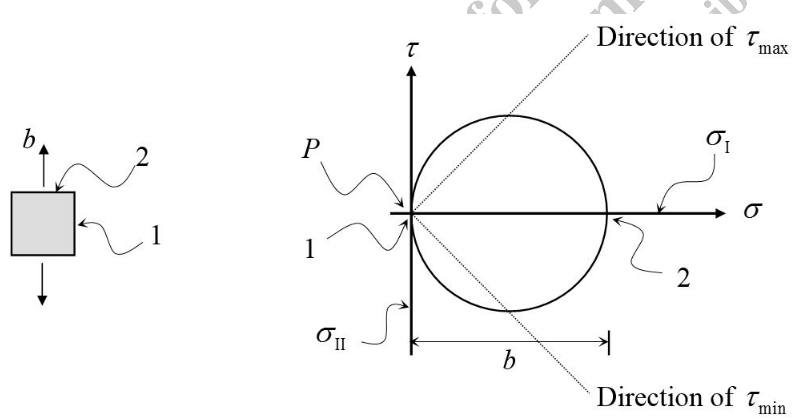
1) $\sigma_x = 0$ *;* $\sigma_y = b > 0$ *;* $\tau_{xy} = 0$

2) $\sigma_x = 0$ *;* $\sigma_y = 0$ *;* $\tau_{xy} = my\ ,\ \ m > 0$

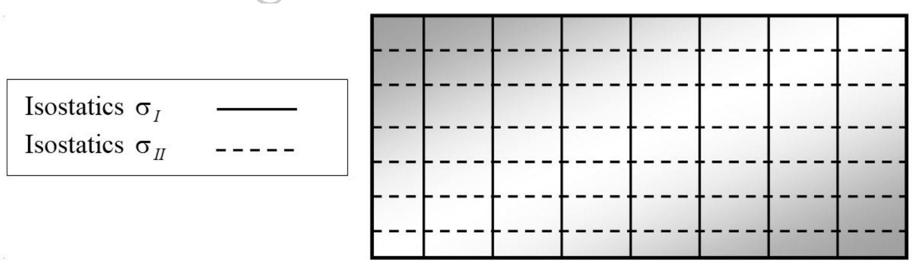
Plot for each state the isostatics and the slip lines, and indicate the singular points.

Solution

1) The Mohr's circle for the stress state $\sigma_x = 0$; $\sigma_y = b > 0$; $\tau_{xy} = 0$ is:



Then, the *isostatics* are:



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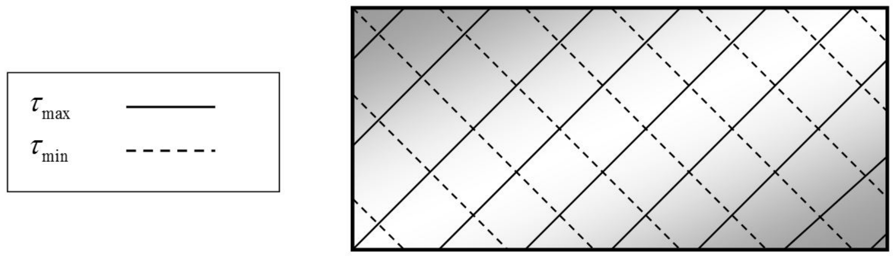
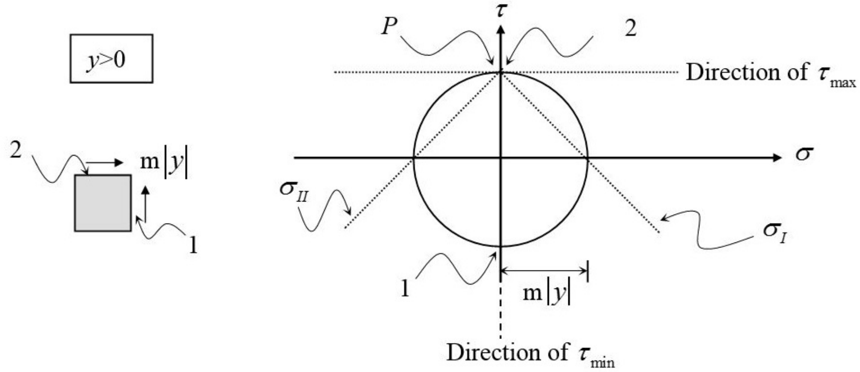
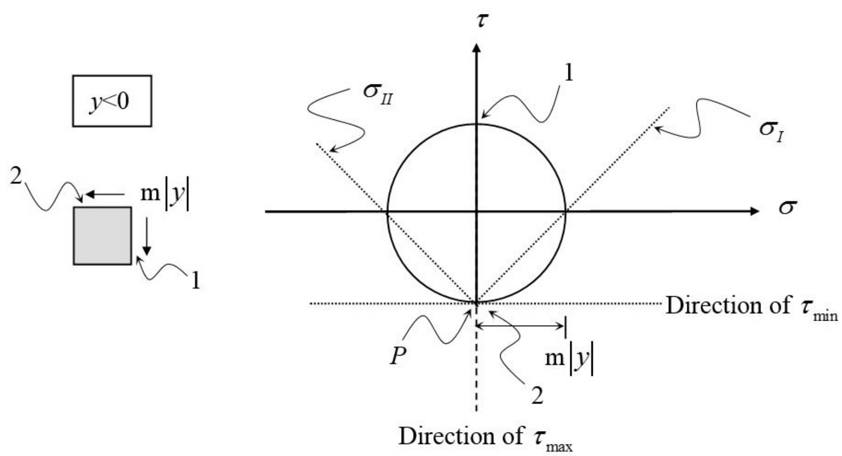
And the *slip lines* are:



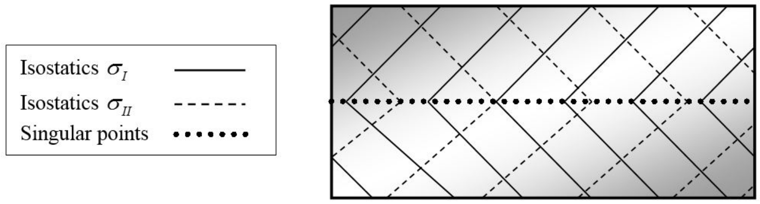
There do not exist singular points for this stress state.

2) The Mohr's circle for the stress state $\sigma_x = 0$; $\sigma_y = 0$; $\tau_{xy} = my$, $m > 0$ is:

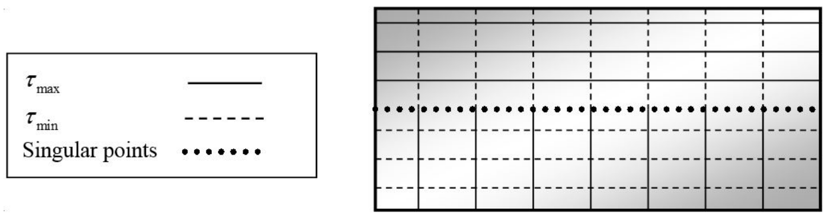




Then, the *isostatics* and singular points are:



And the *slip lines* are:



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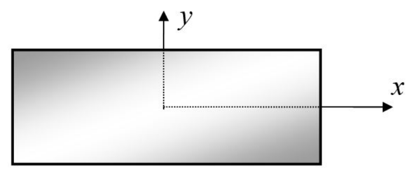
EXERCISES

7.1 – *A rectangular plate is subjected to the following plane strain state:*

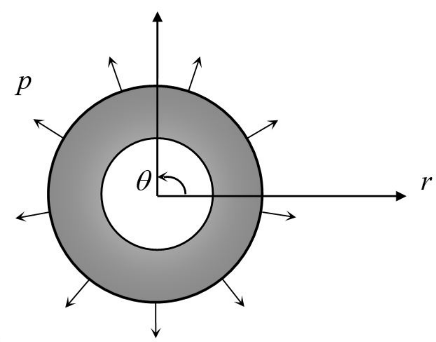
$$\sigma_x = \sigma_y$$
$$\tau_{xy} = ax$$
$$\sigma_y = b$$
$$(a > 0\,,\; b > 0)$$



Plot the isostatics and the slip lines, and indicate the singular points.

7.2 – *Plot the isostatics in the transversal section of the cylindrical shell shown below. Assume a field of the form:*

$$\begin{cases} u_r = Ar + \dfrac{B}{r}\,; \quad A > 0\,,\; B > 0 \\ u_\theta = 0 \\ u_z = 0 \end{cases}$$



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Chapter 8
Plasticity

8.1 Introduction

The elastoplastic models (constitutive equations) are used in continuum mechanics to represent the mechanical behavior of materials whose behavior, once certain limits in the values of the stresses (or strains) are exceeded, is no longer representable by means of simpler models such as the elastic ones. In this chapter, these models will be studied considering, in all cases, that *strains are infinitesimal*.

Broadly speaking, plasticity introduces two important modifications with respect to the lineal elasticity seen in chapters 6 and 7:

1) The *loss of linearity*: stresses cease to be proportional to strains.

2) The concept of *permanent or plastic strain*: a portion of the strain generated during the loading process is not recovered during the unloading process.

8.2 Previous Notions

The concepts in this section are a review of those already studied in Sections 4.4.4 to 4.4.7 of Chapter 4.

8.2.1 Stress Invariants

Consider the Cauchy stress tensor $\boldsymbol{\sigma}$ and its matrix of components in a base associated with the Cartesian axes $\{x,y,z\}$ (see Figure 8.1),

$$[\boldsymbol{\sigma}]_{xyz} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} . \tag{8.1}$$

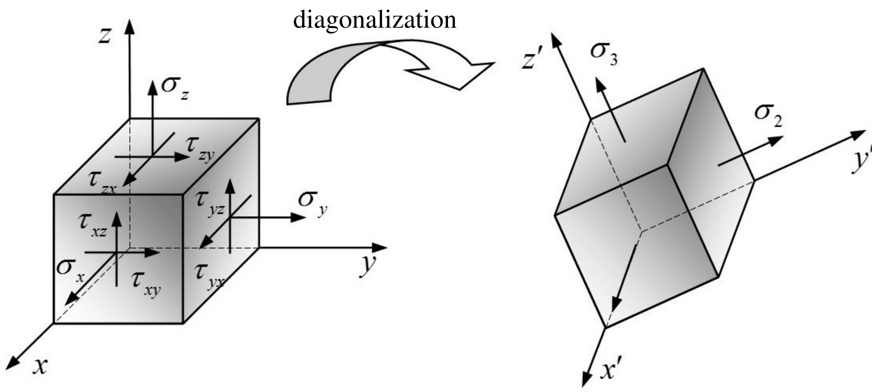

Figure 8.1: Diagonalization of the stress tensor.

Since $\boldsymbol{\sigma}$ is a symmetrical second-order tensor, it will diagonalize in an orthonormal base and all its eigenvalues will be real numbers. Then, consider a system of Cartesian axes $\{x', y', z'\}$ associated with a base in which $\boldsymbol{\sigma}$ diagonalizes. Its matrix of components in this base is

$$[\boldsymbol{\sigma}]_{x'y'z'} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}, \tag{8.2}$$

where $\sigma_1 \geq \sigma_2 \geq \sigma_3$, denoted as *principal stresses*, are the eigenvectors of $\boldsymbol{\sigma}$ and the directions associated with the axes $\{x', y', z'\}$ are named *principal directions* (see Figure 8.1).

To obtain the stresses and the principal directions of $\boldsymbol{\sigma}$, the corresponding eigenvalue and eigenvector problem must be solved:

$$\text{Find } \lambda \text{ and } \mathbf{v} \text{ such that } \boldsymbol{\sigma} \cdot \mathbf{v} = \lambda \mathbf{v} \quad \Longrightarrow \quad (\boldsymbol{\sigma} - \lambda \mathbf{1}) \cdot \mathbf{v} = \mathbf{0} \,, \tag{8.3}$$

where λ corresponds to the eigenvalues and $\mathbf{v}$ to the eigenvectors. The necessary and sufficient condition for (8.3) to have a solution is

$$\det(\boldsymbol{\sigma} - \lambda \mathbf{1}) = |\boldsymbol{\sigma} - \lambda \mathbf{1}| = 0 \,, \tag{8.4}$$

which, in component form, results in

$$\begin{vmatrix} \sigma_x - \lambda & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \lambda & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \lambda \end{vmatrix} = 0 \,. \tag{8.5}$$

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The algebraic development of (8.5), named *characteristic equation*, corresponds to a third-degree polynomial equation in λ, that can be written as

$$\boxed{\lambda^3 - I_1 \lambda^2 - I_2 \lambda - I_3 = 0} \, , \tag{8.6}$$

where the coefficients $I_1(\sigma_{ij})$, $I_2(\sigma_{ij})$ and $I_3(\sigma_{ij})$ are certain functions of the components σ_{ij} of the tensor $\boldsymbol{\sigma}$ expressed in the coordinate system $\{x,y,z\}$. Yet, the solutions to (8.6), which will be a function of its coefficients (I_1, I_2, I_3), are the principal stresses that, on the other hand, are independent of the system of axes chosen to express $\boldsymbol{\sigma}$. Consequently, said coefficients must be *invariant* with respect to any change of base. Therefore, the coefficients I_1, I_2 and I_3 are denoted as *I stress invariants* or fundamental stress invariants and their expression (resulting from the computation of (8.5)) is

$$\textit{I stress invariants} \quad \begin{cases} I_1 = \mathrm{Tr}(\boldsymbol{\sigma}) = \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 = \dfrac{1}{2}\left(\boldsymbol{\sigma} : \boldsymbol{\sigma} - I_1^2\right) = -\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right) \\ I_3 = \det(\boldsymbol{\sigma}) = \sigma_1\,\sigma_2\,\sigma_3 \end{cases} \tag{8.7}$$

Obviously, any scalar function of the stress invariants will also be an invariant and, thus, new invariants can be defined based on the I stress invariants given in (8.7). In particular, the so-called *J stress invariants* are defined as

$$\textit{J stress invariants} \quad \begin{cases} J_1 = I_1 = \sigma_{ii} = \mathrm{Tr}(\boldsymbol{\sigma}) \\[2ex] J_2 = \dfrac{1}{2}\left(I_1^2 + 2I_2\right) = \dfrac{1}{2}\sigma_{ij}\sigma_{ji} = \\ \quad = \dfrac{1}{2}(\boldsymbol{\sigma} : \boldsymbol{\sigma}) = \dfrac{1}{2}\mathrm{Tr}(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) \\[2ex] J_3 = \dfrac{1}{3}\left(I_1^3 + 3I_1I_2 + 3I_3\right) = \dfrac{1}{3}\sigma_{ij}\sigma_{jk}\sigma_{ki} = \\ \quad = \dfrac{1}{3}\mathrm{Tr}(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}) \end{cases} \tag{8.8}$$

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Remark 8.1. Note that

$$I_1 = 0 \quad \Longrightarrow \quad J_i = I_i \qquad i \in \{1,2,3\} \ .$$

Also, the invariants J_i , $i \in \{1,2,3\}$ can be expressed in a unified and compact form by means of

$$J_i = \frac{1}{i}\,\mathrm{Tr}\left(\boldsymbol{\sigma}^i\right) \qquad i \in \{1,2,3\} \ .$$

8.2.2 *Spherical and Deviatoric Components of the Stress Tensor*

Given the stress tensor $\boldsymbol{\sigma}$, the *mean stress* σ_m is defined as

$$\boxed{\sigma_m = \frac{I_1}{3} = \frac{1}{3}\,\mathrm{Tr}(\boldsymbol{\sigma}) = \frac{1}{3}\sigma_{ii} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)} \tag{8.9}$$

and the *mean pressure* $\bar{p}$ as

$$\boxed{\bar{p} = -\sigma_m} \ . \tag{8.10}$$

The Cauchy stress tensor can be decomposed into a *spherical* part (or component), $\boldsymbol{\sigma}_{sph}$, and a *deviatoric* one, $\boldsymbol{\sigma}'$,

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}_{sph} + \boldsymbol{\sigma}'} \ , \tag{8.11}$$

where the spherical part of the stress tensor is defined as

$$\boxed{\begin{array}{l} \boldsymbol{\sigma}_{sph} :\stackrel{def}{=} \dfrac{1}{3}\,\mathrm{Tr}(\boldsymbol{\sigma})\,\mathbf{1} = \sigma_m \mathbf{1} \\ \boldsymbol{\sigma}_{sph} \stackrel{not}{\equiv} \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} \end{array}} \tag{8.12}$$

and, from (8.11) and (8.12), the deviatoric part is given by

$$\boxed{\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \boldsymbol{\sigma}_{sph} \stackrel{not}{\equiv} \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_m \end{bmatrix}} \ . \tag{8.13}$$

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Finally, the I and J invariants of the deviatoric tensor $\boldsymbol{\sigma}'$, named I' and J' invariants, respectively, are derived from (8.7), (8.8), (8.9) and (8.13).

$$
\boxed{\text{J' stress invariants} \quad \begin{cases} J'_1 = I'_1 = 0 \\[2mm] J'_2 = I'_2 = \dfrac{1}{2}\left(\boldsymbol{\sigma}' : \boldsymbol{\sigma}'\right) = \dfrac{1}{2}\sigma'_{ij}\,\sigma'_{ji} \\[2mm] J'_3 = I'_3 = \dfrac{1}{3}\left(\sigma'_{ij}\,\sigma'_{jk}\,\sigma'_{ki}\right) \end{cases}} \tag{8.14}
$$

Remark 8.2. It is easily proven that *the principal directions of* $\boldsymbol{\sigma}$ *coincide with those of* $\boldsymbol{\sigma}'$, that is, that both tensors diagonalize in the same base. In effect, working in the base associated with the principal directions of $\boldsymbol{\sigma}$, i.e., the base in which $\boldsymbol{\sigma}$ diagonalizes, and, given that $\boldsymbol{\sigma}_{sph}$ is a hydrostatic tensor and, thus, is diagonal in any base, then $\boldsymbol{\sigma}'$ also diagonalizes in the same base (see Figure 8.2).

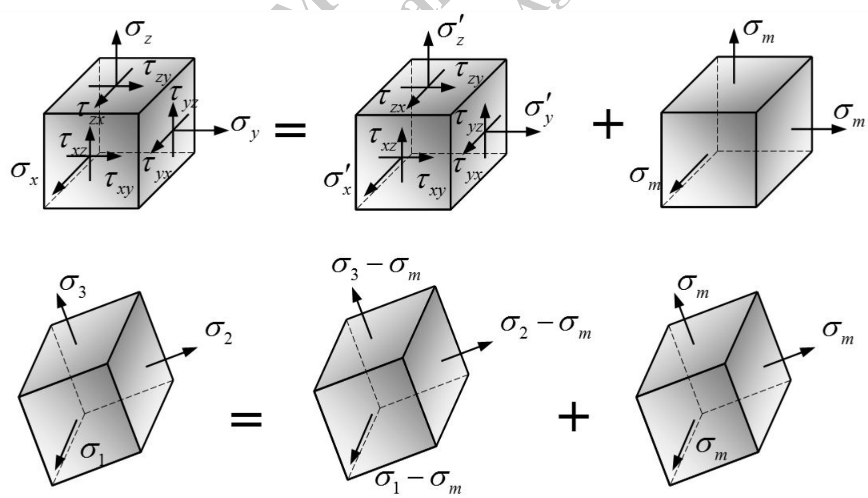

Figure 8.2: Diagonalization of the spherical and deviatoric parts of the stress tensor.

Remark 8.3. The *effective stress* or *equivalent uniaxial stress* $\bar{\sigma}$ is the scalar

$$\bar{\sigma} = \sqrt{3J_2'} = \sqrt{\frac{3}{2}\sigma_{ij}'\sigma_{ji}'} = \sqrt{\frac{3}{2}\boldsymbol{\sigma}':\boldsymbol{\sigma}'} \ .$$

The name of equivalent uniaxial stress is justified because its value for an uniaxial stress state coincides with said uniaxial stress (see Example 8.1).

Example 8.1 – *Compute the value of the equivalent uniaxial stress (or effective stress) $\bar{\sigma}$ for an uniaxial stress state defined by*

$$\boldsymbol{\sigma} \overset{not}{\equiv} \begin{bmatrix} \sigma_u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Solution

The *mean stress* is

$$\sigma_m = \frac{1}{3}\operatorname{Tr}(\boldsymbol{\sigma}) = \frac{\sigma_u}{3} \ .$$

Then, the *spherical component* of the stress tensor is

$$\boldsymbol{\sigma}_{sph} \overset{not}{\equiv} \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} = \begin{bmatrix} \dfrac{\sigma_u}{3} & 0 & 0 \\ 0 & \dfrac{\sigma_u}{3} & 0 \\ 0 & 0 & \dfrac{\sigma_u}{3} \end{bmatrix}$$

and the *deviatoric component* results in

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \boldsymbol{\sigma}_{sph} \overset{not}{\equiv} \begin{bmatrix} \sigma_u - \sigma_m & 0 & 0 \\ 0 & -\sigma_m & 0 \\ 0 & 0 & -\sigma_m \end{bmatrix} = \begin{bmatrix} \dfrac{2}{3}\sigma_u & 0 & 0 \\ 0 & -\dfrac{1}{3}\sigma_u & 0 \\ 0 & 0 & -\dfrac{1}{3}\sigma_u \end{bmatrix} .$$

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Finally, the equivalent uniaxial stress (or effective stress) is obtained,

$$\bar{\sigma} = \sqrt{\frac{3}{2}\,\sigma'_{ij}\sigma'_{ji}} = \sqrt{\frac{3}{2}\,\sigma_u^2\left(\frac{4}{9}+\frac{1}{9}+\frac{1}{9}\right)} = \sqrt{\frac{3}{2}\frac{2}{3}}\,|\sigma_u| = |\sigma_u| \quad \Longrightarrow$$

$$\boxed{\bar{\sigma} = |\sigma_u|}\,.$$

8.3 Principal Stress Space

Consider a system of Cartesian axes in $\mathbb{R}^3\,\{x \equiv \sigma_1,\, y \equiv \sigma_2,\, z \equiv \sigma_3\}$ such that each stress state, characterized by the values of the three principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$, corresponds to a point in this space, which is known as the *principal stress space*[1] (see Figure 8.3).

Definition 8.1. The *hydrostatic stress axis* is the locus of points in the principal stress space that verify the condition $\sigma_1 = \sigma_2 = \sigma_3$ (see Figure 8.3). The points located on the hydrostatic stress axis represent hydrostatic states of stress (see Chapter 4, Section 4.4.5).

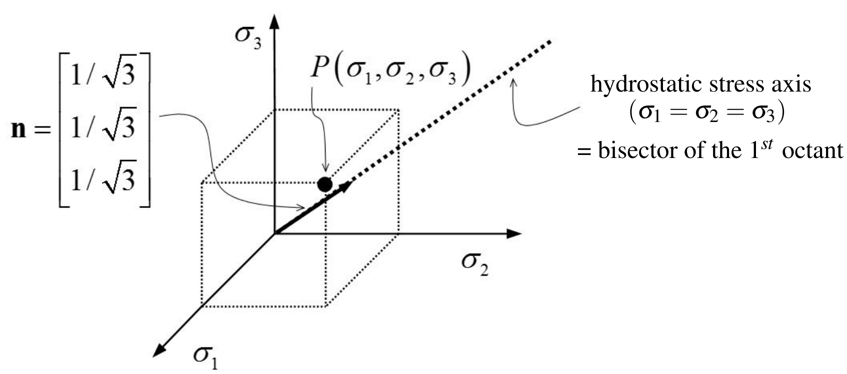

Figure 8.3: The principal stress space.

[1] The principal stress space is also known as the *Haigh-Westergaard stress space*.

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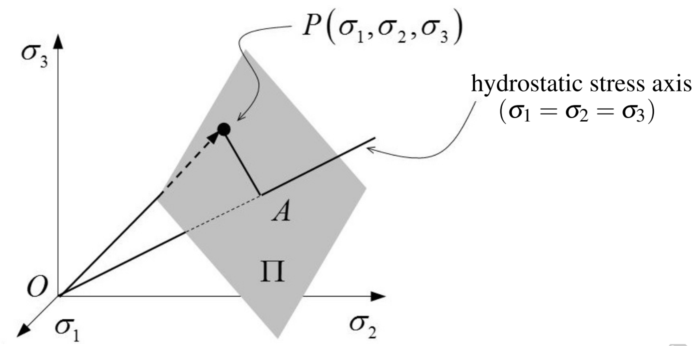

Figure 8.4: The hydrostatic stress axis and the octahedral plane.

Definition 8.2. The *octahedral plane Π* is any of the planes that are perpendicular to the hydrostatic stress axis (see Figure 8.4). The equation of an octahedral plane is

$$\sigma_1 + \sigma_2 + \sigma_3 = const.$$

and the unit normal vector of said plane is

$$\mathbf{n} \stackrel{not}{\equiv} \frac{1}{\sqrt{3}}[1,1,1]^T .$$

8.3.1 Normal and Shear Octahedral Stresses

Consider P is a point in the principal stress space with coordinates $(\sigma_1, \sigma_2, \sigma_3)$. The position vector of this point is defined as $OP \stackrel{not}{\equiv} [\sigma_1, \sigma_2, \sigma_3]^T$ (see Figure 8.5). Now, the octahedral plane Π containing point P is considered. The intersection of the hydrostatic stress axis with said plane defines point A.

Definition 8.3. Based on Figure 8.5, the *normal octahedral stress* is defined as

$$\left|\overline{OA}\right| = \sqrt{3}\,\sigma_{oct}$$

and the *shear or tangential octahedral stress* is

$$\left|\overline{AP}\right| = \sqrt{3}\,\tau_{oct} .$$

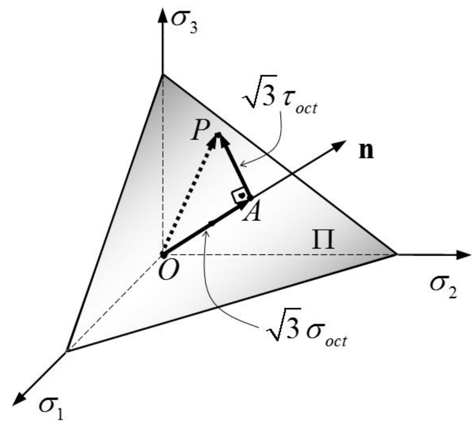

Figure 8.5: Definitions of the normal and shear octahedral stresses.

Remark 8.4. The normal octahedral stress σ_{oct} informs of the distance between the origin O of the principal stress space and the octahedral plane that contains point P. The locus of points in the principal stress space with the same value of σ_{oct} is the octahedral plane placed at a distance $\sqrt{3}\,\sigma_{oct}$ of the origin.
The shear octahedral stress τ_{oct} informs of the distance between point P and the hydrostatic stress axis. It is, thus, a measure of the distance that separates the stress state characterized by point P from a hydrostatic stress state. The locus of points in the principal stress space with the same value of τ_{oct} is a cylinder whose axis is the hydrostatic stress axis and whose radius is $\sqrt{3}\,\tau_{oct}$.

The distance $\left|\overline{OA}\right|$ can be computed as the projection of the vector $\overline{OP}$ on the unit normal vector of the octahedral plane, $\mathbf{n}$,

$$\left.\begin{array}{l} \left|\overline{OA}\right| = \overline{OP}\cdot\mathbf{n} \overset{not}{\equiv} \left[\sigma_1,\,\sigma_2,\,\sigma_3\right]\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3}\end{bmatrix} = \dfrac{\sqrt{3}}{3}\left(\sigma_1+\sigma_2+\sigma_3\right) = \sqrt{3}\sigma_m \\ \\ \left|\overline{OA}\right| = \sqrt{3}\,\sigma_{oct} \end{array}\right\} \Rightarrow \tag{8.15}$$

$$\boxed{\sigma_{oct} = \sigma_m = \frac{I_1}{3}} \tag{8.16}$$

where the definition (8.9) of mean stress σ_m has been taken into account.

The distance $\left|\overline{AP}\right|$ can be obtained solving for the right triangle OAP in Figure 8.5,

$$\left|\overline{AP}\right|^2 = \overline{OP}^2 - \overline{OA}^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2 \ . \tag{8.17}$$

By means of several algebraic operations, this distance can be expressed in terms of the second invariant of the deviatoric stress tensor in (8.14), J_2', as

$$\left.\begin{array}{l} \left|\overline{AP}\right|^2 = 2J_2' \quad \Longrightarrow \quad \left|\overline{AP}\right| = \sqrt{2}\,(J_2')^{1/2} \\ \left|\overline{AP}\right| = \sqrt{3}\,\tau_{oct} \end{array}\right\} \Longrightarrow \tag{8.18}$$

$$\boxed{\tau_{oct} = \sqrt{\frac{2}{3}}\,(J_2')^{1/2}} \tag{8.19}$$

Alternative expressions of τ_{oct} in terms of the value of J_2' in (8.14) are

$$\begin{aligned} \tau_{oct} &= \frac{1}{\sqrt{3}}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)^2\right)^{1/2} \quad \text{and} \\ \tau_{oct} &= \frac{1}{3\sqrt{3}}\left((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right)^{1/2} \ . \end{aligned} \tag{8.20}$$

Remark 8.5. In a pure *spherical* stress state of $\boldsymbol{\sigma}$,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{sph} = \sigma_m \mathbf{1} \ \Leftrightarrow \ \boldsymbol{\sigma}' = \boldsymbol{\sigma} - \boldsymbol{\sigma}_{sph} = \mathbf{0} \ \Leftrightarrow \ J_2' = 0 \ \Leftrightarrow \ \boxed{\tau_{oct} = 0} \, .$$

A spherical stress state is characterized by $\tau_{oct} = 0$ and, thus, is located on the hydrostatic stress axis (see Figure 8.5).

In a pure *deviatoric* stress state of $\boldsymbol{\sigma}$,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' \ \Leftrightarrow \ \sigma_m = \mathrm{Tr}\,(\boldsymbol{\sigma}) = \mathrm{Tr}\left(\boldsymbol{\sigma}'\right) = 0 \ \Leftrightarrow \ \boxed{\sigma_{oct} = 0} \, .$$

A deviatoric stress state is characterized by $\sigma_{oct} = 0$ and, therefore, is located on the octahedral plane containing the origin of the principal stress space.

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Remark 8.6. A point P of the principal stress space is univocally characterized by the three invariants $I_1 \equiv J_1$, J_2' and J_3' (see Figure 8.6):

- The first stress invariant I_1 characterizes the distance ($= \sqrt{3}\,\sigma_{oct}$) from the origin to the octahedral plane Π containing this point through the relation $\sigma_{oct} = I_1/3$. Thus, it places point P in a certain octahedral plane.
- The second deviatoric stress invariant J_2' characterizes the distance ($= \sqrt{3}\,\tau_{oct}$) from the hydrostatic stress axis to the point. Thus, it places point P on a certain circle in the octahedral plane with center in the hydrostatic stress axis and radius $\sqrt{3}\,\tau_{oct} = \sqrt{2}\,(J_2')^{1/2}$.
- The third deviatoric stress invariant J_3' characterizes the position of the point on this circle by means of an angle $\theta\,(J_3')$.

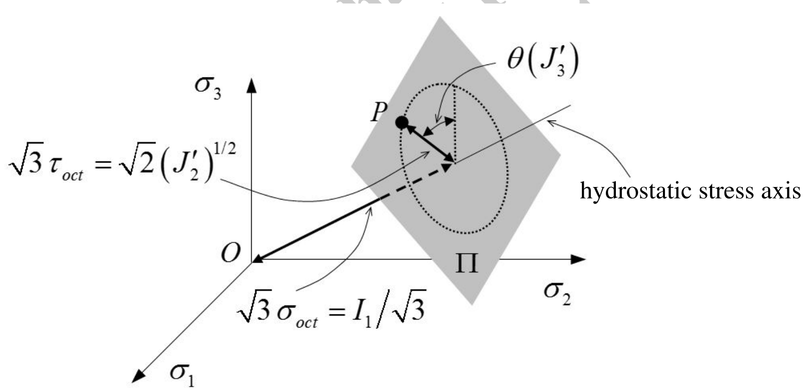

Figure 8.6: Univocal definition of a point by means of the invariants I_1, J_2' and J_3'.

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Remark 8.7. Figure 8.7 shows the projection of the principal stress space on an octahedral plane Π. The division of the stress space into six sectors can be observed in this projection. Each sector is characterized by a different ordering of the principal stresses and the sectors are separated by the projections on the plane of the bisectors $\sigma_2 = \sigma_3$, $\sigma_1 = \sigma_3$ and $\sigma_1 = \sigma_2$.

Selecting the criterion $\sigma_1 \geq \sigma_2 \geq \sigma_3$ automatically reduces the feasible work domain to the sector marked in gray in the figure. The intersection of any surface of the type $f(\sigma_1, \sigma_2, \sigma_3) = 0$ with the plane Π is reduced to a curve in said sector.

This curve can be automatically extended to the rest of sectors, that is, the curve obtained with the same function $f(\sigma_1, \sigma_2, \sigma_3) = 0$ but considering the different orderings of the principal stresses can be easily plotted, by considering the symmetry conditions with respect to the bisector planes. The resulting curve presnts, thus, three axes of symmetry with respect to each of the axis in Figure 8.7.

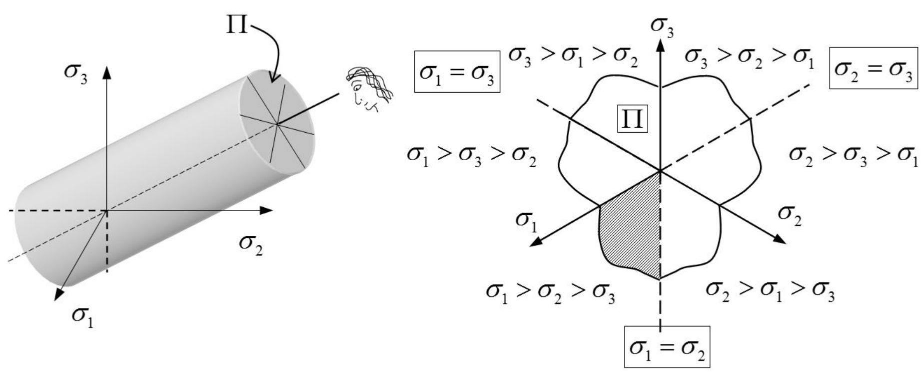

Figure 8.7: Projection on an octahedral plane.

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8.4 Rheological Models

Rheological models are idealizations of mechanical models, constructed as a combination of simple *elements*, whose behavior is easily intuitable, and that allow perceiving more complex mechanical behaviors. Here, as a step previous to the analysis of elastoplastic models, frictional rheological models will be used to introduce the concept of irrecoverable or permanent strain and its consequences.

8.4.1 Elastic Element (Spring Element)

The elastic rheological model is defined by a spring with constant E (see Figure 8.8). The model establishes a proportionality between stress and strain, both in loading and unloading, being the constant E the proportionality factor (see Figure 8.8).

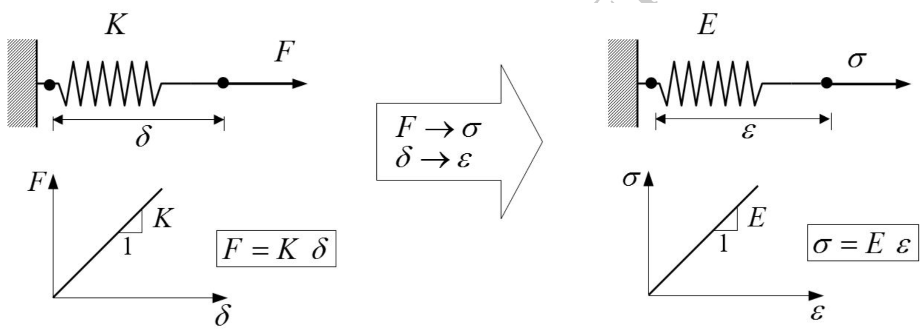

Figure 8.8: Stress-strain relation in an elastic model.

8.4.2 Frictional Element

Consider a solid block placed on a rough surface (see Figure 8.9) and subjected to a vertical compressive load N and a horizontal load F (positive rightward and negative leftward). δ is the horizontal displacement of the block. The Coulomb friction model[2] establishes that the modulus of the reaction force R exerted by the contact surface on the block cannot exceed a certain limit value $F_u = \mu N$, where $\mu \geq 0$ is the *friction coefficient* between the block and the surface. Consequently, while the load F is below said limit value, the block does not move. When the limit value $F_u = \mu N$ is reached, the block starts moving in a quasi-static state (without any acceleration). To maintain the quasi-static regime, this limit value must no be exceeded. These concepts can be mathematically expressed as

[2] The Coulomb friction model is also known as *dry friction model*.

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$$
\begin{aligned}
&|F| < \mu N \quad &\Longleftrightarrow \quad &\delta = 0 \quad &&\text{there is no motion ,}\\
&|F| = \mu N \quad &\Longleftrightarrow \quad &\delta \neq 0 \quad &&\text{there is motion ,}\\
&|F| > \mu N \quad & & &&\text{impossible .}
\end{aligned}
\tag{8.21}
$$

The behavior of the Coulomb friction model, in terms of the force-displacement relation $F - \delta$, is graphically represented in Figure 8.9, both for positive values of the load F (rightward motion) and negative ones (leftward motion).

By analogy with the mechanical friction model, the frictional rheological model in Figure 8.10 is defined, where σ is the stress (analogous to the load F in the Coulomb model) that acts on the device and ε is the strain suffered by this device (analogous to the displacement δ). This rheological model includes a frictional device characterized by a limit value σ_e (analogous to the role of μN in the Coulomb model) whose value cannot be exceeded.

Figure 8.11 shows the stress-strain curve corresponding to the frictional rheological model for a loading-unloading-reloading cycle, which can be split into the following sections.

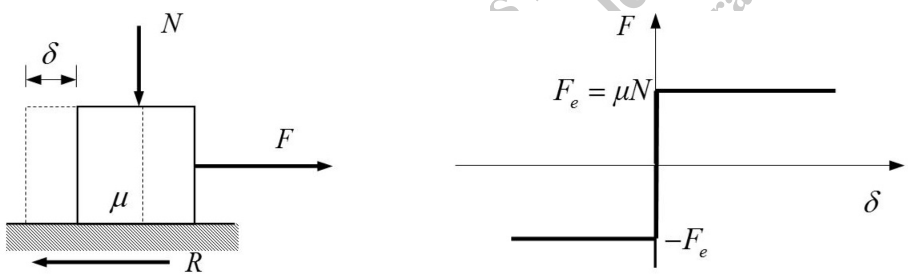
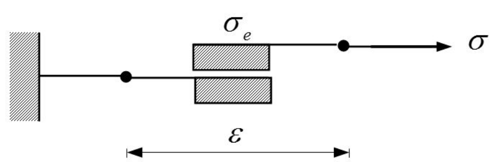


Figure 8.9: Coulomb's law of friction.



$$
\begin{aligned}
|\sigma| < \sigma_e &\;\rightarrow\; \Delta\varepsilon = 0\\
|\sigma| = \sigma_e &\;\rightarrow\; \Delta\varepsilon \neq 0\\
|\sigma| > \sigma_e &\;\rightarrow\; \text{impossible}
\end{aligned}
$$

Figure 8.10: Frictional rheological model.

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Section $0-1$: The (tensile) stress σ increases until the threshold value $\sigma = \sigma_e$ is reached. There is no strain.

Section $1-2$: Once the threshold $\sigma = \sigma_e$ has been reached, stress cannot continue increasing although it can keep its value constant. Then, the frictional element *flows*, generating a strain ε that grows indefinitely while the stress is maintained (*loading* process).

Section $2-3$: At point 2, the tendency of the stress is inverted, stress starts decreasing $(\Delta\sigma < 0)$ and *unloading* begins $(\sigma < \sigma_e)$. Further strain increase is automatically halted $(\Delta\varepsilon = 0)$. This situation is maintained until stress is canceled $(\sigma = 0)$ at point 3. Note that, if the process was to be halted at this point, the initial state of null stress would be recovered but not the initial state of null strain. Instead, a *permanent or residual strain* would be observed $(\varepsilon \neq 0)$. This reveals that, in this model, the trajectory of the stress-strain curve is different in the loading and unloading regimes and that the deformation process is (from a thermodynamic point of view) irreversible in character.

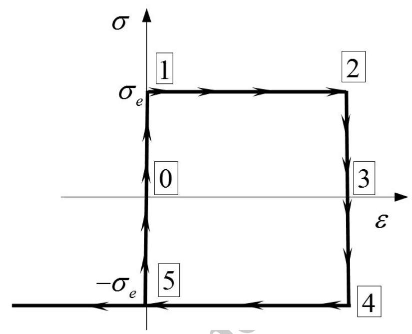

Figure 8.11: Stress-strain curve for a loading-unloading-reloading cycle in a frictional rheological model.

Section $3-4$: Beyond point 3, the sign of the stress is inverted and stress becomes compressive. However, since $|\sigma| < \sigma_e$, no changes in strain are observed $(\Delta\varepsilon = 0)$.

Section $4-5$: At point 4, the criterion $|\sigma| = \sigma_e$ is satisfied and the model enters a loading regime again. The element *flows* at a constant stress value $\sigma = -\sigma_e$, generating negative strain $(\Delta\varepsilon < 0)$, which progressively reduces the accumulated strain. Finally, at point 5, the initial strain state is recovered, but not the original stress state. Beyond this point, if unloading was imposed, there would be a corresponding decrease in stress until the cycle was closed at point 0. Conversely, the loading regime could continue, generating a permanent negative strain.

8.4.3 *Elastic-Frictional Model*

The basic rheological elements, elastic and frictional, can be combined to produce a more complex model, named *elastic-frictional model*, by placing an elastic element, characterized by the parameter E, in series with a frictional element, characterized by the parameter σ_e (denoted as *elastic limit*), as shown in Figure 8.12. Consider σ is the stress that acts on the model and ε is the total strain of this model. Since the basic elements are placed in series, the same stress will act on both of them. On the other hand, the total strain can be decomposed into the sum of the strain experienced by the elastic element (ε^e) plus the strain experienced by the frictional device (ε^f). The same logic can be applied at incremental level.

$$\boxed{\left.\begin{array}{l}\sigma = \sigma^e = \sigma^f \\ \varepsilon = \varepsilon^e + \varepsilon^f = \dfrac{\sigma}{E} + \varepsilon^f \\ \Delta\varepsilon = \Delta\varepsilon^e + \Delta\varepsilon^f\end{array}\right\} \begin{array}{c}\text{Additive decomposition} \\ \text{of strain}\end{array}} \tag{8.22}$$

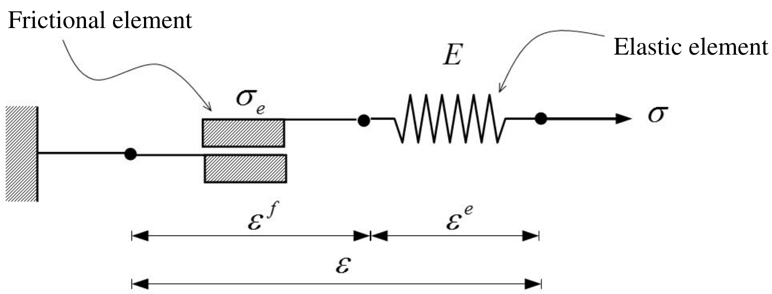

Figure 8.12: Elastic-frictional element.

Taking into account the stress-strain behavior of each basic element that composes the rheological model, the combined model will satisfy:

- $\boxed{|\sigma| < \sigma_e} \Longrightarrow \Delta\varepsilon^f = 0 \Longrightarrow \Delta\varepsilon = \Delta\varepsilon^e \Longrightarrow \boxed{\begin{cases}\Delta\varepsilon = \Delta\varepsilon^e \\ \Delta\sigma = E\Delta\varepsilon\end{cases}}$

  The frictional element does not deform for stresses $|\sigma| < \sigma_e$, therefore *all strains are absorbed by the elastic element*.

- $\boxed{|\sigma| = \sigma_e} \Longrightarrow \Delta\varepsilon^f \neq 0 \Longrightarrow \varepsilon = \dfrac{\sigma}{E} + \varepsilon^f \Longrightarrow$

  $$\boxed{\begin{cases}|\sigma| = \sigma_e \\ \Delta\varepsilon = \Delta\varepsilon^f \Longrightarrow \Delta\varepsilon^e = 0 \Longrightarrow \Delta\sigma = 0\end{cases}}$$

  All strain increments are absorbed by the frictional element with a null increment of stress.

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- $\boxed{|\sigma| > \sigma_e}$

  This is incompatible with the characteristics of the frictional element.

Figure 8.13 shows the stress-strain curve for a loading-unloading-reloading cycle of the elastic-frictional model, which can be decomposed into the following sections.

Section $0-1$:

$$|\sigma| < \sigma_e \implies \Delta\varepsilon^f = 0 \implies \Delta\varepsilon = \Delta\varepsilon^e$$

This section corresponds to the *elastic loading* phase. At the end of the loading, at point 1, the strain is $\varepsilon = \varepsilon^e = \sigma_e/E$. The value of σ_e at the end of this elastic section justifies its denomination as *elastic limit*.

Section $1-2$:

$$|\sigma| = \sigma_e \implies \Delta\varepsilon^f \neq 0 \implies \begin{cases} \varepsilon = \dfrac{\sigma_e}{E} + \varepsilon^f \\ \Delta\varepsilon = \Delta\varepsilon^f > 0 \end{cases}$$

This section corresponds to the *frictional loading* during which no deformation is generated in the elastic element *(no elastic strain is generated)* and all increments of strain are absorbed by the frictional element.

Section $2-3$:

$$|\sigma| < \sigma_e \implies \Delta\varepsilon^f = 0 \implies \Delta\varepsilon = \Delta\varepsilon^e$$

This section corresponds to the *elastic unloading*. At the end of the unloading, at point 3, the initial state of null stress is recovered $(\sigma = 0)$. Consequently, the elastic strain at this point is $\varepsilon^e = \sigma/E = 0$ and, thus, the residual or irrecoverable strain is $\varepsilon = \varepsilon^f \neq 0$. That is, the strain generated by the frictional element during the frictional loading section $1-2$ is not recovered during this phase of stress relaxation to zero. This allows qualifying *the frictional component of strain* ε^f *as an irrecoverable or irreversible strain.*

Section $3-4$:

$$|\sigma| < \sigma_e \implies \Delta\varepsilon^f = 0 \implies \Delta\varepsilon = \Delta\varepsilon^e$$

This section corresponds to the *elastic reloading* phase, similar to section $0-1$ but with a compressive stress $(\sigma < 0)$. The frictional component of strain is not modified during the reloading and the final value, at point 4, of the elastic strain is $\varepsilon^e = -\sigma_e/E$.

Section $4-5$:

$$|\sigma| = \sigma_e \implies \Delta\varepsilon^f \neq 0 \implies \begin{cases} \varepsilon = -\dfrac{\sigma_e}{E} + \varepsilon^f \\ \Delta\varepsilon = \Delta\varepsilon^f < 0 \end{cases}$$

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This section corresponds to the *frictional reloading* during which negative frictional strain is generated $\left(\Delta \varepsilon^f < 0\right)$. Therefore, the total value of the frictional strain decreases until it becomes zero at point 5 (characterized by $\varepsilon = \varepsilon^e = -\sigma_e/E$ and $\varepsilon^f = 0$). An additional elastic unloading at this point would result in recovering the initial state 0.

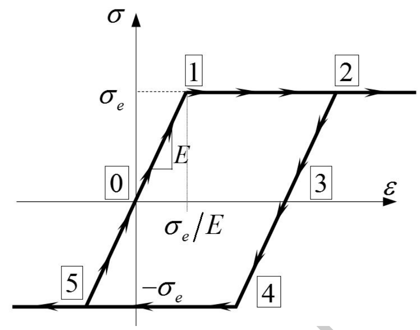

Figure 8.13: Stress-strain curve for a loading-unloading-reloading cycle in an elastic-frictional rheological model.

8.4.4 Frictional Model with Hardening

Consider the rheological model in Figure 8.14 composed of an elastic element (characterized by the parameter H', which will be denoted as *hardening modulus*) and a frictional element (characterized by the *elastic limit* σ_e) placed in parallel. The parallel arrangement results in both rheological elements sharing the same strain, while the total stress in the model is the sum of the stress in the frictional element $(\sigma^{(1)})$ plus the stress in the elastic element $(\sigma^{(2)})$.

$$\boxed{\begin{array}{l} \begin{cases} \sigma = \sigma^{(1)} + \sigma^{(2)} \\ \Delta\sigma = \Delta\sigma^{(1)} + \Delta\sigma^{(2)} \end{cases} \\ \varepsilon = \varepsilon^e = \varepsilon^f \end{array}} \tag{8.23}$$

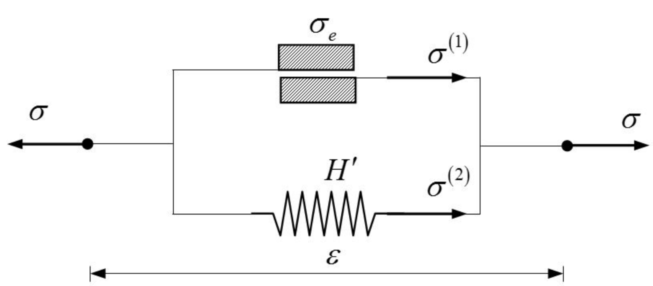

Figure 8.14: Frictional model with hardening.

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Analyzing separately the behavior of each element results in:

a) Frictional element

$$\boxed{\begin{array}{ll} \left|\sigma^{(1)}\right| < \sigma_e & \Delta\varepsilon^f = \Delta\varepsilon = 0 \\ \left|\sigma^{(1)}\right| = \sigma_e & \Delta\varepsilon^f = \Delta\varepsilon \neq 0 \\ \left|\sigma^{(1)}\right| > \sigma_e & \textit{impossible} \end{array}} \tag{8.24}$$

b) Elastic element

$$\boxed{\begin{cases} \sigma^{(2)} = H'\varepsilon^e = H'\varepsilon \\ \Delta\sigma^{(2)} = H'\Delta\varepsilon^e = H'\Delta\varepsilon \end{cases}} \tag{8.25}$$

c) Combining (8.24) and (8.25) leads to

$$\left|\sigma^{(1)}\right| = \left|\sigma - \sigma^{(2)}\right| = \left|\sigma - H'\varepsilon\right| \tag{8.26}$$

In agreement with (8.24) and (8.25), the following situations can be established regarding the rheological model:

- $\left|\sigma^{(1)}\right| < \sigma_e \iff \boxed{\left|\sigma - H'\varepsilon\right| < \sigma_e} \Longrightarrow \begin{cases} \Delta\varepsilon^f = \Delta\varepsilon = 0 \\ \Delta\sigma^{(2)} = H'\Delta\varepsilon^e = H'\Delta\varepsilon = 0 \end{cases}$

  $\Longrightarrow \boxed{\begin{cases} \Delta\sigma = \Delta\sigma^{(1)} \\ \Delta\varepsilon = 0 \end{cases}}$

  All the stress is absorbed by the frictional device and strain is null.

- $\left|\sigma^{(1)}\right| = \sigma_e \iff \boxed{\left|\sigma - H'\varepsilon\right| = \sigma_e} \Longrightarrow \begin{cases} \left|\sigma^{(1)}\right| = \sigma_e \\ \left|\sigma^{(2)}\right| = \left|\sigma - \sigma^{(1)}\right| \end{cases}$

  $\Longrightarrow \boxed{\Delta\sigma^{(2)} = \Delta\sigma = H'\Delta\varepsilon}$

  All stress increments are totally absorbed by the elastic element.

Figure 8.15 shows the stress-strain curve corresponding to this rheological model for a loading-unloading-reloading cycle, which can be decomposed into the following sections.

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Section $0-1$:

$$\left|\sigma^{(1)}\right| < \sigma_e \implies \Delta\varepsilon = 0 \implies \begin{cases} \Delta\sigma^{(2)} = E\Delta\varepsilon = 0 \\ \Delta\sigma^{(1)} = \Delta\sigma \end{cases}$$

In this section all the stress is absorbed by the frictional element. At the end of the section, at point 1, the strain is $\varepsilon = 0$ and the stress is $\sigma = \sigma_e$. This section is characterized by the condition

$$\boxed{\left|\sigma - H'\varepsilon\right| < \sigma_e} \,.$$

Section $1-2$:

$$\left|\sigma^{(1)}\right| = \sigma_e \implies \begin{cases} \sigma = \sigma_e + \sigma^{(2)} \\ \Delta\sigma = \Delta\sigma^{(2)} = H'\Delta\varepsilon \end{cases}$$

This is a loading section in which all stress is absorbed by the elastic element. In global terms, the model increases its capacity to resist stress (the model is said to suffer *hardening*) proportionally to the increment of strain, being the proportionality factor the *hardening modulus* H'. This section is characterized by the condition

$$\boxed{\left|\sigma - H'\varepsilon\right| = \sigma_e} \,.$$

Section $2-3$:

$$\left|\sigma^{(1)}\right| < \sigma_e \implies \Delta\varepsilon = 0 \implies \begin{cases} \Delta\sigma^{(1)} = \Delta\sigma \\ \Delta\sigma^{(2)} = 0 \end{cases}$$

In this section the stress in the frictional element decreases with a null increment of strain and keeping the stress constant in the elastic element. This state is maintained until stress is totally inverted in the frictional element. Thus, at point 3, the stress is $\sigma^{(1)} = -\sigma^e$. This section is characterized by the condition

$$\boxed{\left|\sigma - H'\varepsilon\right| < \sigma_e} \,.$$

Section $3-4$:

$$\Big|\underbrace{\sigma^{(1)}}_{-\sigma^e}\Big| = \sigma_e \implies \begin{cases} \sigma = -\sigma_e + \sigma^{(2)} \\ \Delta\sigma = \Delta\sigma^{(2)} = H'\Delta\varepsilon \end{cases}$$

The situation is symmetrical with respect to section $1-2$, with the elastic element decreasing the stress it can bear, until the stress becomes null at point 3,

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where $\sigma^{(1)} = -\sigma_e$ and $\sigma^{(2)} = 0$. This section is characterized by the condition

$$\boxed{\left|\sigma - H'\varepsilon\right| = \sigma_e} \; .$$

Beyond this point, relaxation of the stress in the frictional element leads to the original state at point 0.

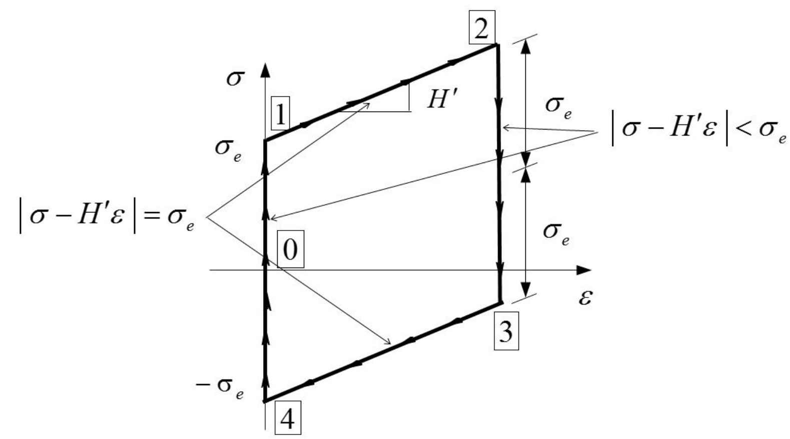

Figure 8.15: Stress-strain curve for a loading-unloading-reloading cycle in a frictional rheological model with hardening.

8.4.5 *Elastic-Frictional Model with Hardening*

Combining now an elastic element, with elastic modulus E, in series with the frictional model introduced in section 8.4.4, which has a hardening modulus H' and an elastic limit σ_e, the elastic-frictional model with hardening shown in Figure 8.16 is obtained.

Applying the stress equilibrium and strain compatibility equations on the model (see Figure 8.16) results in

$$\boxed{\begin{array}{l} \begin{cases} \varepsilon = \varepsilon^e + \varepsilon^f \\ \Delta\varepsilon = \Delta\varepsilon^e + \Delta\varepsilon^f \end{cases} \rightarrow \begin{array}{c}\text{Additive decomposition}\\ \text{of strain}\end{array} \\ \begin{cases} \sigma = \sigma^e = \sigma^f \\ \Delta\sigma = \Delta\sigma^e = \Delta\sigma^f \end{cases} \end{array}} \tag{8.27}$$

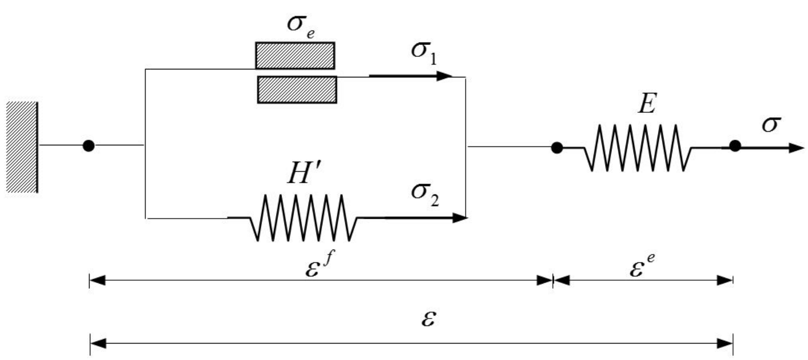

Figure 8.16: Elastic-frictional model with hardening.

where σ^e and σ^f represent, respectively, the stresses sustained by the elastic element and the frictional model with hardening. Combining now the behavior of an elastic element (see Figure 8.8) with that of the frictional model with hardening in Figure 8.14, yields the following situations:

- $\boxed{\left|\sigma - H'\varepsilon_f\right| < \sigma_e} \implies \begin{cases} \Delta\varepsilon^f = 0 \\ \Delta\sigma = \Delta\varepsilon^e \end{cases} \implies \boxed{\Delta\sigma = E\Delta\varepsilon}$

  The frictional element with hardening does not deform and the increment of strain $\Delta\varepsilon$ is completely absorbed by the elastic element. This case is denoted as *elastic process*.

- $\boxed{\left|\sigma - H'\varepsilon_f\right| = \sigma_e}$

  a) $\boxed{\sigma\Delta\sigma > 0} \iff \begin{cases} \sigma > 0 \ \text{ and } \ \Delta\sigma > 0 \\ \qquad\quad \text{or} \\ \sigma < 0 \ \text{ and } \ \Delta\sigma < 0 \end{cases} \implies \begin{cases} \Delta\sigma = \Delta\sigma^f = H'\Delta\varepsilon^f \\ \Delta\sigma = \Delta\sigma^e = E\Delta\varepsilon^e \end{cases}$

  $$\implies \Delta\varepsilon = \Delta\varepsilon^e + \Delta\varepsilon^f = \frac{1}{E}\Delta\sigma + \frac{1}{H'}\Delta\sigma = \frac{E+H'}{EH'}\Delta\sigma$$

  $$\implies \boxed{\begin{cases} \Delta\sigma = E^{ef}\Delta\varepsilon \\ E^{ef} = E\dfrac{H'}{E+H'} \end{cases}}$$

  The strain increment is absorbed by the two elements of the model (the frictional one with hardening and the elastic one). The relation between the stress increment $\Delta\sigma$ and the strain increment $\Delta\varepsilon$ is given by the *elastic-frictional tangent modulus E^{ef}*. This case is called *inelastic loading process*.

$$b)\ \boxed{\sigma\,\Delta\sigma<0} \iff \begin{cases} \sigma>0 \ \text{ and } \ \Delta\sigma<0 \\ \qquad\quad \text{or} \\ \sigma<0 \ \text{ and } \ \Delta\sigma>0 \end{cases}$$

$$\Longrightarrow \Delta\varepsilon^f=0 \Longrightarrow \Delta\varepsilon=\Delta\varepsilon^e \Longrightarrow \boxed{\Delta\sigma=E\,\Delta\varepsilon}$$

Every strain increment $\Delta\varepsilon$ is absorbed by the elastic element. This case is named *elastic unloading process*.

Figure 8.17 shows the stress-strain curve corresponding to the model for a loading-unloading-reloading cycle, in which the following sections can be differentiated.

Section $0-1$ and *section* $2-3$:

$$\left|\sigma-H'\varepsilon_f\right|<\sigma_e \Longrightarrow \boxed{\Delta\sigma=E\Delta\varepsilon}$$

Correspond to *elastic processes*.

Section $1-2$ and *section* $3-4$:

$$\begin{cases} \left|\sigma-H'\varepsilon_f\right|=\sigma_e \\ \sigma\,\Delta\sigma>0 \end{cases} \Longrightarrow \boxed{\Delta\sigma=E^{ef}\Delta\varepsilon}$$

Correspond to *inelastic loading processes*.

Point 2:

$$\begin{cases} \left|\sigma-H'\varepsilon_f\right|=\sigma_e \\ \sigma\,\Delta\sigma<0 \end{cases} \Longrightarrow \boxed{\Delta\sigma=E\Delta\varepsilon}$$

Corresponds to an *elastic unloading process*.

Note that if $H'=0$, then $E^{ef}=0$, and the elastic-frictional model in Figure 8.13 is recovered.

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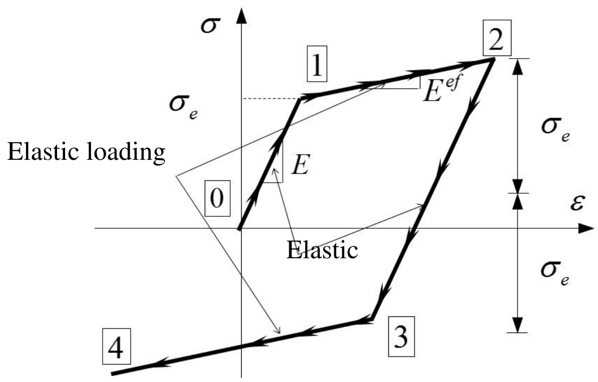

Figure 8.17: Stress-strain hardening curve for a loading-unloading-reloading cycle in an elastic-frictional model with hardening.

8.5 Elastoplastic Phenomenological Behavior

Consider a steel bar of length ℓ and cross-section A subjected to a tensile force F at its extremes. The stress in the bar will be $\sigma = F/A$ (see Figure 8.18) and the corresponding strain can be estimated as $\varepsilon = \delta/\ell$, where δ is the lengthening of the bar. If the bar is subjected to several loading and unloading cycles, the response typically obtained, in terms of stress-strain curve $\sigma - \varepsilon$, is as indicated in Figure 8.19.

Observation of the first cycle reveals that, as long as the stress does not exceed the value σ_e (denoted as *elastic limit*) in point 1, the behavior is linear elastic, characterized by the elastic modulus E $(\sigma = E\varepsilon)$, and there do not exist irrecoverable strains (in a possible posterior unloading, the strain produced during loading would be recovered).

For stress values above σ_e, the behavior ceases to be elastic and part of the strain is no longer recovered during an ensuing unloading to null stress (point 3). There appears, thus, a remaining strain named *plastic strain*, ε^p. However, during the unloading section $2-3$ the behavior is again, in an approximate form, incrementally elastic $(\Delta\sigma = E\,\Delta\varepsilon)$. The same occurs with the posterior reloading $3-2$, which produces an incrementally elastic behavior, until the stress reaches, in point 2, the maximum value it will have achieved during the loading

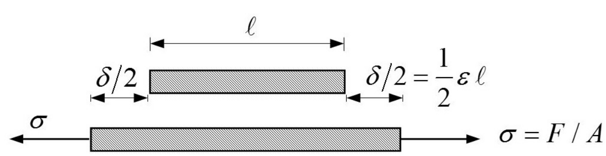

Figure 8.18: Uniaxial tensile loading test.

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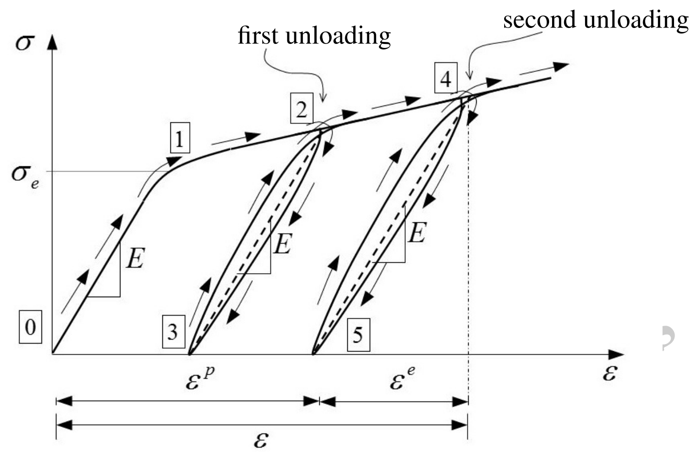

Figure 8.19: Response to loading-unloading-reloading cycles in an uniaxial tensile loading test.

process. From this point on, the behavior is no longer incrementally elastic (as if the material *remembered* the maximum stress to which it has been previously subjected). A posterior loading-unloading-reloading cycle $2-4-5-4$ exposes again that, during section $2-4$, additional plastic strain is generated, which appears in the form of permanent strain in point 5, and, also, additional *elastic strain* ε^e is produced, understood as the part of the strain that can be recovered during the unloading section $4-5$.

8.5.1 Bauschinger Effect

Consider a sample of virgin material (a material that has not suffered previous states of inelastic strain) subjected to an uniaxial *tensile* test and another sample of the same virgin material subjected to an uniaxial *compressive* test. In certain materials, the responses obtained, in terms of the stress-strain curve $\sigma-\varepsilon$ in Figure 8.20, for both tests are symmetrical with respect to the origin. That is, in the tensile test the response is elastic up to a value of $\sigma=\sigma_e$ (*tensile elastic limit*) and in the compressive response the answer is also elastic up to a value of $\sigma=-\sigma_e$ (*compressive elastic limit*), being the rest of both curves (for an assumed regime of monotonous loading) also symmetrical. In this case, the stress-strain curve of the virgin material is said to be symmetrical in tension and compression.

Suppose now that a specimen that has been *previously subjected to a history of plastic strains*[3], for example a tensile loading-unloading cycle such as the $0-1-2-3$ cycle shown in Figure 8.19, undergoes now a compressive test. Consider also that $\sigma_y>\sigma_e$ is the maximum stress the material has been

[3] This procedure is known as *cold hardening* and its purpose is to obtain an apparent elastic limit that is superior to that of the virgin material $\sigma_y>\sigma_e$.

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subjected to during the loading process. An hypothetical *symmetrical* behavior would result in the material having now an elastic behavior in the stress range $[-\sigma_y, \sigma_y]$. However, in certain cases, the elastic behavior in compression ends much earlier (see Figure 8.20). This is the effect known as *Bauschinger effect* or *kinematic hardening*. Note that the stress-strain curve of the elastic-frictional model in Figure 8.17 exhibits this type of hardening.

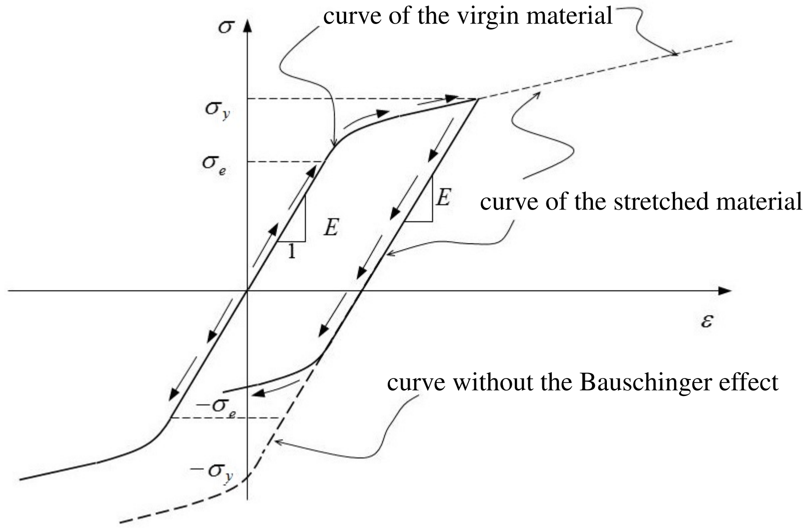

Figure 8.20: Bauschinger effect or kinematic hardening.

Remark 8.8. In view of the phenomenological behavior observed in Figure 8.19 and in Figure 8.20, the elastoplastic behavior is characterized by the following facts:

1) Unlike in the elastic case, there does not exist unicity in the stress-strain relation. A same value of strain can correspond to infinite values of stress and vice-versa. The stress value depends not only on the strain, but also on the loading history.

2) There does not exist a linear relation between stress and strain. At most, this linearity may be incremental in certain sections of the deformation process.

3) Irrecoverable or irreversible strains are produced in a loading-unloading cycle.

8.6 Incremental Theory of Plasticity in 1 Dimension

The elastoplastic behavior analyzed in section 8.5 can be modeled using mathematical models of certain complexity[4]. One of the most popular approximations is the so-called incremental theory of plasticity. In a one-dimensional case, the theory seeks to approximate a stress-strain behavior such as the one observed in Figure 8.19 by means of piece-wise approximations using elastic and inelastic regions such as the ones shown in Figure 8.21. The generalization to several dimensions requires the introduction of more abstract concepts.

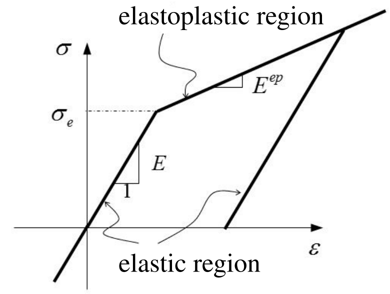

Figure 8.21: Uniaxial stress-strain curve for an elastoplastic model.

8.6.1 Additive Decomposition of Strain. Hardening Variable

The total strain ε is decomposed into the sum of an elastic (or recoverable) strain ε^e, governed by Hooke's law, and a plastic (or irrecoverable) strain ε^p,

$$\boxed{\text{Additive decomposition of strain} \quad \begin{cases} \varepsilon = \varepsilon^e + \varepsilon^p \\ \varepsilon^e = \dfrac{\sigma}{E} \end{cases} \Longrightarrow \begin{cases} d\varepsilon = d\varepsilon^e + d\varepsilon^p \\ d\varepsilon^e = \dfrac{d\sigma}{E} \end{cases}} \tag{8.28}$$

where E is the elastic modulus. In addition, the hardening variable $\alpha(\sigma, \varepsilon^p)$ is defined by means of the evolution equation as follows[5].

$$\boxed{\text{Hardening variable } \alpha \quad \begin{cases} d\alpha = \text{sign}(\sigma)\, d\varepsilon^p \\ d\alpha \geq 0 \\ \alpha\big|_{\varepsilon^p = 0} = 0 \end{cases}} \tag{8.29}$$

[4] Up to a certain point, these models may be inspired, albeit with certain limitations, in the elastic-frictional rheological models described in section 8.4.

[5] Here, the sign operator is used, which is defined as $x \geq 0 \iff \text{sign}(x) = +1$ and $x < 0 \iff \text{sign}(x) = -1$.

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Remark 8.9. Note that the hardening variable α is always positive, in agreement with its definition in (8.29), and, considering the modules of the expression $d\alpha = \text{sign}(\sigma)\, d\varepsilon^p$, results in

$$d\alpha = |d\alpha| = \underbrace{|sign(\sigma)|}_{=1} |d\varepsilon^p| \implies \boxed{d\alpha = |d\varepsilon^p|} \,.$$

Then, for a process with monotonously increasing plastic strains, both variables coincide,

$$d\varepsilon^p \geq 0 \implies \alpha = \int_0^{\varepsilon^p} |d\varepsilon^p| = \int_0^{\varepsilon^p} d\varepsilon^p = \varepsilon^p \,.$$

However, if the process does not involve a monotonous increase, the plastic strain may decrease and its value no longer coincides with that of the hardening variable α.

8.6.2 Elastic Domain. Yield Function. Yield Surface

The *elastic domain in the stress space* is defined as the interior of the domain enclosed by the surface $F(\sigma,\alpha) = 0$,

$$\boxed{\text{Elastic domain:} \quad \mathbb{E}_\sigma := \{\sigma \in \mathbb{R} \ \mid \ F(\sigma,\alpha) < 0\}} \tag{8.30}$$

where the function $F(\sigma,\alpha) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is denoted as *yield function*.

The *initial elastic domain* $\mathbb{E}_\sigma^0$ is defined as the elastic domain corresponding to a null plastic strain ($\varepsilon^p = \alpha = 0$),

$$\boxed{\text{Initial elastic domain:} \quad \mathbb{E}_\sigma^0 := \{\sigma \in \mathbb{R} \ \mid \ F(\sigma,0) < 0\}} \,. \tag{8.31}$$

An additional requirement of the initial elastic domain is that it must contain the null stress state,

$$0 \in \mathbb{E}_\sigma^0 \implies F(0,0) < 0 \,, \tag{8.32}$$

and this is achieved by defining a yield function of the type

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$$\boxed{\text{Yield function:} \quad F(\sigma,\alpha) \equiv |\sigma| - \sigma_y(\alpha)} \tag{8.33}$$

where $\sigma_y(\alpha) > 0$ is known as the *yield stress*. The initial value (for $\alpha = 0$) of the yield stress is the elastic limit σ_e (see Figure 8.22) and the function $\sigma_y(\alpha) : \mathbb{R}^+ \to \mathbb{R}^+$ is named *hardening law*.

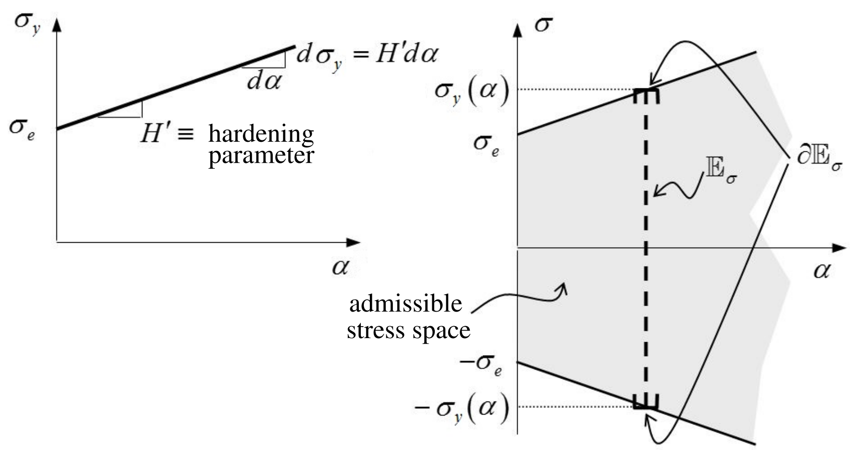

Figure 8.22: Hardening law and admissible stress space.

The *yield surface* is defined as the boundary of the elastic domain.

$$\boxed{\text{Yield surface: } \partial\mathbb{E}_\sigma := \left\{\sigma \in \mathbb{R} \;|\; F(\sigma,\alpha) \equiv |\sigma| - \sigma_y(\alpha) = 0\right\}} \tag{8.34}$$

The elastic domain $\mathbb{E}_\sigma$ together with its boundary $\partial\mathbb{E}_\sigma$ determine the admissible stress space (domain) $\bar{\mathbb{E}}_\sigma$

$$\boxed{\begin{array}{l}\text{Admissible stress space:}\\ \bar{\mathbb{E}}_\sigma = \mathbb{E}_\sigma \bigcup \partial\mathbb{E}_\sigma = \left\{\sigma \in \mathbb{R} \;|\; F(\sigma,\alpha) \equiv |\sigma| - \sigma_y(\alpha) \le 0\right\}\end{array}} \tag{8.35}$$

and it is *postulated* that any feasible (admissible) stress state must belong to the admissible stress space $\bar{\mathbb{E}}_\sigma$. Considering the definitions of elastic domain in (8.30), yield surface in (8.34) and admissible stress space in (8.35), the following is established.

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$$
\boxed{
\begin{array}{l}
F(\sigma,\alpha) < 0 \iff |\sigma| < \sigma_y(\alpha) \iff \begin{cases} \sigma \text{ in the elastic domain} \\ (\sigma \in \mathbb{E}_\sigma) \end{cases} \\
F(\sigma,\alpha) = 0 \iff |\sigma| = \sigma_y(\alpha) \iff \begin{cases} \sigma \text{ on the yield surface} \\ (\sigma \in \partial\mathbb{E}_\sigma) \end{cases} \\
F(\sigma,\alpha) > 0 \iff |\sigma| > \sigma_y(\alpha) \iff \text{non-admissible stress state}
\end{array}
}
\tag{8.36}
$$

Remark 8.10. Note how, in (8.35), the admissible stress space depends on the hardening variable α. The admissible domain evolves with the yield function $\sigma_y(\alpha)$ such that (see Figure 8.22)

$$
\bar{\mathbb{E}}_\sigma \equiv [-\sigma_y(\alpha), \sigma_y(\alpha)] .
$$

8.6.3 Constitutive Equation

To characterize the response of the material, the following situations are defined:

- *Elastic regime*

$$
\sigma \in \mathbb{E}_\sigma \implies \boxed{d\sigma = E d\varepsilon}
\tag{8.37}
$$

- *Elastoplastic regime in unloading*

$$
\left. \begin{array}{l} \sigma \in \partial\mathbb{E}_\sigma \\ dF(\sigma,\alpha) < 0 \end{array} \right\} \implies \boxed{d\sigma = E d\varepsilon}
\tag{8.38}
$$

- *Elastoplastic regime in plastic loading*

$$
\left. \begin{array}{l} \sigma \in \partial\mathbb{E}_\sigma \\ dF(\sigma,\alpha) = 0 \end{array} \right\} \implies \boxed{d\sigma = E^{ep} d\varepsilon}
\tag{8.39}
$$

where E^{ep} is denoted as *elastoplastic tangent modulus*.

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Remark 8.11. The situation $\sigma \in \partial\mathbb{E}_\sigma$ and $dF(\sigma,\alpha) > 0$ cannot occur since, if $\sigma \in \partial\mathbb{E}_\sigma$, from (8.34) results

$$F(\sigma,\alpha) \equiv |\sigma| - \sigma_y(\alpha) = 0 \, .$$

If, in addition, $dF(\sigma,\alpha) > 0$ then,

$$F(\sigma + d\sigma, \alpha + d\alpha) = \underbrace{F(\sigma,\alpha)}_{=0} + \underbrace{dF(\sigma,\alpha)}_{>0} > 0$$

and, in agreement with (8.36), the stress state $\sigma + d\sigma$ is not admissible.

8.6.4 Hardening Law. Hardening Parameter

The hardening law provides the evolution of the yield stress $\sigma_y(\alpha)$ in terms of the hardening variable α (see Figure 8.22). Even though the aforementioned hardening law may be of a more general nature, it is common (and often sufficient) to consider a *linear* hardening law of the type

$$\sigma_y = \sigma_e + H'\alpha \implies \boxed{d\sigma_y(\alpha) = H' d\alpha} \, , \tag{8.40}$$

where H' is known as the *hardening parameter*.

8.6.5 Elastoplastic Tangent Modulus

The value of the elastoplastic tangent modulus E^{ep} introduced in (8.39) is calculated in the following manner. Consider an elastoplastic regime in plastic loading. Then, from (8.39)[6],

$$\left.\begin{array}{l} \sigma \in \partial\mathbb{E}_\sigma \implies F(\sigma,\alpha) \equiv |\sigma| - \sigma_y(\alpha) = 0 \\ dF(\sigma,\alpha) = 0 \end{array}\right\} \implies$$
$$d|\sigma| - d\sigma_y(\alpha) = 0 \implies \operatorname{sign}(\sigma)\, d\sigma - H' d\alpha = 0 \, , \tag{8.41}$$

where (8.40) has been taken into account. Introducing the first expression of (8.29) in (8.41) yields

$$\operatorname{sign}(\sigma)\, d\sigma - H' \operatorname{sign}(\sigma)\, d\varepsilon^p = 0 \implies \boxed{d\varepsilon^p = \frac{1}{H'} d\sigma} \, . \tag{8.42}$$

[6] The property $d|x|/dx = \operatorname{sign}(x)$ is used here.

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Consider now the additive decomposition of strain defined in (8.28), which together with (8.42) results in

$$\left.\begin{array}{l} d\varepsilon = d\varepsilon^e + d\varepsilon^p \\ d\varepsilon^e = \dfrac{1}{E} d\sigma \\ d\varepsilon^p = \dfrac{1}{H'} d\sigma \end{array}\right\} \Longrightarrow d\varepsilon = \frac{1}{E} d\sigma + \frac{1}{H'} d\sigma = \left(\frac{1}{E} + \frac{1}{H'}\right) d\sigma \Longrightarrow$$
$$d\sigma = \frac{1}{\dfrac{1}{E} + \dfrac{1}{H'}} d\varepsilon \Longrightarrow \boxed{\begin{cases} d\sigma = E^{ep}\, d\varepsilon \\ E^{ep} = E \dfrac{H'}{E + H'} \end{cases}} . \tag{8.43}$$

8.6.6 Uniaxial Stress-Strain Curve

The constitutive equation defined by expressions (8.37) to (8.39) allows obtaining the corresponding stress-strain curve for an uniaxial process of loading-unloading-reloading (see Figure 8.23) in which the following sections are observed.

Section $0-1$:

$$|\sigma| < \sigma_e \Longrightarrow \sigma \in \mathbb{E}_\sigma \Longrightarrow \textit{Elastic regime}$$

From (8.37), $d\sigma = E d\varepsilon$ and the behavior is linear elastic, defining an elastic region in the stress-strain curve.

Section $1-2-4$:

$$\left.\begin{array}{l} F(\sigma, \alpha) \equiv |\sigma| - \sigma_y(\alpha) = 0 \Longrightarrow \sigma \in \partial\mathbb{E}_\sigma \\ dF(\sigma, \alpha) = 0 \end{array}\right\} \Longrightarrow \begin{array}{c} \text{Elastoplastic regime} \\ \text{in plastic loading} \end{array}$$

From (8.39), $d\sigma = E^{ep} d\varepsilon$, defining an elastoplastic region.

Section $2-3-2$:

$$F(\sigma, \alpha) \equiv |\sigma| - \sigma_y(\alpha) < 0 \Longrightarrow \sigma \in \partial\mathbb{E}_\sigma \Longrightarrow \textit{Elastic regime}$$

From (8.37), $d\sigma = E d\varepsilon$ and the behavior is linear elastic, defining an elastic region in the stress-strain curve.

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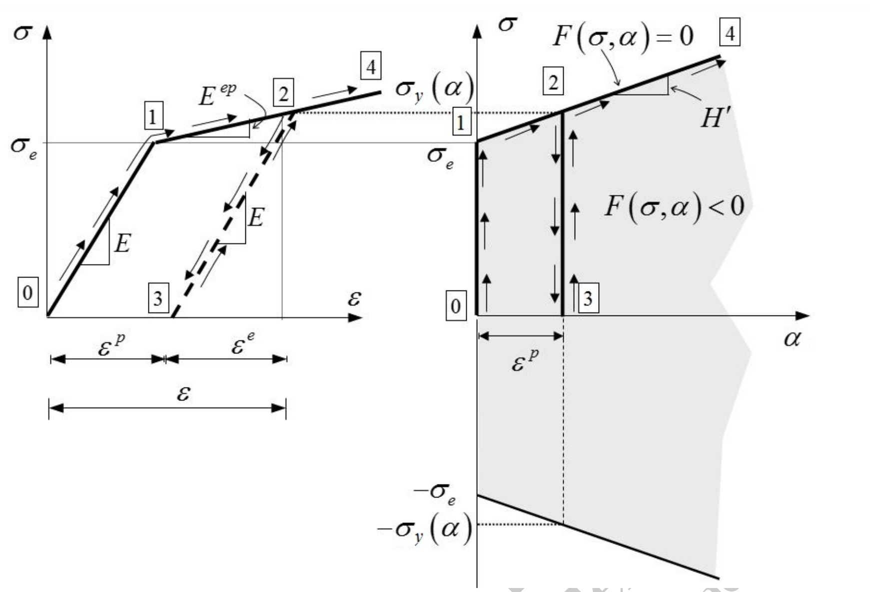

Figure 8.23: Uniaxial stress-strain curve for a loading-unloading-reloading cycle considering the incremental theory of plasticity.

Remark 8.12. In point 2 of Figure 8.23 the following two processes are distinguished:

$$\left.\begin{array}{l} F(\sigma,\alpha) \equiv |\sigma| - \sigma_y(\alpha) = 0 \implies \sigma \in \partial\mathbb{E}_\sigma \\ dF(\sigma,\alpha) < 0 \end{array}\right\} \begin{array}{c}\text{Elastic unloading} \\ \text{in section } 2-3\end{array}$$

$$\left.\begin{array}{l} F(\sigma,\alpha) \equiv |\sigma| - \sigma_y(\alpha) = 0 \implies \sigma \in \partial\mathbb{E}_\sigma \\ dF(\sigma,\alpha) = 0 \end{array}\right\} \begin{array}{c}\text{Plastic loading in} \\ \text{section } 2-4\end{array}$$

Remark 8.13. Note that plastic strain is only generated during the plastic loading process in the elastoplastic region (see Figure 8.24).

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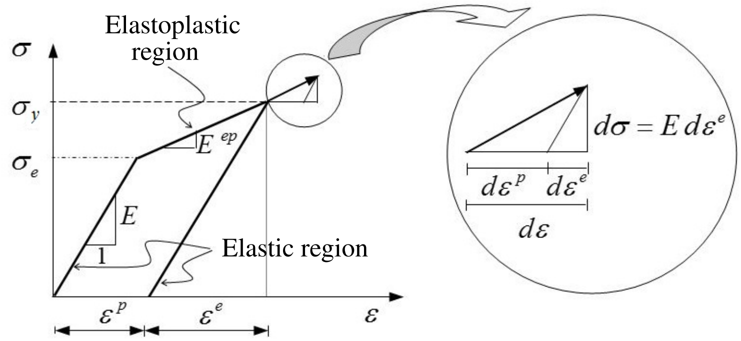

Figure 8.24: Generation of plastic strain in the elastoplastic region.

Remark 8.14. Note the similarity between the stress-strain curve in Figure 8.23 and the one obtained with the elastic-frictional rheological model with hardening in section 8.4.5 (Figure 8.17). The friction strain in said model is equivalent to the plastic strain in the incremental theory of plasticity.

Remark 8.15. The hardening parameter H' plays a fundamental role in the definition of the slope E^{ep} of the elastoplastic region. Following (8.43),

$$\boxed{E^{ep} = E\frac{H'}{E+H'}}$$

and, depending on the value of H', different situations arise (see Figure 8.25):

- $H' > 0 \Longrightarrow E^{ep} > 0 \rightarrow$ *Plasticity with strain hardening*. The limit case $H' = \infty \Rightarrow E^{ep} = E$ recovers the linear elastic behavior.
- $H' = 0 \Longrightarrow E^{ep} = 0 \rightarrow$ *Perfect plasticity*.
- $H' < 0 \Longrightarrow E^{ep} < 0 \rightarrow$ *Plasticity with strain softening*[7]. The limit case corresponds to $H' = -E \Rightarrow E^{ep} = -\infty$.

[7] Plasticity with strain softening presents a specific problematic regarding the uniqueness of the solution to the elastoplastic problem, which is beyond the scope of this text.

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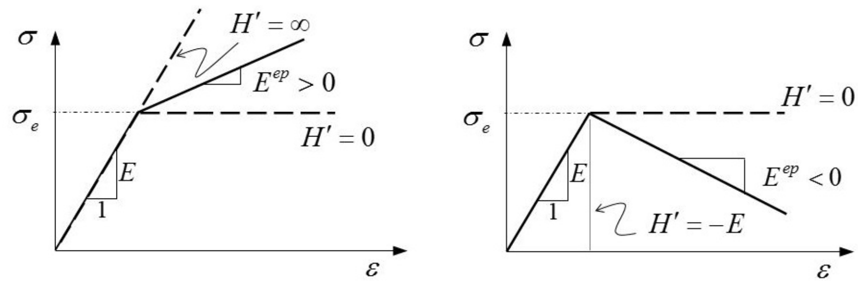

Figure 8.25: Role of the hardening parameter H' in the definition of the slope E^{ep}.

8.7 Plasticity in 3 Dimensions

The incremental theory of plasticity developed in one dimension in section 8.6 can be generalized to a multiaxial stress state (three dimensions) using the same ingredients, that is:

1) Additive decomposition of strain

$$\text{Additive decomposition of strain} \quad \begin{cases} \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p \\ \boldsymbol{\varepsilon}^e = \mathbb{C}^{-1} : \boldsymbol{\sigma} \end{cases} \Longrightarrow \begin{cases} d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^e + d\boldsymbol{\varepsilon}^p \\ d\boldsymbol{\varepsilon}^e = \mathbb{C}^{-1} : d\boldsymbol{\sigma} \end{cases} \tag{8.44}$$

where $\mathbb{C}^{-1}$ is now the (constant) constitutive elastic tensor defined in chapter 6.

2) Hardening variable α and flow rule (evolution equations)

$$\text{Flow rule} \begin{cases} d\varepsilon^p = \lambda \dfrac{\partial G(\boldsymbol{\sigma}, \alpha)}{\partial \boldsymbol{\sigma}} \\ d\alpha = \lambda \quad \alpha \in [0, \infty) \end{cases} \tag{8.45}$$

where λ is the *plastic multiplier* and $G(\boldsymbol{\sigma}, \alpha)$ is the *plastic potential function*.

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3) Yield function. Elastic domain and yield surface

$$
\begin{array}{ll}
\text{Yield function} & \begin{cases} F(\boldsymbol{\sigma},\alpha) \equiv \phi(\boldsymbol{\sigma}) - \sigma_y(\alpha) \\ \sigma_y(\alpha) = \sigma_e + H'\alpha \ \text{(hardening law)} \end{cases} \\
\text{Elastic domain} & \mathbb{E}_\sigma := \{\boldsymbol{\sigma} \mid F(\boldsymbol{\sigma},\alpha) < 0\} \\
\text{Initial elastic domain} & \mathbb{E}^0_\sigma := \{\boldsymbol{\sigma} \mid F(\boldsymbol{\sigma},0) < 0\} \\
\text{Yield surface} & \partial\mathbb{E}_\sigma := \{\boldsymbol{\sigma} \mid F(\boldsymbol{\sigma},\alpha) = 0\} \\
\text{Admissible stress state} & \bar{\mathbb{E}}_\sigma = \mathbb{E}_\sigma \bigcup \partial\mathbb{E}_\sigma = \{\boldsymbol{\sigma} \mid F(\boldsymbol{\sigma},\alpha) \leq 0\}
\end{array}
\tag{8.46}
$$

where $\phi(\boldsymbol{\sigma}) \geq 0$ is denoted as the *equivalent uniaxial stress*, σ_e is the elastic limit obtained in an uniaxial test of the material (it is a material property) and $\sigma_y(\alpha)$ is the yield stress. The hardening parameter H' plays the same role as in the uniaxial case and determines the expansion or contraction of the elastic domain $\mathbb{E}_\sigma$, in the stress space, as α grows. Consequently,

$$
\begin{array}{llll}
H' > 0 & \Longrightarrow & \text{Expansion of } \mathbb{E}_\sigma \text{ with } \alpha & \rightarrow \text{Plasticity with hardening} \\
H' < 0 & \Longrightarrow & \text{Contraction of } \mathbb{E}_\sigma \text{ with } \alpha & \rightarrow \text{Plasticity with softening} \\
H' = 0 & \Longrightarrow & \text{Constant elastic domain } \left(\mathbb{E}_\sigma = \mathbb{E}^0_\sigma\right) & \rightarrow \text{Perfect plasticity}
\end{array}
\tag{8.47}
$$

4) Loading-unloading conditions (Karush-Kuhn-Tucker conditions) *and consistency condition*

$$
\begin{array}{ll}
\text{Loading-unloading conditions} & \rightarrow \ \lambda \geq 0\,; \ F(\boldsymbol{\sigma},\alpha) \leq 0\,; \ \lambda F(\boldsymbol{\sigma},\alpha) = 0 \\
\text{Consistency condition} & \rightarrow \ \text{If } F(\boldsymbol{\sigma},\alpha) = 0 \ \Rightarrow \ \lambda F(\boldsymbol{\sigma},\alpha) = 0
\end{array}
\tag{8.48}
$$

The loading-unloading conditions and the consistency condition are additional ingredients, with respect to the unidimensional case, which allow obtaining, after certain algebraic manipulation, the plastic multiplier λ introduced in (8.45).

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8.7.1 Constitutive Equation

Similarly to the uniaxial case, the following situations are differentiated in relation to the constitutive equation:

- *Elastic regime*

$$\boldsymbol{\sigma} \in \mathbb{E}_\sigma \implies \boxed{d\boldsymbol{\sigma} = \mathbb{C} : d\boldsymbol{\varepsilon}} \tag{8.49}$$

- *Elastoplastic regime in unloading*

$$\left.\begin{array}{l} \boldsymbol{\sigma} \in \partial\mathbb{E}_\sigma \\ dF(\boldsymbol{\sigma},\alpha) < 0 \end{array}\right\} \implies \boxed{d\boldsymbol{\sigma} = \mathbb{C} : d\boldsymbol{\varepsilon}} \tag{8.50}$$

- *Elastoplastic regime in plastic loading*

$$\left.\begin{array}{l} \boldsymbol{\sigma} \in \partial\mathbb{E}_\sigma \\ dF(\boldsymbol{\sigma},\alpha) = 0 \end{array}\right\} \implies \boxed{d\boldsymbol{\sigma} = \mathbb{C}^{ep} : d\boldsymbol{\varepsilon}} \tag{8.51}$$

where $\mathbb{C}^{ep}$ is known as the *elastoplastic constitutive tensor* which, after certain algebraic operations considering (8.44) to (8.48), is defined as

$$\boxed{\begin{array}{l} \mathbb{C}^{ep}(\boldsymbol{\sigma},\alpha) = \mathbb{C} - \dfrac{\mathbb{C} : \dfrac{\partial G}{\partial \boldsymbol{\sigma}} \otimes \dfrac{\partial F}{\partial \boldsymbol{\sigma}} : \mathbb{C}}{H' + \dfrac{\partial F}{\partial \boldsymbol{\sigma}} : \mathbb{C} : \dfrac{\partial G}{\partial \boldsymbol{\sigma}}} \\ \\ \mathbb{C}^{ep}_{ijkl} = \mathbb{C}_{ijkl} - \dfrac{\mathbb{C}_{ijpq} \dfrac{\partial G}{\partial \sigma_{pq}} \dfrac{\partial F}{\partial \sigma_{rs}} \mathbb{C}_{rskl}}{H' + \dfrac{\partial F}{\partial \sigma_{pq}} \mathbb{C}_{pqrs} \dfrac{\partial G}{\partial \sigma_{rs}}} \quad i,j,k,l \in \{1,2,3\} \end{array}} \tag{8.52}$$

8.8 Yield Surfaces. Failure Criteria

A fundamental ingredient in the theory of plasticity is the existence of an initial elastic domain $\mathbb{E}^0_\sigma$ (see Figure 8.26) which can be written as

$$\mathbb{E}^0_\sigma := \{ \boldsymbol{\sigma} \mid F(\boldsymbol{\sigma}) \equiv \phi(\boldsymbol{\sigma}) - \sigma_e < 0 \} \tag{8.53}$$

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and determines a domain in the stress space delimited by the initial yield surface $\partial\mathbb{E}_{\sigma}^{0}$,

$$\partial\mathbb{E}_{\sigma}^{0} := \{ \boldsymbol{\sigma} \mid F(\boldsymbol{\sigma}) \equiv \phi(\boldsymbol{\sigma}) - \sigma_e = 0 \} \ . \tag{8.54}$$

Given that the initial elastic domain contains the origin of the stress space ($\boldsymbol{\sigma} = \mathbf{0}$), every loading process in any point of the medium will include an elastic regime (as long as the trajectory of the stresses remains inside $\mathbb{E}_{\sigma}^{0}$, see Figure 8.26) that will end at the instant in which said trajectory reaches the yield surface $\partial\mathbb{E}_{\sigma}^{0}$. The initial yield surface plays then the role of indicating the instant of *failure* (understood as the end of the elastic behavior) independently of the possible post-failure (plastic) behavior that initiates beyond this instant. Thus, the importance of the initial yield surface and the interest in formulating the mathematical equations that adequately determine this surface for the different materials of interest in engineering.

With the aim of defining the yield surface independently of the reference system (isotropic material)[8], even if formulated in the principal stress space, its mathematical equation is typically defined in terms of the stress invariants,

$$F(\boldsymbol{\sigma}) \equiv \mathcal{F}\left(I_1, J_2', J_3'\right) \ , \tag{8.55}$$

and, since the criterion $\sigma_1 \geq \sigma_2 \geq \sigma_3$ is adopted, its definition only affects the first sector of the principal stress space and can be automatically extended, due to symmetry conditions (see Remark 8.7), to the rest of sectors in Figure 8.7.

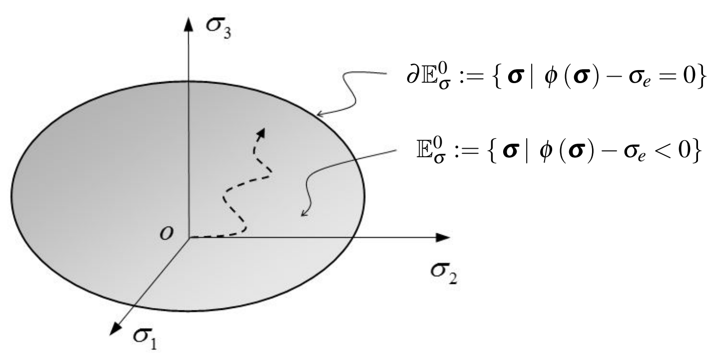

Figure 8.26: Initial elastic domain and initial yield surface.

[8] An isotropic elastoplastic behavior is characterized by the fact that the yield surface, understood as an additional ingredient of the constitutive equation, is independent of the reference system.

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8.8.1 Von Mises Criterion

In the von Mises criterion the yield surface is defined as

$$\boxed{\text{Von Mises criterion:} \quad F(\boldsymbol{\sigma}) \equiv \bar{\sigma}(\boldsymbol{\sigma}) - \sigma_e = \sqrt{3J_2'} - \sigma_e = 0} \tag{8.56}$$

where $\bar{\sigma}(\boldsymbol{\sigma}) = \sqrt{3J_2'}$ is the *effective stress* (see Remark 8.3). An alternative expression is obtained taking (8.19) and (8.20) and replacing them in (8.56), which produces

$$F(\boldsymbol{\sigma}) \equiv \frac{1}{\sqrt{2}} \Big((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 \Big)^{1/2} - \sigma_e = 0\,. \tag{8.57}$$

The graphical representation of the von Mises yield surface is shown in Figure 8.27.

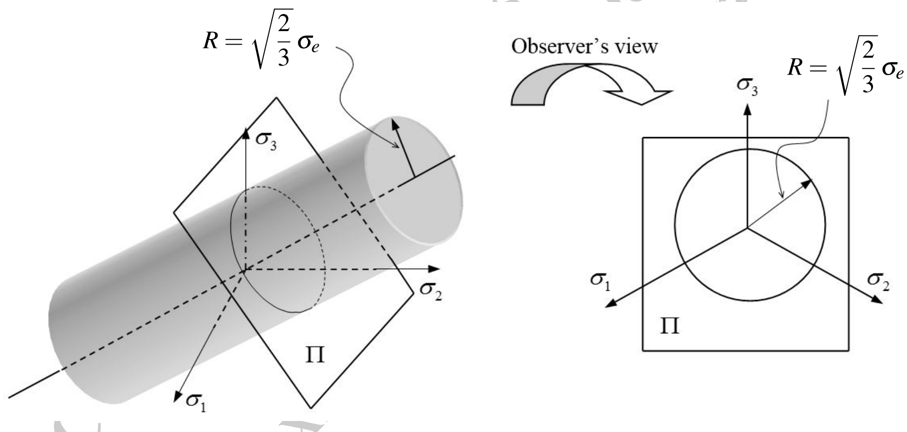

Figure 8.27: Von Mises criterion in the principal stress space.

> *Remark 8.16.* Equation (8.56) highlights the dependency of the von Mises yield surface solely on the second stress invariant J_2'. Consequently, all the points of the surface are characterized by the same value of J_2', which defines a cylinder whose axis is the hydrostatic stress axis.

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Remark 8.17. The von Mises criterion is adequate as a failure criterion in metals, in which, typically, hydrostatic stress states (both in tensile and compressive loading) have an elastic behavior and failure is due to the presence of deviatoric stress components.

Example 8.2 – *Compute the expression of the von Mises criterion for an uniaxial tensile loading case.*

Solution

An uniaxial tensile loading case is characterized by the stress state

$$\boldsymbol{\sigma} \stackrel{not}{\equiv} \begin{bmatrix} \sigma_u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

The effective stress is known to be $\bar{\sigma} = |\sigma_u|$ (see Example 8.1) and, replacing in the expression of the von Mises criterion (8.56), yields

$$F(\boldsymbol{\sigma}) \equiv \bar{\sigma}(\boldsymbol{\sigma}) - \sigma_e = |\sigma_u| - \sigma_e .$$

Thus, the initial elastic domain is characterized in the same way as in unidimensional plasticity seen in Section 8.6.2, by the condition

$$\boxed{F(\boldsymbol{\sigma}) < 0 \implies |\sigma_u| < \sigma_e} .$$

Example 8.3 – *Compute the expression of the von Mises criterion for a stress state representative of a beam under composed flexure.*

Solution

The stress state for a beam under composed flexure is

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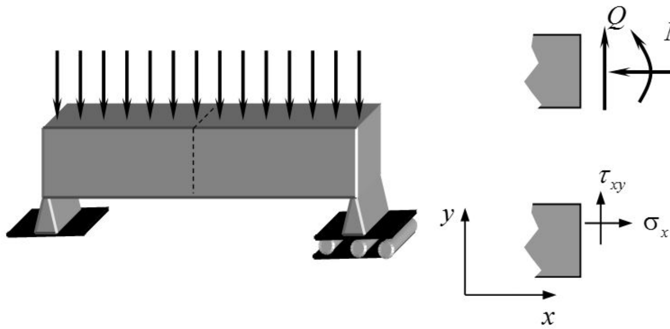

$$\boldsymbol{\sigma} \overset{not}{\equiv} \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad \sigma_m = \frac{1}{3}\sigma_x \quad \Longrightarrow$$

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \frac{1}{3}\sigma_x \mathbf{1} \overset{not}{\equiv} \begin{bmatrix} \frac{2}{3}\sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & -\frac{1}{3}\sigma_x & 0 \\ 0 & 0 & -\frac{1}{3}\sigma_x \end{bmatrix} .$$

Then, the second stress invariant J_2' is computed as

$$J_2' = \frac{1}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}' = \frac{1}{2}\left(\frac{4}{9}\sigma_x^2 + \frac{1}{9}\sigma_x^2 + \frac{1}{9}\sigma_x^2 + \tau_{xy}^2 + \tau_{xy}^2\right) = \frac{1}{3}\sigma_x^2 + \tau_{xy}^2 .$$

And the effective stress is obtained for the von Mises criterion,

$$\bar{\sigma} = \sqrt{3J_2'} = \sqrt{\sigma_x^2 + 3\tau_{xy}^2} \quad \Longrightarrow \quad F(\boldsymbol{\sigma}) < 0 \quad \Longrightarrow \quad \bar{\sigma} < \sigma_e \quad \Longrightarrow$$

$$\boxed{\sigma_{co} = \sqrt{\sigma_x^2 + 3\tau_{xy}^2} < \sigma_e} \; .$$

The comparison stress, $\sigma_{co} = \sqrt{\sigma_{xx}^2 + 3\tau_{xy}^2}$, which can be regarded as a scalar for comparison with the uniaxial elastic limit σ_e, is commonly used in the design standards of metallic structures.

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8.8.2 Tresca Criterion or Maximum Shear Stress Criterion

The Tresca criterion, also known as the *maximum shear stress criterion*, states that the elastic domain ends, for a certain point in the medium, when the maximum shear stress acting on any of the planes containing this point, $\tau_{\max}$, reaches half the value of the uniaxial elastic limit σ_e,

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_e}{2} \, . \tag{8.58}$$

Figure 8.28 illustrates the failure situation in terms of Mohr's circle in three dimensions. In a loading process in which this circle increases starting from the origin, the elastic behavior ends when the circle with radius $\tau_{\max}$ becomes tangent to the straight line $\tau = \tau_{\max} = \sigma_e/2$.

It follows from (8.58) that the Tresca criterion can be written as

$$\boxed{\text{Tresca criterion:} \qquad F(\boldsymbol{\sigma}) \equiv (\sigma_1 - \sigma_3) - \sigma_e = 0} \tag{8.59}$$

Remark 8.18. It can be verified that the Tresca criterion is written in an unequivocal form as a function of J_2' and J_3' and does not depend on the first stress invariant I_1.

$$\text{Tresca criterion:} \qquad F(\boldsymbol{\sigma}) \equiv (\sigma_1 - \sigma_3) - \sigma_e \equiv \mathcal{F}\left(J_2', J_3'\right)$$

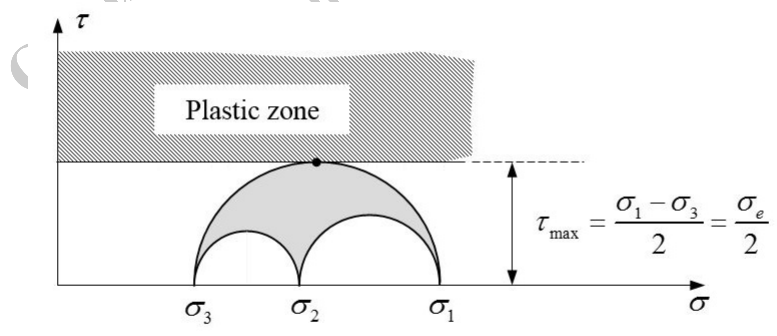

Figure 8.28: Representation of the Tresca criterion using Mohr's circle in three dimensions.

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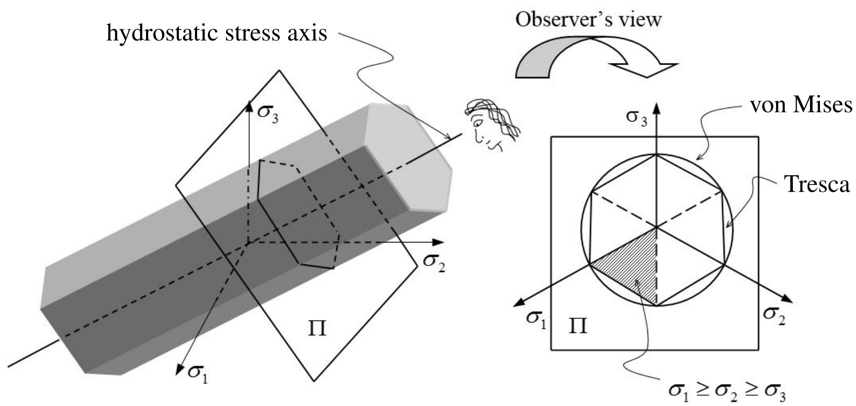

Figure 8.29: Tresca criterion in the principal stress space.

Figure 8.29 shows the yield surface corresponding to the Tresca criterion in the principal stress space, which results in an hexahedral prism whose axis is the hydrostatic stress axis.

Remark 8.19. Since the Tresca criterion does not depend on the first stress invariant (and, therefore, on the stress σ_{oct}, see (8.16)), the corresponding yield surface does not depend on the distance from the origin to the octahedral plane containing the point (see Remark 8.4). Thus, if a point in the stress space, characterized by its stress invariants (I_1, J_2', J_3'), is on said yield surface, all the points in the stress space with the same values of J_2' and J_3' will also be on this surface. This circumstance qualifies the yield surface as a prismatic surface whose axis is the hydrostatic stress axis.
On the other hand, the dependency on the two invariants J_2' and J_3', prevents (unlike in the case of the von Mises criterion) the surface from being cylindrical. In short, the symmetry conditions establish that the surface of the Tresca criterion be an hexagonal prism inscribed in the von Mises cylinder (see Figure 8.29).

Remark 8.20. The Tresca criterion is used to model the behavior of metals, in a similar manner to the case of the von Mises criterion (see Remark 8.17).

Example 8.4 – *Compute the expression of the Tresca criterion for an uniaxial tensile loading case.*

Solution

An uniaxial tensile load case is characterized by the stress state

$$\boldsymbol{\sigma} \stackrel{not}{\equiv} \begin{bmatrix} \sigma_u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

For the case $\sigma_u \geq 0$,

$$\left. \begin{array}{l} \sigma_1 = \sigma_u \\ \sigma_3 = 0 \end{array} \right\} \Longrightarrow F(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1 - \sigma_3) - \sigma_e = \sigma_u - \sigma_e = |\sigma_u| - \sigma_e .$$

For the case $\sigma_u < 0$,

$$\left. \begin{array}{l} \sigma_1 = 0 \\ \sigma_3 = \sigma_u \end{array} \right\} \Longrightarrow F(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1 - \sigma_3) - \sigma_e = -\sigma_u - \sigma_e = |\sigma_u| - \sigma_e .$$

And the initial elastic domain is then characterized in the same way as in the one-dimensional plasticity seen in Section 8.6.2, by the condition

$$\boxed{F(\boldsymbol{\sigma}) < 0 \implies |\sigma_u| < \sigma_e} .$$

8.8.3 Mohr-Coulomb Criterion

The Mohr-Coulomb criterion can be viewed as generalization of the Tresca criterion, in which the maximum shear stress sustained depends on the own stress state of the point (see Figure 8.30). The yield line, in the space of Mohr's circle, is a straight line characterized by the cohesion c and the internal friction angle ϕ, both of which are considered to be material properties,

$$\tau = c - \sigma \tan \phi . \tag{8.60}$$

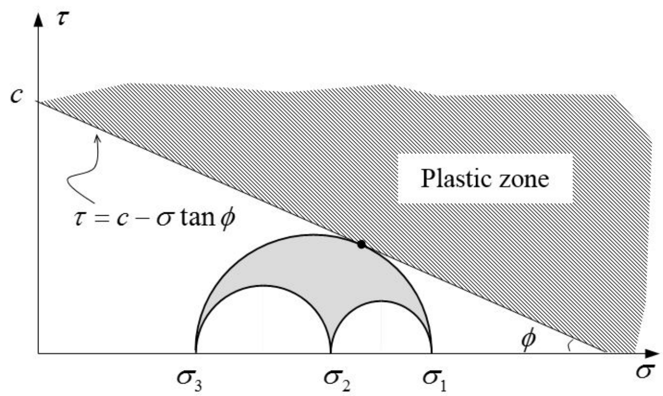

Figure 8.30: Representation of the Mohr-Coulomb criterion using Mohr's circle in three dimensions.

The end of the elastic behavior (failure) in an increasing load process takes place when the first point in the Mohr's circle (corresponding to a certain plane) reaches the aforementioned yield line.

The shear stress in this plane, τ, becomes smaller as the normal stress σ in the plane increases. It therefore becomes obvious that the behavior of the model under tensile loading is considerably different to the behavior under compressive loading. As can be observed in Figure 8.30, the yield line crosses the normal stress axis in the positive side of these stresses, limiting thus the material's capacity to withstand tensile loads.

To obtain the mathematical expression of the yield surface, consider a stress state for which plasticization initiates. In such case, the corresponding Mohr's circle is defined by the major and minor principal stresses and is tangent to the yield line at point A (see Figure 8.31), verifying

$$R = \frac{\sigma_1 - \sigma_3}{2} \quad \Longrightarrow \quad \begin{cases} \tau_A = R\cos\phi = \dfrac{\sigma_1 - \sigma_3}{2}\cos\phi \\ \sigma_A = \dfrac{\sigma_1 + \sigma_3}{2} + R\sin\phi = \dfrac{\sigma_1 + \sigma_3}{2} + \dfrac{\sigma_1 - \sigma_3}{2}\sin\phi \end{cases} \tag{8.61}$$

and, replacing (8.61) in (8.60), results in

$$\begin{gathered} \tau_A = c - \sigma_A \tan\phi \quad \Longrightarrow \quad \tau_A + \sigma_A \tan\phi - c = 0 \quad \Longrightarrow \\ \frac{\sigma_1 - \sigma_3}{2}\cos\phi + \left(\frac{\sigma_1 + \sigma_3}{2} + \frac{\sigma_1 - \sigma_3}{2}\sin\phi\right)\tan\phi - c = 0 \quad \Longrightarrow \\ (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3)\sin\phi - 2c\cos\phi = 0\,. \end{gathered} \tag{8.62}$$

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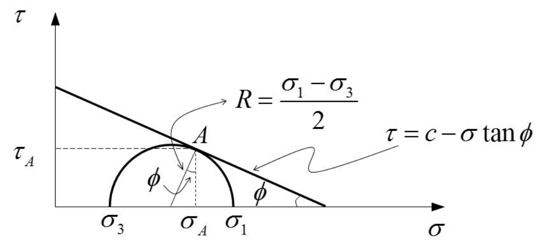

Figure 8.31: Deduction of the expression for the Mohr-Coulomb criterion using Mohr's circle.

Mohr-Coulomb criterion:

$$F(\boldsymbol{\sigma}) \equiv (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3)\sin\phi - 2c\cos\phi = 0 \tag{8.63}$$

Remark 8.21. The equation

$$F(\boldsymbol{\sigma}) \equiv (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3)\sin\phi - 2c\cos\phi = 0 \, ,$$

which is linear in σ_1 and σ_3, defines *a plane* in the principal stress space that is restricted to the sector $\sigma_1 \geq \sigma_2 \geq \sigma_3$. Extension, taking into account symmetry conditions, to the other five sectors (see Remark 8.7) defines six planes that constitute a pyramid of indefinite length whose axis is the hydrostatic stress axis (see Figure 8.32). The distance from the origin of the principal stress space to the vertex of the pyramid is $d = \sqrt{3}\, c \cot\phi$.

Remark 8.22. The particularization $\phi = 0$ and $c = \sigma_e/2$ reduces the Mohr-Coulomb criterion to the Tresca criterion (see (8.59) and (8.63)).

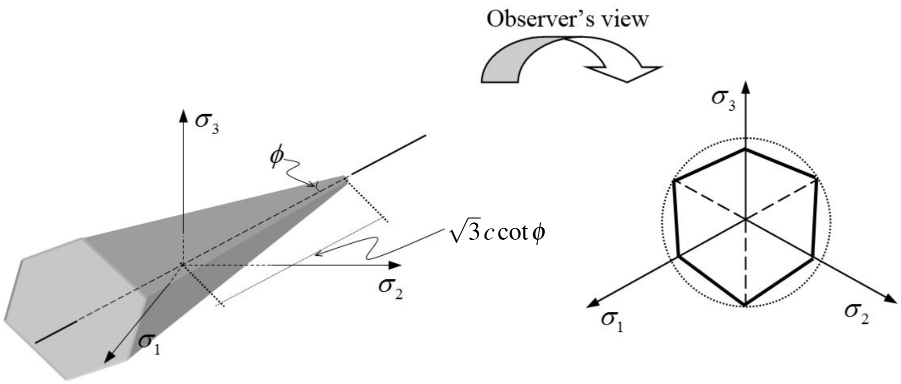

Figure 8.32: Mohr-Coulomb criterion in the principal stress space.

> *Remark 8.23.* In soil mechanics, the sign criterion of the normal stresses is the opposite to the one used in continuum mechanics ($\sigma \equiv -\sigma$, see Chapter 4) and, thus, $\sigma_1 \equiv -\sigma_3$ and $\sigma_3 \equiv -\sigma_1$. Then, the Mohr-Coulomb criterion in (8.63) becomes
>
> $$F(\boldsymbol{\sigma}) \equiv (\sigma_1 - \sigma_3) - (\sigma_1 + \sigma_3)\sin\phi - 2c\cos\phi \ .$$
>
> The corresponding graphical representations are shown in Figures 8.33 and 8.34.

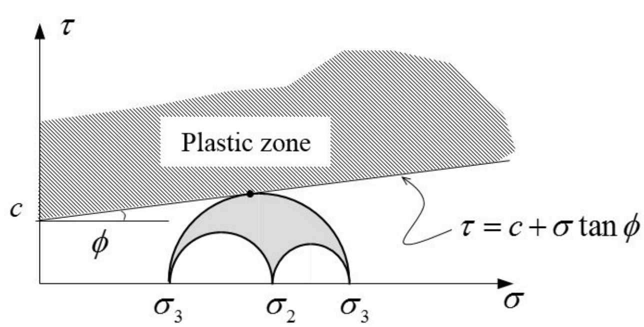

Figure 8.33: Representation of the Mohr-Coulomb criterion using Mohr's circle in three dimensions and soil mechanics sign criterion.

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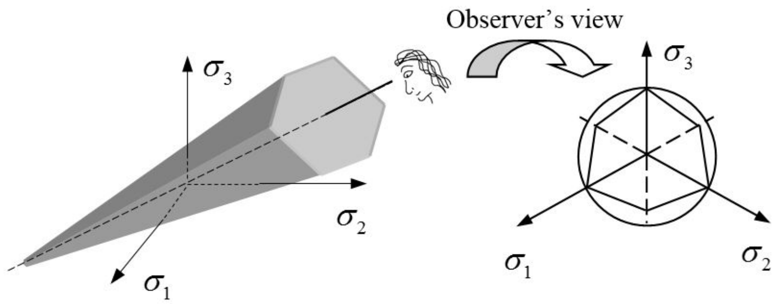

Figure 8.34: Mohr-Coulomb criterion in the principal stress space, using soil mechanics sign criterion.

Remark 8.24. Following certain algebraic operations, the Mohr-Coulomb criterion can be written in terms of the three stress invariants.

$$\text{Mohr-Coulomb criterion:} \quad F(\boldsymbol{\sigma}) \equiv \mathcal{F}\left(I_1, J_2', J_3'\right)$$

Remark 8.25. The Mohr-Coulomb criterion is especially adequate for cohesive-frictional materials (concrete, rocks and soils), which are known to exhibit considerably different uniaxial elastic limits under tensile and compressive loadings.

8.8.4 Drucker-Prager Criterion

The yield surface defined by the Drucker-Prager criterion is given by

$$\boxed{\text{Drucker-Prager criterion:} \quad F(\boldsymbol{\sigma}) \equiv 3\,\alpha\sigma_m + \left(J_2'\right)^{1/2} - \beta = 0} \tag{8.64}$$

where

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$$\alpha = \frac{2\sin\phi}{\sqrt{3}\,(3-\sin\phi)}\ ,\ \beta = \frac{6\,c\cos\phi}{\sqrt{3}\,(3-\sin\phi)} \text{ and } \sigma_m = \frac{\sigma_1+\sigma_2+\sigma_3}{3} = \frac{I_1}{3}\ , \quad (8.65)$$

being c and ϕ the cohesion and the internal friction angle, respectively, which are considered to be material properties. Considering (8.16) and (8.18), the criterion can be rewritten as

$$F(\boldsymbol{\sigma}) \equiv \alpha I_1 + \left(J_2'\right)^{1/2} - \beta = 3\,\alpha\sigma_{oct} + \sqrt{\frac{3}{2}}\,\tau_{oct} - \beta = \mathcal{F}\left(I_1, J_2'\right) = 0\ . \quad (8.66)$$

Remark 8.26. The independence on the third stress invariant J_3' establishes that, if a certain point in the stress space belongs to the yield surface, all the other points with the same value of the stress invariants I_1 and J_2' also belong to this surface, independently of the value of the third stress invariant J_3'. Given that the constant values of these invariants correspond to points of the octahedral plane placed at a same distance from the hydrostatic stress axis (see Figure 8.6), it can be concluded that the yield surface is a surface of revolution around this axis.
In addition, because the relation between σ_{oct} and τ_{oct} in (8.66) is lineal, the surface is a conical surface whose axis is the hydrostatic stress axis (see Figure 8.5 and Figure 8.35). The distance from the origin of the principal stress space to the vertex of the cone is $d = \sqrt{3}\,c\cot\phi$. It can be verified that the Drucker-Prager surface has the Mohr-Coulomb surface with the same values of cohesion, c, and internal friction angle, ϕ, semi-inscribed in it.

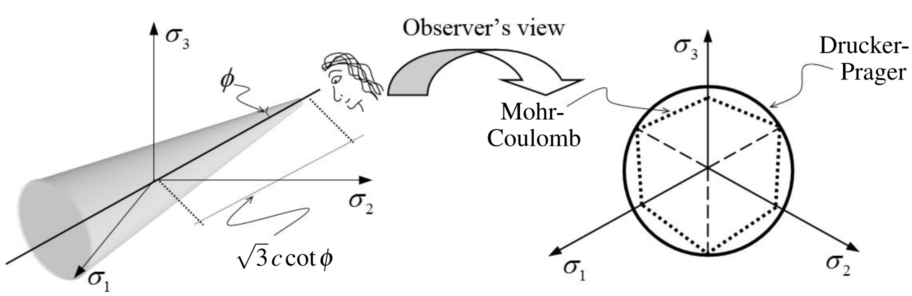

Figure 8.35: Drucker-Prager criterion in the principal stress space.

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Remark 8.27. The position of the vertex of the Drucker-Prager cone in the positive side of the hydrostatic stress axis establishes a limitation in the elastic behavior range for hydrostatic stress states in tensile loading (while there is no limitation in the elastic limit for the hydrostatic compression case). This situation, which also occurs in the Mohr-Coulomb criterion, is typically observed in cohesive-frictional materials (concrete, rocks and soils), for which these two criteria are especially adequate.

Remark 8.28. In soil mechanics, where the sign criterion for the normal stresses is inverted, the yield surface for the Drucker-Prager criterion is as indicated in Figure 8.36.

Remark 8.29. The particularization $\phi = 0$ and $c = \sigma_e/2$ reduces the Drucker-Prager criterion to the von Mises criterion (see (8.56), (8.64) and (8.65)).

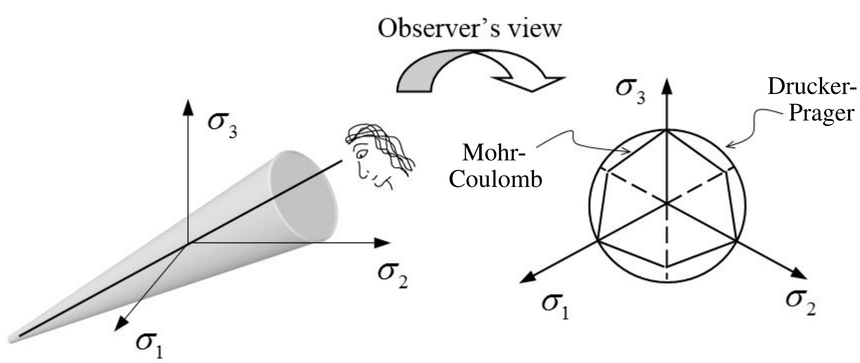

Figure 8.36: Drucker-Prager criterion in the principal stress space, using soil mechanics sign criterion.

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PROBLEMS

Problem 8.1 – *Justify the shape the yield surface will have in the principal stress space for each of the following cases:*

a) $f\left(I_1^2\right) = 0$

b) $f\left(J'_2\right) = 0$

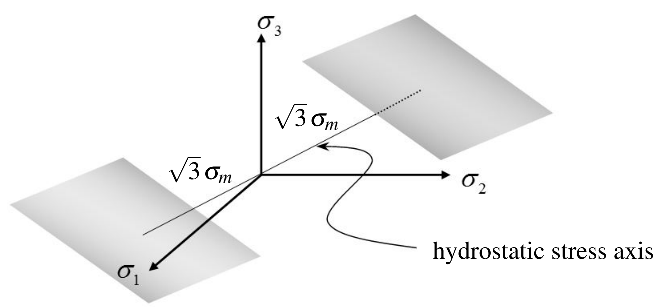
c) $a I_1^2 + b\,\tau_{oct}^2 = c$ *with a, b and c strictly positive*

Solution

a) In this case, there is a condition on the mean stress since

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_m \ .$$

Then, the yield surface is an octahedral plane whose distance to the origin is imposed by the first stress invariant. However, because this invariant is squared, there are two octahedral planes, one in each direction of the hydrostatic stress axis.



b) Here, the distance between a given stress state and an hydrostatic stress state is imposed. So, the yield surface is a cylinder with circular section in the octahedral planes,

$$J'_2 = \frac{3}{2}\tau_{oct}^2 \quad \Longrightarrow \quad \text{distance} = \sqrt{3}\,\tau_{oct} \ .$$

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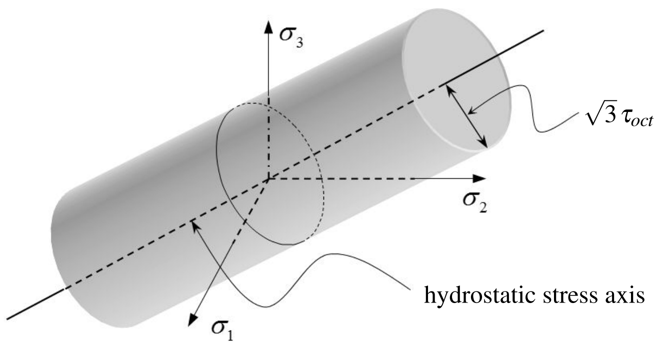
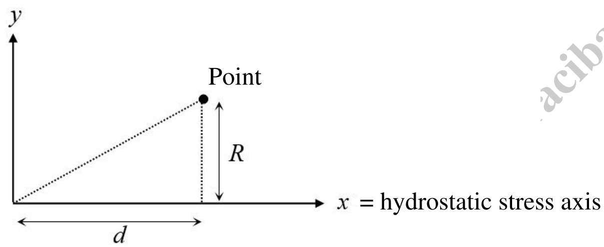


c) The representation of a plane defined by a given point of the yield surface and the hydrostatic stress axis is:



Then, the relations

$$\left.\begin{array}{l} d = x = \sqrt{3}\,\sigma_{oct} = \dfrac{\sqrt{3}}{3} I_1 \\ R = y = \sqrt{3}\,\tau_{oct} \end{array}\right\} \implies \left\{\begin{array}{l} I_1 = \sqrt{3}\,x \\ \tau_{oct} = \dfrac{R}{\sqrt{3}} \end{array}\right.$$

are deduced and replacing these values in the given expression of the yield surface results in

$$a I_1^2 + b\,\tau_{oct}^2 = c \implies 3ax^2 + \frac{by^2}{3} = c \implies \left(\frac{x}{\sqrt{\dfrac{c}{3a}}}\right)^2 + \left(\frac{y}{\sqrt{\dfrac{3c}{b}}}\right)^2 = 1\ .$$

This is the mathematical description of an ellipse in the $x-y$ plane previously defined. In addition, since the third stress invariant does not intervene in the definition of the yield surface, the hydrostatic stress axis is an axis of radial symmetry and, thus, the rotation of the ellipse about the x-axis ($\equiv$ hydrostatic stress axis) defines the final surface.
In conclusion, if the axes considered are the axes x ($\equiv$ hydrostatic stress axis), y and z, the yield surface is defined by

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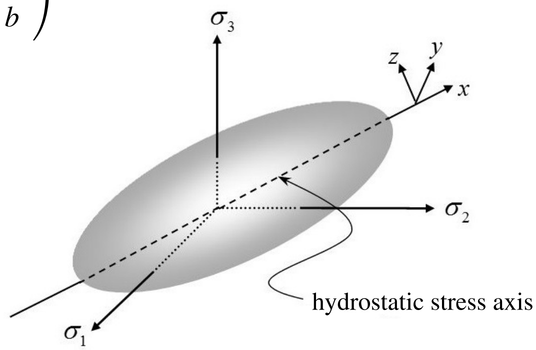
$$\left(\frac{x}{\sqrt{\dfrac{c}{3a}}}\right)^2+\left(\frac{y}{\sqrt{\dfrac{3c}{b}}}\right)^2+\left(\frac{z}{\sqrt{\dfrac{3c}{b}}}\right)^2=1 \, .$$

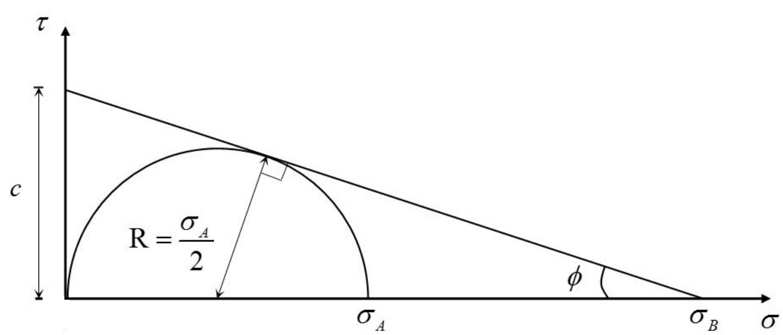


Problem 8.2 – *Graphically determine, indicating the most significant values, the cohesion and internal friction angle of an elastoplastic material that follows the Mohr-Coulomb yield criterion using the following information:*

1) *In an uniaxial tensile stress state* $(\sigma_1=\sigma\ ,\ \sigma_2=\sigma_3=0)$*, the material plasticizes at* $\sigma=\sigma_A$*.*
2) *In a triaxial isotensile test of the same material* $(\sigma_1=\sigma_2=\sigma_3=\sigma)$*, it plasticizes at* $\sigma=\sigma_B$*.*

Solution

In the uniaxial tensile stress state, the Mohr's circle will cross the origin and the value $\sigma=\sigma_A$ in the horizontal axis. However, for the triaxial isotensile stress state, the Mohr's circle will degenerate to a point in this axis, $\sigma=\sigma_B$. Thus, the following graph is plotted



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which allows establishing the relations

$$\tan\phi = \frac{c}{\sigma_B} \qquad \text{and} \qquad \sin\phi = \frac{\sigma_A/2}{\sigma_B - \sigma_A/2} \ .$$

Finally, the cohesion and internal friction angle are

$$\boxed{\phi = \arcsin\frac{\sigma_A/2}{\sigma_B - \sigma_A/2} \quad \text{and} \quad c = \sigma_B \tan\phi} \ .$$

Problem 8.3 – *The following properties of a certain material have been experimentally determined:*

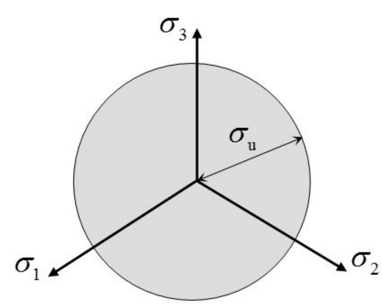
1) *In a hydrostatic compressive regime, the material never plasticizes.*
2) *In a hydrostatic tensile regime, the virgin material plasticizes for a value of the mean stress $\sigma_m = \sigma^*$.*
3) *In an uniaxial tensile regime, the virgin material plasticizes for a tensile stress value σ_u .*
4) *In other cases, plasticization occurs when the norm of the deviatoric stresses varies linearly with the mean stress,*

$$|\boldsymbol{\sigma}'| = \sqrt{\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} = a\,\sigma_m + b \ .$$

Plot the yield surface, indicating the most significant values, and calculate the values a and b in terms of σ^* and σ_u.

Solution

Property *1)* and *2)* indicate that the yield surface is closed in the tensile part of the hydrostatic stress axis but open in the compressive part. In addition, property *3)* indicates that the octahedral plane that contains the origin will have the shape shown in the figure to the right. Since property *4)* indicates that the deviatoric stresses vary linearly with the mean stress (as is the case for the Drucker-Prager criterion), then the yield surface is necessarily a right circular cone whose axis is the hydrostatic



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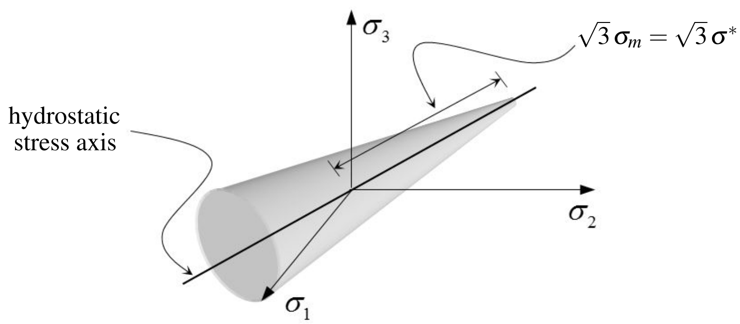
stress axis and whose vertex is in the tensile part of this axis:



To calculate the values of a and b, the yield criterion $|\boldsymbol{\sigma}'| = \sqrt{\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} = a\,\sigma_m + b$ is applied on the vertex of the cone, which corresponds with the hydrostatic tensile case and, thus, has no deviatoric stresses,

$$|\boldsymbol{\sigma}'| = 0 \quad \Longrightarrow \quad a\sigma_m\Big|_{\sigma_m=\sigma^*} + b = 0 \quad \Longrightarrow \quad a\sigma^* + b = 0\,. \qquad [1]$$

The procedure is repeated for the uniaxial tensile case, whose deviatoric stresses are now

$$\boldsymbol{\sigma} \overset{not}{\equiv} \begin{bmatrix} \sigma_u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \boldsymbol{\sigma}_{sph} = \frac{1}{3}\sigma_u \mathbf{1} \quad \Longrightarrow \quad \boldsymbol{\sigma}' \overset{not}{\equiv} \frac{\sigma_u}{3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then, applying the yield criterion $|\boldsymbol{\sigma}'| = a\,\sigma_m + b$ produces

$$|\boldsymbol{\sigma}'| = \sqrt{\frac{2}{3}}\,\sigma_u \quad \Longrightarrow \quad \sqrt{\frac{2}{3}}\,\sigma_u = a\left(\frac{1}{3}\sigma_u\right) + b\,. \qquad [2]$$

Equations [1] and [2] allow determining the desired values of a and b as

$$\boxed{a = \frac{\sqrt{\dfrac{2}{3}}\,\sigma_u}{\dfrac{\sigma_u}{3} - \sigma^*} \qquad \text{and} \qquad b = -\frac{\sqrt{\dfrac{2}{3}}\,\sigma_u \sigma^*}{\dfrac{\sigma_u}{3} - \sigma^*}}\,.$$

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Problem 8.4 – *The metallic component PQRS has a thickness "e" and is composed of two different materials, (1) and (2), considered to be perfect elasto-plastic materials. The component is subjected to a pure shear test by means of the machine shown in Figure A, such that the uniform stress and strain states produced are*

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = 0 \ , \quad \gamma_{xz} = \gamma_{yz} = 0 \ , \quad \gamma_{xy} = \gamma = \frac{\delta}{\mathrm{h}} \ ,$$
$$\sigma_x = \sigma_y = \sigma_z = 0 \ , \quad \tau_{xz} = \tau_{yz} = 0 \quad \textit{and} \quad \tau_{xy} = \tau \neq 0 \ .$$

When a component exclusively composed of one of the materials is tested separately, a $\tau - \gamma$ *curve of the type shown in Figure B is obtained for both materials. Determine:*

a) *The elastic limit that will be obtained in separate uniaxial tensile tests of each material, assuming they follow the von Mises criterion.*

When the component composed of the two materials is tested, the $P - \delta$ *curve shown in Figure C is obtained. Determine:*

b) *The values of the elastic load and displacement,* P_e *and* δ_e.
c) *The values of the plastic load and displacement,* P_p *and* δ_{p}.
d) *The coordinates* $P - \delta$ *of points C and D in Figure C.*

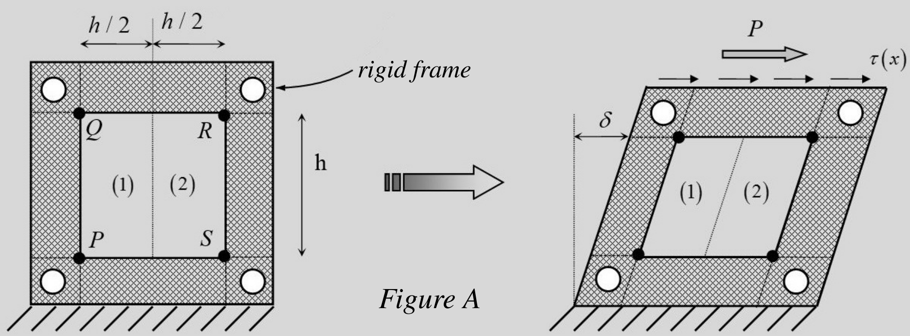

Figure A

HYPOTHESES:
Material (1)
$G = G$ *and* $\tau_e = \tau^*$
Material (2)
$G = G$ *and* $\tau_e = 2\,\tau^*$

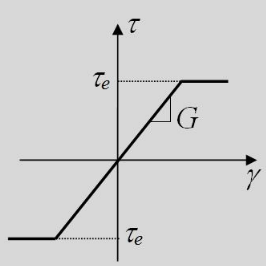

Figure B

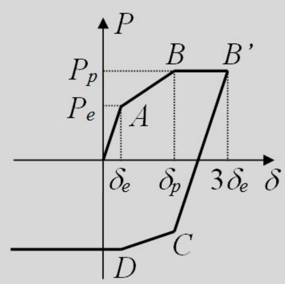

Figure C

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Solution

a) In an uniaxial state of stress, plasticization according to the von Mises criterion is known to begin when (see Example 8.2)

$$\bar{\sigma} = \sigma_e \, ,$$

where $\bar{\sigma}$ is the effective stress and σ_e is the elastic limit. In addition, the following relations seen in this chapter, are known to hold.

$$\bar{\sigma} = (3J_2')^{\frac{1}{2}} \qquad J_2' = \frac{1}{2}\,\mathrm{Tr}\left(\boldsymbol{\sigma}'^2\right)$$
$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \boldsymbol{\sigma}_{sph} \qquad \boldsymbol{\sigma}_{sph} = \sigma_m \mathbf{1} \qquad \sigma_m = \frac{1}{3}\,\mathrm{Tr}\,(\boldsymbol{\sigma})$$

For this problem in particular,

$$\boldsymbol{\sigma} \overset{not}{\equiv} \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

so $\sigma_m = 0$ and, therefore, $\boldsymbol{\sigma}_{sph} = \mathbf{0}$, leading to $\boldsymbol{\sigma}' = \boldsymbol{\sigma}$. Then,

$$\left(\boldsymbol{\sigma}'\right)^2 \overset{not}{\equiv} \begin{bmatrix} \tau^2 & 0 & 0 \\ 0 & \tau^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies J'_2 = \tau^2 \implies \bar{\boldsymbol{\sigma}} = \sqrt{3}\,\tau \, .$$

Considering that material (1) plasticizes when $\tau_e = \tau^*$ and material (2), when $\tau_e = 2\tau^*$, then

$$\boxed{\begin{aligned} &\text{Material 1} \implies \sigma_e = \sqrt{3}\,\tau^* \, , \\ &\text{Material 2} \implies \sigma_e = 2\sqrt{3}\,\tau^* \, . \end{aligned}}$$

b) The elastic load P_e and the elastic displacement δ_e determine the end of the elastic regime in the component. The statement of the problem indicates that when the materials are tested separately, the $\tau - \gamma$ curve in Figure B is obtained, where $\tau_e = \tau^*$ in material (1) and $\tau_e = 2\tau^*$ in material (2). It is also known that G is the same in both materials, that is, they have the same slope in their respective $\tau - \gamma$ curves.
Now, to determine the combined behavior of these materials in the metallic component, one can assume that the behavior will be elastic in this component as long as both materials are in their corresponding elastic domain. Therefore,

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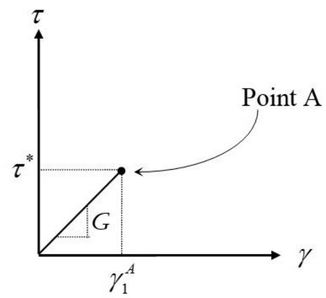
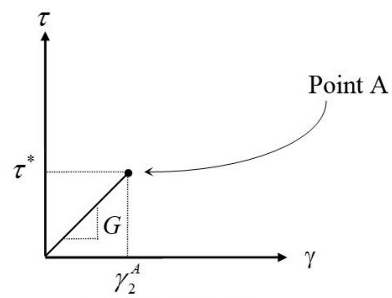
since the elastic interval of material (1) is smaller, then this material will define the elastic domain of the whole component (up to point A in Figure C).





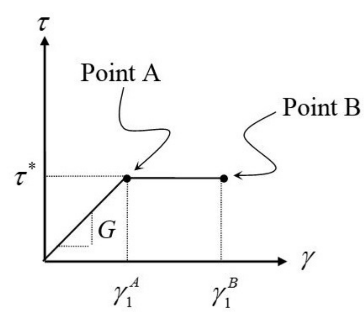
To obtain the value of the elastic force, equilibrium of forces is imposed for the force P_e and the stresses each material has at point A. Note that equilibrium is imposed on forces, therefore, stresses must be multiplied by the surface on which they act, considering the magnitude perpendicular to the plane of the paper as the unit value.

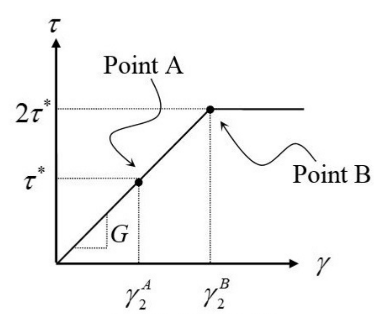
$$P_e = \frac{h}{2}\tau_1^A + \frac{h}{2}\tau_2^A = \frac{h}{2}\tau^* + \frac{h}{2}\tau^* \quad \Longrightarrow \quad \boxed{P_e = h\,\tau^*}$$

The elastic displacement is obtained imposing kinematic compatibility of the two materials,

$$\delta_e = \gamma_1^A h = \gamma_2^A h \quad \Longrightarrow \quad \boxed{\delta_e = \frac{\tau^*}{G}\,h}\ .$$

c) To obtain the plastic values P_p and δ_p one must take into account that, at point A, material (1) begins to plasticize, while material (2) initiates plasticization at point B. Therefore, the behavior of the complete component will be perfectly plastic starting at point B, but elastoplastic between points A and B. To determine the coordinates of point B, the same procedure as before is used. Plotting the $\tau - \gamma$ curves of each separate material up to point B results now in





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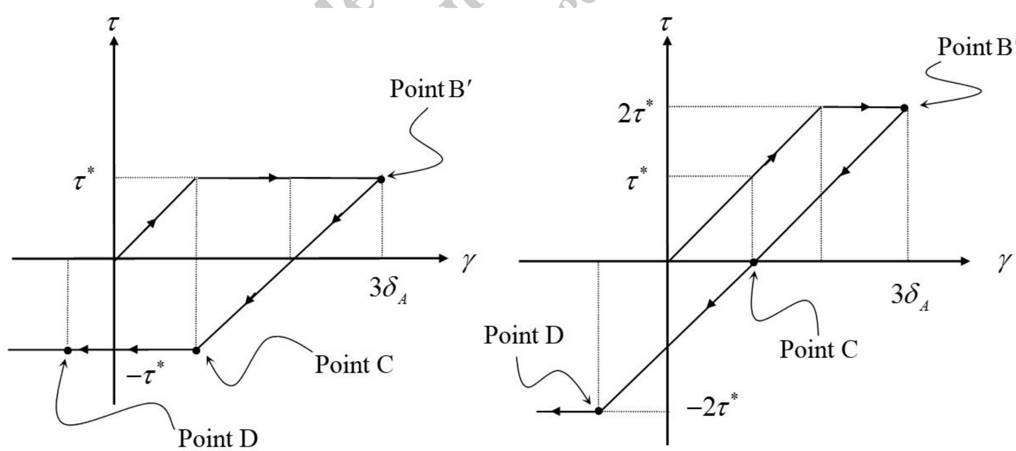
and, imposing the equilibrium and compatibility equations, yields the values of P_p and δ_p.

$$\left.\begin{aligned} P_p &= \frac{h}{2}\tau_1^B + \frac{h}{2}\tau_2^B = \frac{h}{2}\tau^* + \frac{h}{2}2\tau^* \\ \delta_p &= \gamma_1^B h = \gamma_2^B h = \frac{2\tau^*}{G} h \end{aligned}\right\} \quad \Longrightarrow \quad \boxed{\begin{aligned} &P_p = \frac{3}{2}\tau^* h \quad \text{and} \\ &\delta_p = 2\frac{\tau^* h}{G} = 2\delta_e \ . \end{aligned}}$$

d) The coordinates of points A and B have already been obtained. The statement of the problem gives the value of point B', which corresponds to a deformation of $3\delta_e$ when the plastic load P_p is maintained constant (perfectly plastic regime). Consider first the material (1). Unloading takes place starting at B' and, according to the information given, this material plasticizes when it reaches a value of $-\tau^*$. The slope of the curve is still the value of the material parameter G since this is independent of the material being under loading or unloading conditions. Thus, to determine point C it is enough to draw a straight line that crosses point B' and is parallel to OA, until the value $-\tau^*$ is reached.
The same occurs in the case of material (2), with the difference that when the line parallel to OA is drawn to cross point B', this line must be extended to the value $-2\tau^*$ (which corresponds to point D).



MATERIAL (1) MATERIAL (2)

Then, the load and displacement values at point B' are

$$\delta_{B'} = 3\delta_A = \frac{3\tau^* h}{G} = 3\delta_e \quad \text{and} \quad P_{B'} = P_B = \frac{3}{2}\tau^* h \ .$$

To obtain the load and displacement values at point C, the equilibrium and compatibility equations are imposed. Taking into account the τ and γ values obtained at point C in the curves above yields

$$\left.\begin{aligned} & P_C = \frac{h}{2}\tau_1^C + \frac{h}{2}\tau_2^C = \frac{h}{2}(-\tau^*) + \frac{h}{2}(0) = -\frac{\tau^* h}{2} \\ & \delta_C = \gamma_1^C\, h = \left(\frac{\tau^*}{G}\right) h \end{aligned}\right\} \quad \Longrightarrow \quad \boxed{\begin{aligned} & P_C = -\frac{\tau^* h}{2} \quad \text{and} \\ & \delta_C = \frac{\tau^* h}{G} = \delta_e \ . \end{aligned}}$$

Repeating the procedure for point D results in

$$\left.\begin{aligned} & P_D = \frac{h}{2}\tau_1^D + \frac{h}{2}\tau_2^D = \frac{h}{2}(-\tau^*) + \frac{h}{2}(-2\tau^*) = -\frac{3\tau^* h}{2} \\ & \delta_D = \gamma_2^D\, h = \left(-\frac{\tau^*}{G}\right) h \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} & P_D = -\frac{3\tau^* h}{2} \quad \text{and} \\ & \delta_D = -\frac{\tau^* h}{G} = -\delta_e \ . \end{aligned}}$$

Problem 8.5 – *Consider the solid cylinder shown in Figure A, which is fully fixed at its base and has a torsional moment M applied on its top end. The cylinder is composed of two materials, (1) and (2), which have an elastoplastic tangent stress-strain behavior, as shown in Figure B. Assume the following displacement field in cylindrical coordinates (Coulomb torque),*

$$\mathbf{u}(r,\theta,z) \stackrel{not}{\equiv} [u_r\,,\ u_\theta\,,\ u_z]^T = \left[0\,,\ \frac{\theta}{h} rz\,,\ 0\right]^T ,$$

where ϕ is the rotation of the section at the free end of the cylinder. Assuming infinitesimal strains, determine:

a) *The strain and stress tensors, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$, in cylindrical coordinates and elastic regime. Plot, indicating the most significant values, the $\sigma_{rr} - r$ and $\tau_{\theta z} - r$ curves for a cross-section of the cylinder at height z. Schematically represent the stress distribution of $\tau_{\theta z}$ in this cross-section.*
b) *The value of $\phi = \phi_e$ (see Figure C) for which plasticization begins in at least one point of the cylinder, indicating where it begins and the corresponding value of the moment $M = M_e$.*
   NOTE: $M = \int_S r\,\tau_{\theta z}\, dS$
c) *The minimum value of $\phi = \phi_1$ for which material (1) has totally plasticized and the corresponding value of $M = M_1$ (see Figure C). Schematically represent the stress distribution in a cross-section at this instant.*

d) *The minimum value of $\phi = \phi_2$ for which material (2) has totally plasticized and the corresponding value of $M = M_2$ (see Figure C). Schematically represent the stress distribution in a cross-section at this instant.*

e) *The asymptotic value of $M = M_p$ (= plastic moment) corresponding to the plasticization of the complete cross-section. Schematically represent the stress distribution in a cross-section at this instant.*

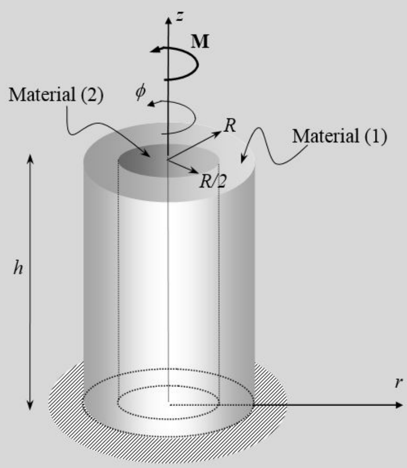


Figure A

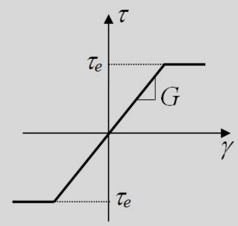


Figure B

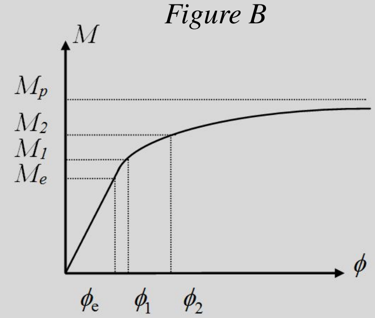


Figure C

HYPOTHESES:
Material (1): $G = G$ *and* $\tau_e = \tau^*$.
Material (2): $G = G$ *and* $\tau_e = 2\,\tau^*$.

Solution

a) The infinitesimal strain tensor is calculated directly from the given displacement field, both in cylindrical coordinates,

$$\boxed{\boldsymbol{\varepsilon} \stackrel{not}{\equiv} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{\phi r}{2h} \\ 0 & \dfrac{\phi r}{2h} & 0 \end{bmatrix}} \; .$$

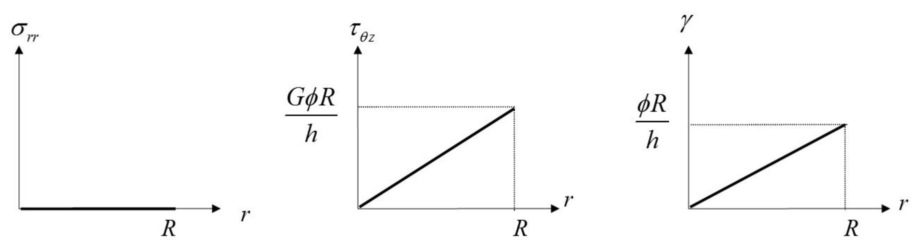
To compute the stress tensor, the constitutive equation of an isotropic elastic material is used. Note that the two materials composing the cylinder have the same parameter G, then

$$\boldsymbol{\sigma} = \lambda \operatorname{Tr}(\boldsymbol{\varepsilon})\,\mathbf{1} + 2\mu\boldsymbol{\varepsilon} \quad \text{and}$$
$$\left.\begin{array}{c} \operatorname{Tr}(\boldsymbol{\varepsilon}) = 0 \\ \mu = G \end{array}\right\} \quad \Longrightarrow \quad \boldsymbol{\sigma} = 2G\boldsymbol{\varepsilon}\ .$$

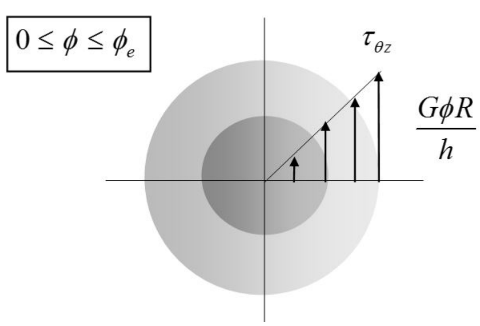
The stress tensor results in

$$\boxed{\boldsymbol{\sigma} \stackrel{not}{\equiv} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{G\phi r}{h} \\ 0 & \dfrac{G\phi r}{h} & 0 \end{bmatrix}}\ .$$

Plotting the σ_{rr} and $\tau_{\theta z}$ components of the stress tensor in terms of the radius r yields:



The stresses are linear and do not depend on the z-coordinate of the cross-section considered. Thus, the distribution of stresses in any cross-section ($z = const.$) of the cylinder is:



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b) Given the stress distribution $\tau = (G\phi r/h) \le \phi \le \phi_e$, the moment acting on the cylinder is

$$M = \int_S r\,\tau(r)\,dS = \int_0^{2\pi}\int_0^R r\left(\frac{G\phi r}{h}\right) r\,dr\,d\theta = 2\pi\int_0^R \frac{G\phi r^3}{h}dr = \frac{\pi G R^4}{2h}\phi. \quad [1]$$

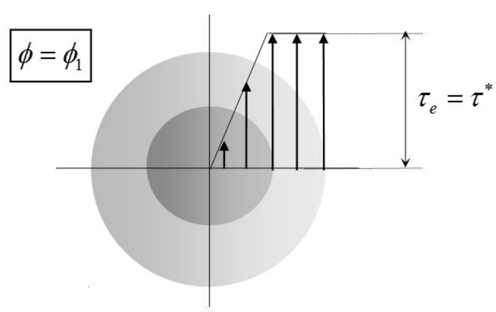
This is the relation between the moment and the rotation angle $(M-\phi)$ at the free end of the cylinder when the two materials behave elastically.
Material (1) starts to plasticize first at $\tau_e = \tau^*$, since material (2) plasticizes at a higher stress, $\tau_e = 2\tau^*$. In addition, the external surface of the cylinder $(r = R)$ suffers larger stresses, and this surface is composed of material (1). Therefore, plasticization will initiate when

$$\tau\Big|_{r=R;\,\phi=\phi_e} = \tau^* \quad \Longrightarrow \quad \frac{G\phi_e R}{h} = \tau^* \quad \Longrightarrow \quad \boxed{\phi_e = \frac{\tau^* h}{GR}}$$

is satisfied. This is the value of the rotation angle at the free end of the cylinder required for plasticization to initiate in the exterior material points of the cylinder (material (1)). The corresponding moment is obtained by replacing ϕ_e in [1],

$$M_e = M(\phi_e) = \frac{\pi G R^4}{2h}\phi_e \quad \Longrightarrow \quad \boxed{M_e = \frac{\pi\tau^* R^3}{2}}\,.$$

c) If the material were elastic, the slope of the stresses τ would increase with ϕ (remaining, though, linear with r), but since the material is now elastoplastic, stresses cannot exceed the value τ_e , which corresponds to the onset of plasticity. Then, the limit value is obtained for $\tau_e = \tau^*$ when $\phi = \phi_1$ for $(R/2) \le r \le R$. That is, material (1) has a perfectly plastic distribution of stresses while material (2) remains elastic.



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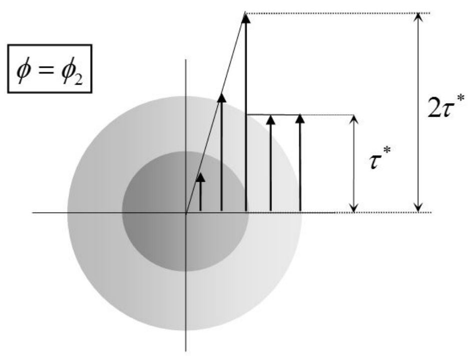
The following condition is imposed to compute this rotation ϕ_1.

$$\tau\Big|_{r=R/2;\ \phi=\phi_1} = \tau^* \qquad \Longrightarrow \qquad \frac{G\phi_1 R}{2h} = \tau^* \qquad \Longrightarrow \qquad \boxed{\phi_1 = \frac{2\tau^* h}{GR}}\ .$$

This is the minimum value of the rotation angle at the free end of the cylinder required for material (1) to be completely plasticized.
In order to compute the corresponding moment, relation [1] between M and ϕ is no longer valid here because material (1) behaves elastoplastically while material (2) behaves completely elastically. The moment acting on the cylinder is now

$$\begin{aligned} M_1 &= \int_0^{2\pi}\int_{R/2}^{R} r\tau^* \, r\,dr\,d\theta + \int_0^{2\pi}\int_0^{R/2} r\left(\frac{G\phi_1 r}{h}\right) r\,dr\,d\theta = \\ &= 2\pi\tau^* \int_{R/2}^{R} r^2\,dr + 2\pi G\frac{\phi_1}{h}\int_0^{R/2} r^3\,dr \quad \Longrightarrow \quad \boxed{M_1 = \frac{31}{48}\pi\tau^* R^3}\ . \end{aligned}$$

d) Material (2) starts plasticizing for $\tau_e = 2\tau^*$, which does not correspond with the end of plasticization in material (1) at $\tau_e = \tau^*$. Then, the stress distribution for $\phi = \phi_2$ (onset of plasticization in material (2)) is



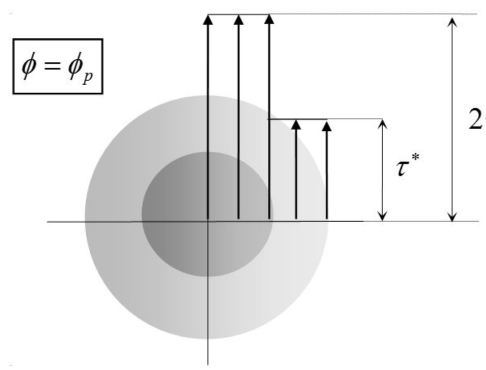
The following condition is imposed to obtain the value of the rotation angle.

$$\tau\Big|_{r=R/2;\ \phi=\phi_2} = 2\tau^* \qquad \Longrightarrow \qquad \frac{G\phi_2 R}{2h} = 2\tau^* \qquad \Longrightarrow \qquad \boxed{\phi_2 = \frac{4\tau^* h}{GR}}\ .$$

The corresponding moment is

$$M_2 = \int_0^{2\pi}\int_{R/2}^{R} r\,\tau^*\,r\,dr\,d\theta + \int_0^{2\pi}\int_0^{R/2} r\left(\frac{G\phi_2 r}{h}\right) r\,dr\,d\theta \quad \Longrightarrow \quad \boxed{M_2 = \frac{17}{24}\pi\tau^* R^3}\ .$$

e) The asymptotic value of $M\,(M_p)$ corresponds to the total plasticization of the cylinder. The stress distribution in this case is:



Through integration, the corresponding moment is obtained,

$$M_p = \int_0^{2\pi}\int_{R/2}^{R} r\,\tau^*\,r\,dr\,d\theta + \int_0^{2\pi}\int_0^{R/2} r\,2\tau^*\,r\,dr\,d\theta \quad \Longrightarrow \quad \boxed{M_p = \frac{3}{4}\pi\tau^* R^3}\ .$$

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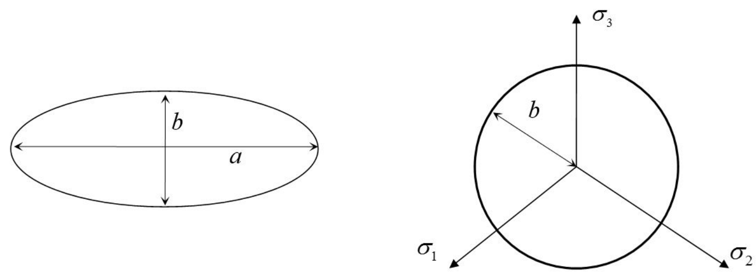
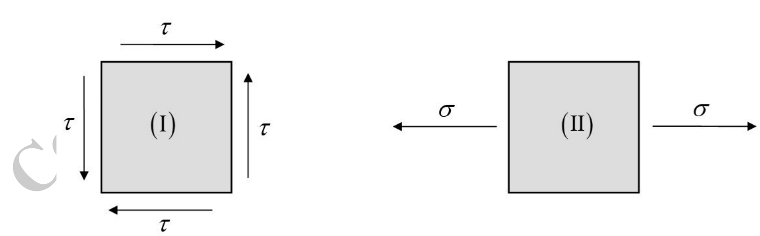
EXERCISES

8.1 – *Formulate in terms of the stress invariants* I_1, J'_2 *and* J'_3 *the equation of the yield surface that, in the principal stress space, is a spheroid (ellipsoid of revolution) with semi-axes a and b.*



Intersection with octahedral plane at $(0,0,0)$

8.2 – *An elastoplastic material is subjected to a pure shear test* (I) *and an uniaxial tensile test* (II). *Plasticization occurs, respectively, at* $\tau = \mathrm{a}$ *and* $\sigma = \mathrm{b}$. *Determine the values of the cohesion and internal friction angle assuming a Mohr-Coulomb yield criterion.*



8.3 – *A component* ABCD *of a perfectly elastoplastic material is tested in the machine illustrated in Figure A. The action-response curve* $(P-\delta)$ *obtained is shown in Figure B. An uniaxial stress-strain state is assumed such that*

$$\begin{aligned} &\varepsilon_x = \frac{\delta}{hL} y \quad \text{and} \quad \varepsilon_y = \varepsilon_z = \gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0 \,, \\ &\sigma_x \neq 0 \qquad \text{and} \quad \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \,. \end{aligned}$$

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Determine the following values, indicated in the curve of Figure B:

a) *The elastic load P_e and the corresponding displacement δ_e.*
b) *The ultimate plastic loads for tensile and compressive loadings, P_p and P_q, respectively.*
c) *The values of P and δ at points (1) and (2).*

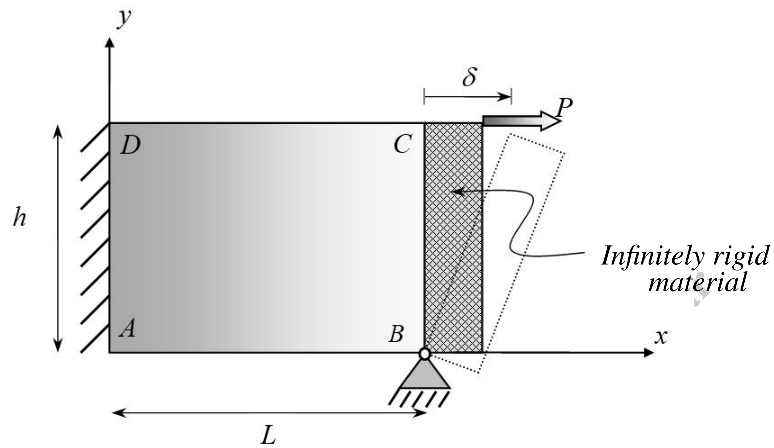

Figure A

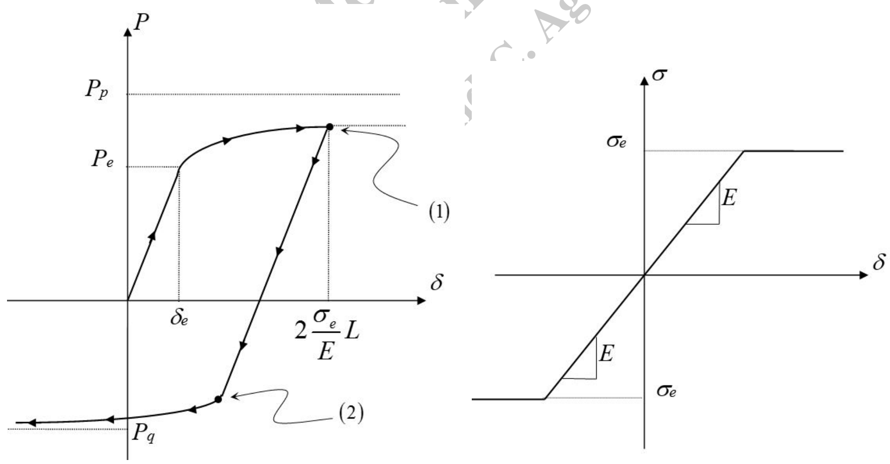

Figure B

Additional hypotheses:

1) *Young's modulus, E, and Poisson's coefficient, ν.*
2) *Elastic limit, σ_e.*
3) *Thickness of the component, b.*

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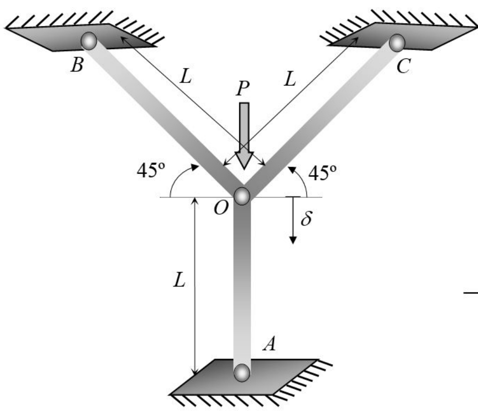
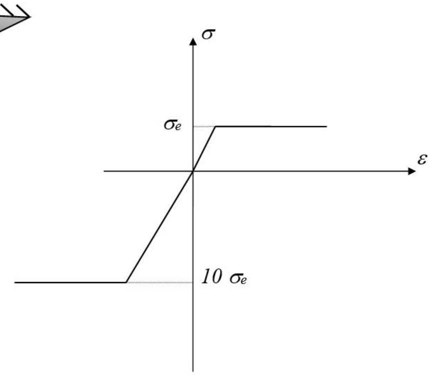
8.4 – *The truss structure OA, OB and OC is composed of concrete, which is assumed to behave as a perfectly elastoplastic material with a tensile elastic limit σ_e and a compressive elastic limit $10\,\sigma_e$. An increasing vertical load P is applied at point O, starting at $P=0$, until a vertical displacement $\delta = 20\,\sigma_e\,L/E$ is reached at this point. Then, the load is decreased back to $P=0$.*

a) *Draw the $P-\delta$ diagram of the process, indicating the most significant values and the state of plasticization of the bars at each instant.*
b) *Calculate the displacement value at point O at the end of the process.*

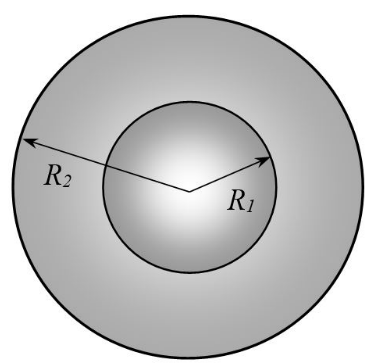




8.5 – *Consider a solid sphere with radius R_1 encased inside a spherical shell with interior radius R_1 and exterior radius R_2. The sphere and the shell are composed of the same material and are initially in contact without exerting any pressure on each other. At a certain moment, the interior sphere is heated up to a temperature increment $\Delta\theta$.*

Determine:

a) *The value of the exterior pressure required on the shell for said shell to keep a constant value (infinitesimal strain hypothesis).*
b) *The displacement, strain and stress fields in both the sphere and the shell under these conditions.*
c) *The minimum value of $\Delta\theta$ for which plasticization initiates in some point, assuming the aforementioned conditions and considering a von Mises criterion.*



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Additional hypotheses:

1) *Material properties:*
– *Young's modulus, E, and Poisson's coefficient, $\nu = 0$.*
– *Thermal constant, α.*
– *Yield stress, σ_y.*
– *Radii, $R_1 = 1$ and $R_2 = 3$.*
2) *The body forces are negligible.*
3) *The displacement and stress fields of a spherical shell with interior radius R_i and exterior radius R_e subjected to an interior pressure P_i and an exterior pressure P_e are, for $\nu = 0$:*

$$\mathbf{u} = \begin{bmatrix} u_r(r) \\ 0 \\ 0 \end{bmatrix} \qquad u_r = Cr + \frac{C_1}{r^2}; \quad C = \frac{P_i R_i^3 - P_e R_e^3}{E\left(R_e^3 - R_i^3\right)}; \quad C_1 = \frac{P_i - P_e}{2E} \frac{R_i^3 R_e^3}{R_e^3 - R_i^3}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\phi\phi} \end{bmatrix} \qquad \sigma_{rr} = E\left(C - \frac{2C_1}{r^3}\right); \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = E\left(C + \frac{C_1}{r^3}\right)$$

8.6 – *Consider a solid sphere with radius R_1 and composed of material (1), encased inside a spherical shell with interior radius R_1, exterior radius R_2 and composed of material (2). The sphere and the shell are initially in contact without exerting any pressure on each other. An exterior pressure P is applied simultaneously with a temperature increment $\Delta\theta$.*

a) *Determine the possible values of $\Delta\theta$ and P (positive or negative) for which the contact (without exerting any pressure) between the sphere and the shell is maintained. Plot the corresponding $P - \Delta\theta$ curve.*
b) *Obtain the stress state of the shell and the sphere for these values.*
c) *Under these conditions, compute, for each value of the pressure P, the value of $\Delta\theta^*$ for which plasticization initiates at some point of the sphere or the shell, according to the von Mises and Mohr-Coulomb criteria. Plot the corresponding $P - \Delta\theta^*$ curves (interaction graphs).*

Additional hypotheses:

1) *Material properties:*
– *Young's moduli, $E^{(1)} = E^{(2)} = E$, and Poisson's coefficients, $\nu^{(1)} = \nu^{(2)} = 0$.*
– *Thermal constants, $\alpha^{(1)} = 2\alpha$ and $\alpha^{(2)} = \alpha$.*
– *Yield stresses, $\sigma_y^{(1)} = \sigma_y^{(2)} = \sigma_y$.*

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– *Cohesion values,* $\mathrm{C}^{(1)} = \mathrm{C}^{(2)} = \mathrm{C}$*, and internal friction angles,* $\phi^{(1)} = \phi^{(2)} = 30°$.
– *Radii,* $\mathrm{R}_1 = 1$ *and* $\mathrm{R}_2 = 2$.

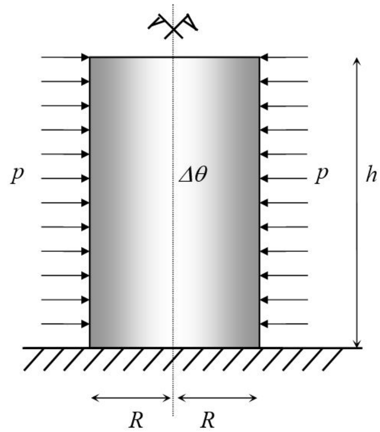
2) *The displacement and stress fields of a spherical shell with interior radius* R_i *and exterior radius* R_e *subjected to an interior pressure* P_i *and an exterior pressure* P_e *are, for* $\nu = 0$*:*

$$\mathbf{u} = \begin{bmatrix} u_r(r) \\ 0 \\ 0 \end{bmatrix} \qquad u_r = Cr + \frac{C_1}{r^2}; \quad C = \frac{P_i R_i^3 - P_e R_e^3}{E\left(R_e^3 - R_i^3\right)}; \quad C_1 = \frac{P_i - P_e}{2E}\frac{R_i^3 R_e^3}{R_e^3 - R_i^3}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & \sigma_{\phi\phi} \end{bmatrix} \qquad \sigma_{rr} = E\left(C - \frac{2C_1}{r^3}\right); \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = E\left(C + \frac{C_1}{r^3}\right)$$

8.7 – *A cylinder of radius R and height h is subjected to an exterior load P and a uniform temperature increment* $\Delta\theta$.

a) *Determine the displacement, strain and tensor fields in terms of the integration constants.*
b) *Determine the integration constants and the corresponding displacement, strain and tensor fields.*
c) *Given* $p = p^* > 0$*, determine the corresponding value of* $\Delta\theta^*$ *such that there are no horizontal displacements.*
d) *Under the conditions described in* c)*, determine the value of* p^* *for which the cylinder begins to plasticize according to the Mohr-Coulomb criterion.*



Additional hypotheses:

1) *Material properties:*
– *Cohesion value,* C*, and internal friction angle,* $\phi = 30°$.
– *Thermal constant,* β.
– *Lamé parameters,* $\lambda = \mu$.
2) *The body forces are negligible.*
3) *The friction between the cylinder and the ground can be neglected.*

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Chapter 9
Constitutive Equations in Fluids

9.1 Concept of Pressure

Several concepts of pressure are used in continuum mechanics (hydrostatic pressure, mean pressure and thermodynamic pressure) which, in general, do not coincide.

9.1.1 Hydrostatic Pressure

> **Definition 9.1. Pascal's law**
> In a confined fluid at rest, the stress state on any plane containing a given point is the same and is characterized by a compressive normal stress.

In accordance with Pascal's law, the stress state of a fluid at rest is characterized by a stress tensor of the type

$$\boxed{\begin{array}{l} \boldsymbol{\sigma} = -p_0\,\mathbf{1} \\ \sigma_{ij} = -p_0\,\delta_{ij} \qquad i,j \in \{1,2,3\} \end{array}}\,, \tag{9.1}$$

where p_0 is denoted as *hydrostatic pressure* (see Figure 9.1).

> **Definition 9.2.** The *hydrostatic pressure* is the compressive normal stress, constant on any plane, that acts on a fluid at rest.

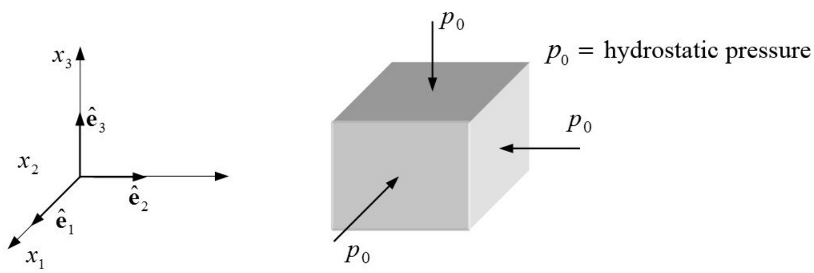

Figure 9.1: Stress state of a fluid at rest.

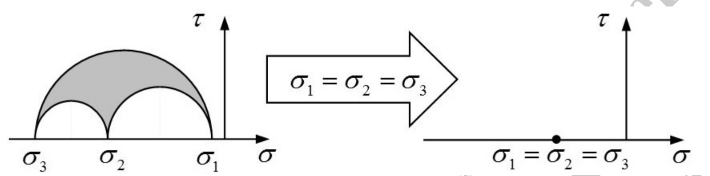

Figure 9.2: Mohr's circle of the stress tensor of a fluid at rest.

Remark 9.1. The stress tensor of a fluid at rest is a *spherical* tensor and its representation in the Mohr's plane is a point (see Figure 9.2). Consequently, any direction is a principal stress direction and the stress state is constituted by the state defined in Section 4.8 of Chapter 4 as *hydrostatic stress state*.

9.1.2 Mean Pressure

Definition 9.3. The *mean stress* σ_m is defined as

$$\sigma_m = \frac{1}{3}\,\mathrm{Tr}(\boldsymbol{\sigma}) = \frac{1}{3}\sigma_{ii}\ .$$

The *mean pressure* $\bar{p}$ is defined as minus the mean stress,

$$\bar{p} \overset{def}{=} \text{mean pressure} = -\sigma_m = -\frac{1}{3}\,\mathrm{Tr}(\boldsymbol{\sigma}) = -\frac{1}{3}\sigma_{ii}\ .$$

Remark 9.2. In a *fluid at rest*, the mean pressure $\bar{p}$ coincides with the hydrostatic pressure p_0,

$$\boldsymbol{\sigma} = -p_0\mathbf{1} \quad \Longrightarrow \quad \sigma_m = \frac{1}{3}(-3p_0) = -p_0 \quad \Longrightarrow \quad \bar{p} = p_0 \, .$$

Generally, in a fluid in motion the mean pressure and the hydrostatic pressure do not coincide.

Remark 9.3. The trace of the Cauchy stress tensor is a stress invariant. Consequently, the mean stress and the mean pressure are also stress invariants and, therefore, their values do not depend on the Cartesian coordinate system used.

9.1.3 Thermodynamic Pressure. Kinetic Equation of State

A new thermodynamic pressure variable, named *thermodynamic pressure* and denoted as p, intervenes in the constitutive equations of fluids or gases.

Definition 9.4. The *thermodynamic pressure* is the pressure variable that intervenes in the constitutive equations of fluids and gases, and is related to the density ρ and the absolute temperature θ by means of the *kinetic equation of state*, $F(p,\rho,\theta) = 0$.

Example 9.1

The ideal gas law is a typical example of kinetic equation of state:

$$F(p,\rho,\theta) \equiv p - \rho R \theta = 0 \quad \Longrightarrow \quad p = \rho R \theta \, ,$$

where p is the thermodynamic pressure and R is the universal gas constant.

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Remark 9.4. In a *fluid at rest*, the hydrostatic pressure p_0, the mean pressure $\bar{p}$ and the thermodynamic pressure p coincide.

$$\text{Fluid at rest}: \quad p_0 = \bar{p} = p$$

Generally, in a fluid in motion the hydrostatic pressure, the mean pressure and the thermodynamic pressure do not coincide.

Remark 9.5. A *barotropic fluid* is defined by a kinetic equation of state in which the temperature does not intervene.

$$\text{Barotropic fluid}: \quad F(p,\rho) = 0 \implies p = f(\rho) \implies \rho = g(p)$$

Remark 9.6. An *incompressible fluid* is a particular case of barotropic fluid in which density is constant ($\rho(\mathbf{x},t) = k = const.$). In this case, the kinetic equation of state can be written as

$$F(p,\rho,\theta) \equiv \rho - k = 0$$

and does not depend on the pressure or the temperature.

9.2 Constitutive Equations in Fluid Mechanics

Here, the set of equations, generically named *constitutive equations*, that must be added to the balance equations to formulate a problem in fluid mechanics (see Section 5.13 in Chapter 5) is considered. These equations can be grouped as follows:

a) *Thermo-mechanical constitutive equation*

This equation expresses the Cauchy stress tensor in terms of the other thermodynamic variables, typically the thermodynamic pressure p, the strain rate tensor $\mathbf{d}$ (which can be considered an implicit function of the velocity, $\mathbf{d}(\mathbf{v}) = \nabla^S \mathbf{v}$), the density ρ and the absolute temperature θ.

$$\boxed{\begin{array}{ll}\text{Thermo-mechanical} & \\ \text{constitutive equation:} & \boldsymbol{\sigma} = -p\,\mathbf{1} + \mathbf{f}(\mathbf{d},\rho,\theta)\end{array}} \quad \text{6 equations} \tag{9.2}$$

b) Entropy constitutive equation

An algebraic equation that provides the specific entropy s in terms of the strain rate tensor, the density and the absolute temperature.

$$\boxed{\begin{array}{ll}\text{Entropy} & \\ \text{constitutive equation:} & s = s(\mathbf{d},\rho,\theta)\end{array}} \quad \text{1 equation} \tag{9.3}$$

c) Thermodynamic constitutive equations or equations of state

These are typically the *caloric equation of state*, which defines the specific internal energy u, and the *kinetic equation of state*, which provides an equation for the thermodynamic pressure.

$$\boxed{\begin{array}{ll}\text{Caloric equation of state:} & u = g(\rho,\theta) \\ \text{Kinetic equation of state:} & F(\rho,p,\theta) = 0\end{array}} \quad \text{2 equations} \tag{9.4}$$

d) Thermal constitutive equations

The most common one is *Fourier's law*, which defines the heat flux by conduction $\mathbf{q}$ as

$$\boxed{\text{Fourier's law:} \quad \begin{cases}\mathbf{q} = -\mathbf{k}\cdot\nabla\theta \\ q_i = k_{ij}\dfrac{\partial\theta}{\partial x_j} \quad i\in\{1,2,3\}\end{cases}} \quad \text{3 equations} \tag{9.5}$$

where $\mathbf{k}$ is the (symmetrical second-order) tensor of thermal conductivity, which is a property of the fluid. For the isotropic case, the thermal conductivity tensor is a spherical tensor $\mathbf{k} = k\,\mathbf{1}$ and depends on the scalar parameter k, which is the thermal conductivity of the fluid.

9.3 Constitutive Equation in Viscous Fluids

The general form of the thermo-mechanical constitutive equation (see (9.2)) for a viscous fluid is

$$\boxed{\begin{array}{l}\boldsymbol{\sigma} = -p\,\mathbf{1} + \mathbf{f}(d,\rho,\theta) \\ \sigma_{ij} = -p\,\delta_{ij} + f_{ij}(d,\rho,\theta) \qquad i,j\in\{1,2,3\}\end{array}}\,, \tag{9.6}$$

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where $\mathbf{f}$ is a symmetrical tensor function. According to the character of the function $\mathbf{f}$, the following models of fluids are defined:

a) Stokesian or Stokes fluid: the function $\mathbf{f}$ is a non-linear function of its arguments.

b) Newtonian fluid: the function $\mathbf{f}$ is a *linear* function of its arguments.

c) Perfect fluid: the function $\mathbf{f}$ is null. In this case, the mechanical constitutive equation is $\boldsymbol{\sigma} = -p\mathbf{1}$.

In the rest of this chapter, only the cases of Newtonian and perfect fluids will be considered.

> *Remark 9.7.* The perfect fluid hypothesis is frequently used in hydraulic engineering, where the fluid under consideration is water.

9.4 Constitutive Equation in Newtonian Fluids

The mechanical constitutive equation[1] for a Newtonian fluid is

$$\boxed{\begin{array}{l} \boldsymbol{\sigma} = -p\,\mathbf{1} + \mathbb{C} : \mathbf{d} \\ \sigma_{ij} = -p\,\delta_{ij} + \mathbb{C}_{ijkl}\, d_{kl} \qquad i,j \in \{1,2,3\} \end{array}} \,, \tag{9.7}$$

where $\mathbb{C}$ is a *constant* fourth-order (viscosity) constitutive tensor. A linear dependency of the stress tensor $\boldsymbol{\sigma}$ on the strain rate tensor $\mathbf{d}$ is obtained as a result of (9.7). For an *isotropic Newtonian fluid*, the constitutive tensor $\mathbb{C}$ is an *isotropic* fourth-order tensor.

$$\begin{cases} \mathbb{C} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I} \\ \mathbb{C}_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + \mu \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \qquad i,j,k,l \in \{1,2,3\} \end{cases} \tag{9.8}$$

Replacing (9.8) in the mechanical constitutive equation (9.7) yields

$$\boldsymbol{\sigma} = -p\,\mathbf{1} + (\lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}) : \mathbf{d} = -p\,\mathbf{1} + \lambda\,\mathrm{Tr}(\mathbf{d})\,\mathbf{1} + 2\mu\,\mathbf{d}\,, \tag{9.9}$$

which corresponds to the constitutive equation of an isotropic Newtonian fluid.

[1] Note that the thermal dependencies of the constitutive equation are not considered here and, thus, the name *mechanical* constitutive equations.

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$$\text{Constit. eqn. of an isotropic Newtonian fluid} \quad \begin{cases} \boldsymbol{\sigma} = -p\,\mathbf{1} + \lambda\,\mathrm{Tr}(\mathbf{d})\,\mathbf{1} + 2\mu\,\mathbf{d} \\ \sigma_{ij} = -p\,\delta_{ij} + \lambda\,d_{ll}\,\delta_{ij} + 2\mu\,d_{ij} \quad i,j \in \{1,2,3\} \end{cases} \tag{9.10}$$

Remark 9.8. Note the parallelism that can be established between the constitutive equation of a Newtonian fluid and that of a linear elastic solid (see Chapter 6):

Newtonian fluid	*Linear elastic solid*
$\begin{cases} \boldsymbol{\sigma} = -p\,\mathbf{1} + \mathbb{C} : \mathbf{d} \\ \sigma_{ij} = -p\,\delta_{ij} + \mathbb{C}_{ijkl}\,d_{kl} \end{cases}$	$\begin{cases} \boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} \\ \sigma_{ij} = \mathbb{C}_{ijkl}\,\varepsilon_{kl} \end{cases}$

Remark 9.9. The parameters λ and μ physically correspond to the viscosities, which are understood as material properties. In the most general case, they may not be constant and can depend on other thermodynamic variables,

$$\lambda = \lambda(\rho,\theta) \quad \text{and} \quad \mu = \mu(\rho,\theta)\ .$$

A typical example is the dependency of the viscosity on the temperature in the form $\mu(\theta) = \mu_0\,\mathrm{e}^{-\alpha(\theta-\theta_0)}$, which establishes that the fluid's viscosity decreases as temperature increases (see Figure 9.3).

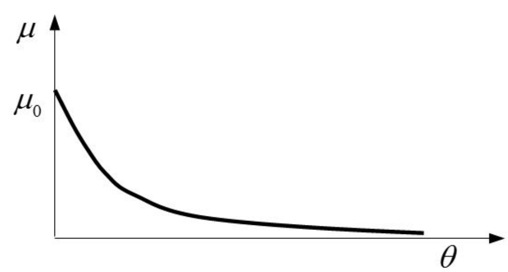

Figure 9.3: Possible dependency of the viscosity μ on the absolute temperature θ.

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9.4.1 Relation between the Thermodynamic and Mean Pressures

In general, the thermodynamic pressure p and the mean pressure $\bar{p}$ in a Newtonian fluid in motion will be different but are related to each other. From the (mechanical) constitutive equation of a Newtonian fluid (9.10),

$$
\begin{gathered}
\boldsymbol{\sigma} = -p\,\mathbf{1} + \lambda\,\mathrm{Tr}(\mathbf{d})\,\mathbf{1} + 2\mu\,\mathbf{d} \quad \Longrightarrow \\
\underbrace{\mathrm{Tr}(\boldsymbol{\sigma})}_{-3\bar{p}} = -p\,\mathrm{Tr}(\mathbf{1}) + \lambda\,\mathrm{Tr}(\mathbf{d})\,\mathrm{Tr}(\mathbf{1}) + 2\mu\,\mathrm{Tr}(\mathbf{d}) = -3p + (3\lambda + 2\mu)\,\mathrm{Tr}(\mathbf{d}) \Longrightarrow \\
p = \bar{p} + \underbrace{\left(\lambda + \frac{2}{3}\mu\right)}_{\mathcal{K}} \mathrm{Tr}(\mathbf{d}) = \bar{p} + \mathcal{K}\,\mathrm{Tr}(\mathbf{d})
\end{gathered}
\tag{9.11}
$$

where $\mathcal{K}$ is denoted as *bulk viscosity*.

$$
\boxed{\text{Bulk viscosity}: \quad \mathcal{K} = \lambda + \frac{2}{3}\mu} \tag{9.12}
$$

Using the mass continuity equation (5.24), results in

$$
\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0 \quad \Longrightarrow \quad \nabla\cdot\mathbf{v} = -\frac{1}{\rho}\frac{d\rho}{dt} \tag{9.13}
$$

Then, considering the relation

$$
\mathrm{Tr}(\mathbf{d}) = d_{ii} = \frac{\partial \mathrm{v}_i}{\partial x_i} = \nabla\cdot\mathbf{v} \tag{9.14}
$$

and replacing in (9.11), yields

$$
\boxed{p = \bar{p} + \mathcal{K}\nabla\cdot\mathbf{v} = \bar{p} - \frac{\mathcal{K}}{\rho}\frac{d\rho}{dt}} \tag{9.15}
$$

which relates the mean and thermodynamic pressures.

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Remark 9.10. In accordance with (9.15), the thermodynamic pressure and the mean pressure in a Newtonian fluid will coincide in the following cases:

- *Fluid at rest*: $\mathbf{v} = 0 \implies p = \bar{p} = p_0$
- *Incompressible fluid*: $\dfrac{d\rho}{dt} = 0 \implies p = \bar{p}$
- *Fluid with null bulk viscosity* $\mathcal{K}$ (Stokes' condition[2]):

$$\mathcal{K} = 0 \implies \lambda = -\frac{2}{3}\mu \implies p = \bar{p}$$

9.4.2 Constitutive Equation in Spherical and Deviatoric Components

Spherical part

From (9.15), the following relation is deduced.

$$\boxed{\bar{p} = p - \mathcal{K}\,\nabla \cdot \mathbf{v} = p - \mathcal{K}\ \mathrm{Tr}(\mathbf{d})} \tag{9.16}$$

Deviatoric part

Using the decomposition of the stress tensor $\boldsymbol{\sigma}$ and the strain rate tensor $\mathbf{d}$ in its spherical and deviator components, and replacing in the constitutive equation (9.10), results in

$$\begin{gathered}
\boldsymbol{\sigma} = \frac{1}{3}\underbrace{\mathrm{Tr}(\boldsymbol{\sigma})}_{-3\bar{p}}\mathbf{1} + \boldsymbol{\sigma}' = -\bar{p}\,\mathbf{1} + \boldsymbol{\sigma}' = -p\,\mathbf{1} + \lambda\,\mathrm{Tr}(\mathbf{d})\,\mathbf{1} + 2\mu\,\mathbf{d} \implies \\
\boldsymbol{\sigma}' = \underbrace{(\bar{p} - p)}_{-\mathcal{K}\,\mathrm{Tr}(\mathbf{d})}\ \mathbf{1} + \lambda\,\mathrm{Tr}(\mathbf{d})\,\mathbf{1} + 2\mu\mathbf{d} = \big(\lambda - \underbrace{\mathcal{K}}_{\lambda + \frac{2}{3}\mu}\big)\,\mathrm{Tr}(\mathbf{d})\,\mathbf{1} + 2\mu\mathbf{d} \implies \\
\boldsymbol{\sigma}' = -\frac{2}{3}\mu\,\mathrm{Tr}(\mathbf{d})\,\mathbf{1} + 2\mu\mathbf{d} = 2\mu\underbrace{\left(\mathbf{d} - \frac{1}{3}\mathrm{Tr}(\mathbf{d})\,\mathbf{1}\right)}_{\mathbf{d}'} \implies
\end{gathered} \tag{9.17}$$

[2] Stokes' condition is assumed in certain cases because the results it provides match the experimental observations.

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$$\boxed{\boldsymbol{\sigma}' = 2\mu \mathbf{d}'} \tag{9.18}$$

where (9.16) and (9.12) have been taken into account.

9.4.3 Stress Power, Recoverable Power and Dissipative Power

Using again the decomposition of the stress and strain rate tensors in their spherical and deviatoric components yields

$$\boldsymbol{\sigma} = -\bar{p}\,\mathbf{1} + \boldsymbol{\sigma}' \qquad \text{and} \qquad \mathbf{d} = \frac{1}{3}\,\mathrm{Tr}\,(\mathbf{d})\,\mathbf{1} + \mathbf{d}' , \tag{9.19}$$

and replacing in the expression of the stress power density (stress power per unit of volume) $\boldsymbol{\sigma} : \mathbf{d}$, results in[3]

$$\begin{aligned}
\boldsymbol{\sigma} : \mathbf{d} &= (-\bar{p}\,\mathbf{1} + \boldsymbol{\sigma}') : \left(\frac{1}{3}\,\mathrm{Tr}\,(\mathbf{d})\,\mathbf{1} + \mathbf{d}'\right) = \\
&= -\frac{1}{3}\bar{p}\ \mathrm{Tr}\,(\mathbf{d})\,\underbrace{\mathbf{1} : \mathbf{1}}_{3} + \boldsymbol{\sigma}' : \mathbf{d}' - \bar{p}\,\underbrace{\mathbf{1} : \mathbf{d}'}_{\mathrm{Tr}\,(\mathbf{d}') = 0} + \frac{1}{3}\,\mathrm{Tr}\,(\mathbf{d})\,\underbrace{\boldsymbol{\sigma}' : \mathbf{1}}_{\mathrm{Tr}\,(\boldsymbol{\sigma}') = 0} = \\
&= -\bar{p}\ \mathrm{Tr}\,(\mathbf{d}) + \boldsymbol{\sigma}' : \mathbf{d}' .
\end{aligned} \tag{9.20}$$

Replacing (9.16) and (9.17) in (9.20) produces

$$\boldsymbol{\sigma} : \mathbf{d} = -\Big(p - \mathcal{K}\ \mathrm{Tr}\,(\mathbf{d})\Big)\,\mathrm{Tr}\,(\mathbf{d}) + 2\mu\,\mathbf{d}' : \mathbf{d}' . \tag{9.21}$$

$$\boxed{\boldsymbol{\sigma} : \mathbf{d} = \underbrace{-p\ \mathrm{Tr}\,(\mathbf{d})}_{\substack{\text{recoverable power} \\ W_R}} + \underbrace{\mathcal{K}\,\mathrm{Tr}^2\,(\mathbf{d}) + 2\mu\,\mathbf{d}' : \mathbf{d}'}_{\substack{\text{dissipative power} \\ 2W_D}} = W_R + 2W_D} \tag{9.22}$$

$$\boxed{\begin{array}{ll}
\text{Recoverable power density:} & W_R = -p\ \mathrm{Tr}\,(\mathbf{d}) \\
\text{Dissipative power density:} & 2W_D = \mathcal{K}\,\mathrm{Tr}^2\,(\mathbf{d}) + 2\mu\,\mathbf{d}' : \mathbf{d}'
\end{array}} \tag{9.23}$$

[3] The property that the trace of a deviator tensor is null is used here.

Associated with the concepts of recoverable and dissipative powers, the recoverable and dissipative parts of the stress tensor, $\boldsymbol{\sigma}_R$ and $\boldsymbol{\sigma}_D$, respectively, are defined as

$$\boldsymbol{\sigma} = -\underbrace{p\,\mathbf{1}}_{\boldsymbol{\sigma}_R} + \underbrace{\lambda\,\mathrm{Tr}(\mathbf{d})\,\mathbf{1} + 2\mu\,\mathbf{d}}_{\boldsymbol{\sigma}_D} \quad \Longrightarrow \quad \boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}_R + \boldsymbol{\sigma}_D}\ . \tag{9.24}$$

Using the aforementioned notation, the recoverable, dissipative and total power densities can be rewritten as

$$\begin{cases} W_R = -p\ \mathrm{Tr}(\mathbf{d}) = -p\,\mathbf{1} : \mathbf{d} = \boldsymbol{\sigma}_R : \mathbf{d}\,, \\ 2W_D = \mathcal{K}\,\mathrm{Tr}^2(\mathbf{d}) + 2\mu\,\mathbf{d}' : \mathbf{d}' = \boldsymbol{\sigma}_D : \mathbf{d}\,, \end{cases} \tag{9.25}$$

$$\boldsymbol{\sigma} : \mathbf{d} = (\boldsymbol{\sigma}_R + \boldsymbol{\sigma}_D) : \mathbf{d} = \boldsymbol{\sigma}_R : \mathbf{d} + \boldsymbol{\sigma}_D : \mathbf{d} = W_R + 2W_D\ .$$

Remark 9.11. In an incompressible fluid, the recoverable power is null. In effect, since the fluid is incompressible, $d\rho/dt = 0$, and considering the mass continuity equation (5.24),

$$\nabla \cdot \mathbf{v} = -\frac{1}{\rho}\frac{d\rho}{dt} = 0 = \mathrm{Tr}(\mathbf{d}) \quad \Longrightarrow \quad W_R = -p\ \mathrm{Tr}(\mathbf{d}) = 0\,.$$

Remark 9.12. Introducing the decomposition of the stress power (9.25), the balance of mechanical energy (5.73) becomes

$$P_e = \frac{d\mathcal{K}}{dt} + \int_V \boldsymbol{\sigma} : \mathbf{d}\,dV = \frac{d\mathcal{K}}{dt} + \int_V \boldsymbol{\sigma}_R : \mathbf{d}\,dV + \int_V \boldsymbol{\sigma}_D : \mathbf{d}\,dV$$

$$\boxed{P_e = \frac{d\mathcal{K}}{dt} + \int_V W_R\,dV + \int_V 2W_D\,dV}\ ,$$

which indicates that the mechanical power entering the fluid P_e is invested in modifying the kinetic energy $\mathcal{K}$ and creating recoverable power and dissipative power.

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9.4.4 Thermodynamic Considerations

1) It can be proven that, under general conditions, the specific recoverable power (recoverable power per unit of mass) is an exact differential

$$\frac{1}{\rho} W_R = \frac{1}{\rho} \boldsymbol{\sigma}_R : \mathbf{d} = \frac{dG}{dt} \,. \tag{9.26}$$

In this case, the recoverable work per unit of mass performed in a closed cycle will be null (see Figure 9.4),

$$\int_A^{B\equiv A} \frac{1}{\rho} W_R \, dt = \int_A^{B\equiv A} \frac{1}{\rho} \boldsymbol{\sigma}_R : \mathbf{d} \, dt = \int_A^{B\equiv A} dG = G_{B\equiv A} - G_A = 0 \,, \tag{9.27}$$

which justifies the denomination of W_R as *recoverable power*.

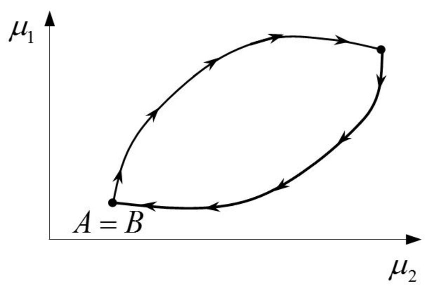


Figure 9.4: Closed cycle.

2) The second law of thermodynamics allows proving that the dissipative power $2W_D$ in (9.25) is always non-negative,

$$2W_D \geq 0 \quad ; \qquad 2W_D = 0 \quad \Longleftrightarrow \quad \mathbf{d} = \mathbf{0} \tag{9.28}$$

and, therefore, in a closed cycle the work performed per unit of mass by the dissipative stresses will, in general, not be null,

$$\int_A^B \frac{1}{\rho} \underbrace{\boldsymbol{\sigma}_D : \mathbf{d}}_{2W_D > 0} \, dt > 0 \,. \tag{9.29}$$

This justifies the denomination of $2W_D$ as (non-recoverable) *dissipative power*. The dissipative power is responsible for the *dissipation* (or loss of energy) phenomenon in fluids.

Example 9.2 – *Explain why an incompressible Newtonian fluid in motion that is not provided with external power (work per unit of time) tends to reduce its velocity to a complete stop.*

Solution

The recoverable power in an incompressible fluid is null (see Remark 9.11). In addition, the dissipative power $2W_D$ is known to be always non-negative (see (9.28)). Finally, applying the balance of mechanical energy (see Remark 9.12) results in

$$0 = P_e = \frac{d\mathcal{K}}{dt} + \int_V \underbrace{W_R}_{=0} \, dV + \int_V 2W_D \, dV \quad \Longrightarrow$$

$$\frac{d\mathcal{K}}{dt} = \frac{d}{dt}\int_V \frac{1}{2}\rho \mathrm{v}^2 dV = -\int_V 2 \underbrace{W_D}_{>0} \, dV < 0$$

and, therefore, the fluid looses (*dissipates*) kinetic energy and the velocity of its particles decreases.

9.4.5 Limitations in the Viscosity Values

Due to thermodynamic considerations, the dissipative power $2W_D$ in (9.25) has been seen to always be non-negative,

$$\boxed{2W_D = \mathcal{K}\,\mathrm{Tr}^2(\mathbf{d}) + 2\mu\,\mathbf{d}' : \mathbf{d}' \geq 0} \;. \tag{9.30}$$

This thermodynamic restriction introduces limitations in the admissible values of the viscosity parameters $\mathcal{K}$, λ and μ of the fluid. In effect, given a certain fluid, the aforementioned restriction must be verified for all motions (that is, for all velocity fields $\mathbf{v}$) that the fluid may possibly have. Therefore, it must be verified for any arbitrary value of the strain rate tensor $\mathbf{d} = \nabla^S(\mathbf{v})$. Consider, in particular, the following cases:

a) *The strain rate tensor* $\mathbf{d}$ *is a spherical tensor.*

In this case, from (9.30) results

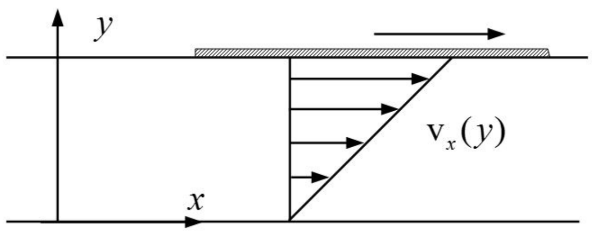
$$\left.\begin{array}{l} \mathrm{Tr}(\mathbf{d}) \neq 0\,;\; \mathbf{d}' = \mathbf{0} \quad \Longrightarrow \quad 2W_D = \mathcal{K}\,\mathrm{Tr}^2(\mathbf{d}) \geq 0 \quad \Longrightarrow \\ \boxed{\mathcal{K} = \lambda + \frac{2}{3}\mu \geq 0} \end{array}\right. \tag{9.31}$$

such that only the non-negative values of the bulk viscosity $\mathcal{K}$ are feasible.

b) The strain rate tensor **d** *is a deviatoric tensor.*

This type of flow is schematically represented in Figure 9.5. In this case, from (9.30) results

$$\operatorname{Tr}(\mathbf{d}) = 0\,;\ \mathbf{d}' \neq \mathbf{0} \implies 2W_D = 2\mu\,\mathbf{d}' : \mathbf{d}' = 2\mu\,\underbrace{d'_{ij} : d'_{ij}}_{>0} \geq 0 \implies \boxed{\mu \geq 0} \tag{9.32}$$



$$\mathbf{v}(x,y) = \begin{bmatrix} v_x(y) \\ 0 \\ 0 \end{bmatrix} \quad ; \quad \mathbf{d} = \begin{bmatrix} 0 & \dfrac{1}{2}\dfrac{\partial v_x}{\partial y} & 0 \\ \dfrac{1}{2}\dfrac{\partial v_x}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{d}'$$

Figure 9.5: Flow characterized by a deviatoric strain rate tensor.

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Chapter 10
Fluid Mechanics

10.1 Governing Equations

A fluid is a particular case of continuous medium that is characterized by its specific set of constitutive equations. Consequently, the fluid mechanics problem is defined by the following equations:

a) **Balance Equations**

1) *Mass continuity equation*

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 \qquad \text{(1 equation)} \tag{10.1}$$

2) *Balance of linear momentum*

$$\nabla \cdot \boldsymbol{\sigma} + \rho\, \mathbf{b} = \rho \frac{d\mathbf{v}}{dt} \qquad \text{(3 equations)} \tag{10.2}$$

3) *Energy balance*

$$\rho \frac{du}{dt} = \boldsymbol{\sigma} : \mathbf{d} + \rho\, r - \nabla \cdot \mathbf{q} \qquad \text{(1 equation)} \tag{10.3}$$

4) *Restrictions imposed by the second law of thermodynamics*

$$\begin{array}{ll} \text{Clausius-Planck inequality} & -\rho \left(\dfrac{du}{dt} - \theta \dfrac{ds}{dt} \right) + \boldsymbol{\sigma} : \mathbf{d} \geq 0 \\ \text{Heat conduction inequality} & -\dfrac{1}{\rho \theta^2}\, \mathbf{q} \cdot \nabla \theta \geq 0 \end{array} \tag{10.4}$$

b) **Constitutive Equations**

5) *Thermo-mechanical constitutive equation*

$$\boldsymbol{\sigma} = -p\,\mathbf{1} + \lambda\,\mathrm{Tr}(\mathbf{d})\,\mathbf{1} + 2\mu\,\mathbf{d} \qquad \text{(6 equations)} \tag{10.5}$$

6) *Entropy constitutive equation*

$$s = s(\mathbf{d}, \rho, \theta) \qquad \text{(1 equation)} \tag{10.6}$$

7) *Law of heat conduction*

$$\mathbf{q} = -K\,\nabla\theta \qquad \text{(1 equation)} \tag{10.7}$$

c) **Thermodynamic equations of state**

8) *Caloric equation of state*

$$u = u(\rho, \theta) \qquad \text{(1 equation)} \tag{10.8}$$

9) *Kinetic equation of state*

$$F(\rho, p, \theta) = 0 \qquad \text{(1 equation)} \tag{10.9}$$

The unknowns[1] of these governing equations are

$$\left.\begin{array}{lcl} \rho & \rightarrow & 1 \text{ unknown} \\ \mathbf{v} & \rightarrow & 3 \text{ unknowns} \\ \boldsymbol{\sigma} & \rightarrow & 6 \text{ unknowns} \\ u & \rightarrow & 1 \text{ unknown} \\ \mathbf{q} & \rightarrow & 3 \text{ unknowns} \\ \theta & \rightarrow & 1 \text{ unknown} \\ s & \rightarrow & 1 \text{ unknown} \\ p & \rightarrow & 1 \text{ unknown} \end{array}\right\} \rightarrow 17 \text{ unknowns}\,. \tag{10.10}$$

The system is formed by a total of 17 PDEs and 17 unknowns which, in general, should be solved together, that is, in a coupled form. However, as noted in Section 5.13.1 of Chapter 5, under certain hypotheses or situations a reduced system

[1] Note that the strain rate tensor $\mathbf{d}$ is not considered an unknown since it is an implicit function of the velocity field $\mathbf{v}$.

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of equations, denoted as the mechanical problem, may be posed and solved separately for a reduced number of unknowns (mechanical variables).

Consider the case of a *barotropic fluid*, which is characterized by the fact that the temperature does not intervene in the kinetic equation of state. Then,

$$\begin{array}{c}\text{Kinetic equation}\\ \text{of state}\end{array} \qquad F(\rho, p) = 0 \quad \Longrightarrow \quad \rho = \rho(p) \ , \tag{10.11}$$

which establishes that the density may be described solely by means of the thermodynamic pressure (see Figure 10.1). Assuming, in addition, that the temperature does not intervene in the thermo-mechanical constitutive equation (10.5), the *governing equations of the (uncoupled) mechanical problem in a Newtonian fluid* are defined as

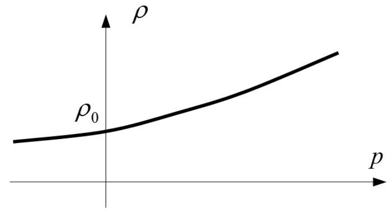

Figure 10.1: Density depends on the thermodynamic pressure in a barotropic fluid.

1) *Mass continuity equation*

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 \qquad \text{(1 equation)} \tag{10.12}$$

2) *Cauchy's equation*

$$\nabla \cdot \boldsymbol{\sigma} + \rho \, \mathbf{b} = \rho \frac{d\mathbf{v}}{dt} \qquad \text{(3 equations)} \tag{10.13}$$

3) *Mechanical constitutive equation*

$$\boldsymbol{\sigma} = -p \, \mathbf{1} + \lambda \operatorname{Tr}(\mathbf{d}) \, \mathbf{1} + 2\mu \, \mathbf{d} \qquad \text{(6 equations)} \tag{10.14}$$

4) *Kinetic equation of state*

$$\rho = \rho(p) \qquad \text{(1 equation)} \tag{10.15}$$

The unknowns of the problem posed by the equations above are

$$\left.\begin{array}{lll} \rho & \rightarrow & 1 \text{ unknown} \\ \mathbf{v} & \rightarrow & 3 \text{ unknowns} \\ \boldsymbol{\sigma} & \rightarrow & 6 \text{ unknowns} \\ p & \rightarrow & 1 \text{ unknown} \end{array}\right\} \rightarrow \ 11 \text{ unknowns} \ . \tag{10.16}$$

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A reduced system of 11 equations and 11 unknowns (mechanical problem) is obtained, which may be solved uncoupled from the rest of the problem (thermal problem).

10.2 Hydrostatics. Fluids at Rest

Consider the following particular cases in terms of a fluid's velocity:

a) Uniform velocity: $\mathbf{v}(\mathbf{x},t) \equiv \mathbf{v}(t)$

In this case, the spatial description of the velocity does not depend on the spatial point being considered and is only a function of time. Therefore,

$$\mathbf{d} = \nabla^S \mathbf{v} = \frac{1}{2}(\mathbf{v}\otimes\nabla + \nabla\otimes\mathbf{v}) = \mathbf{0}\,. \tag{10.17}$$

Then, the constitutive equation (10.14) is reduced to

$$\boldsymbol{\sigma} = -p\,\mathbf{1} + \lambda\,\mathrm{Tr}\,\underbrace{(\mathbf{d})}_{=\mathbf{0}}\,\mathbf{1} + 2\,\mu\,\underbrace{\mathbf{d}}_{=\mathbf{0}} \quad\Longrightarrow\quad \boxed{\boldsymbol{\sigma} = -p\mathbf{1}}\,, \tag{10.18}$$

which indicates that the stress state is hydrostatic (see Figure 10.2). In addition, the mean pressure $\bar{p}$ and the thermodynamic pressure p coincide,

$$\mathrm{Tr}(\boldsymbol{\sigma}) = -3\bar{p} = -3p \quad\Longrightarrow\quad \boxed{\bar{p} = p}\,. \tag{10.19}$$

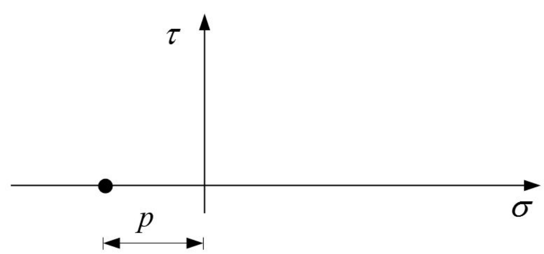

Figure 10.2: Mohr's circle for a fluid with uniform velocity.

b) Uniform and stationary velocity: $\mathbf{v}(\mathbf{x},t) \equiv const.$

A fluid with uniform and stationary velocity is characterized, in addition of (10.17), by

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$$\left.\begin{array}{l} \mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \mathbf{0} \\ \boldsymbol{\sigma} = -p_0\mathbf{1} \quad \Longrightarrow \quad \bar{p} = p = p_0 \end{array}\right\} \begin{array}{c}\text{hydrostatic}\\\text{case}\end{array} \tag{10.20}$$

This is the most general case of *hydrostatics*, which is characterized by a null acceleration (the velocity of each particle is constant, although not necessarily null) and the three pressures (thermodynamic p, mean $\bar{p}$, and hydrostatic p_0) coincide.

c) *Fluid at rest:* $\mathbf{v}(\mathbf{x},t) \equiv const. = \mathbf{0}$
A particular case of hydrostatics is that of a fluid at rest with null velocity.

10.2.1 Hydrostatic Equations

The hydrostatic problem is governed by the following equations:

1) *Constitutive equation*

$$\begin{array}{l} \boldsymbol{\sigma} = -p_0\mathbf{1} \\ \sigma_{ij} = -p_0\,\delta_{ij} \qquad i,j \in \{1,2,3\} \end{array} \tag{10.21}$$

where p_0 is the hydrostatic pressure.

Remark 10.1. Pascal's Principle states that, in a fluid at rest, the pressure is the same in every direction.
This classic fluid mechanics postulate is guaranteed by the spherical structure of the stress tensor in (10.21), which ensures that all directions are principal stress directions (see Figure 10.3).

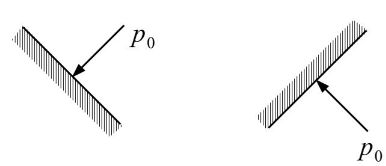

Figure 10.3: Pascal's Principle.

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2) *Mass continuity equation*

$$\left.\begin{array}{l}\dfrac{d\rho}{dt}+\rho\nabla\cdot\mathbf{v}=\mathbf{0}\\ \mathbf{v}=const. \quad\Rightarrow\quad \nabla\cdot\mathbf{v}=0\end{array}\right\}\Rightarrow\quad \frac{d\rho}{dt}=0 \quad\Rightarrow\quad \boxed{\rho\,(\mathbf{X},t)=\rho_0\,(\mathbf{X})=const.} \tag{10.22}$$

and the density of a same particle does not change along time.

3) *Cauchy's equation*

$$\nabla\cdot\boldsymbol{\sigma}+\rho\,\mathbf{b}=\rho\,\frac{d\mathbf{v}}{dt} \tag{10.23}$$

Introducing (10.21) and (10.22) in (10.23),

$$\left\{\begin{array}{l}\nabla\cdot\boldsymbol{\sigma}=\nabla\cdot(-p_0\,\mathbf{1})=-\nabla p_0\\ \left[\nabla\cdot\boldsymbol{\sigma}\right]_j=\dfrac{\partial\sigma_{ij}}{\partial x_i}=\dfrac{\partial}{\partial x_i}(-p_0\,\delta_{ij})=-\dfrac{\partial p_0}{\partial x_j}=-\left[\nabla p_0\right]_j \qquad j\in\{1,2,3\}\end{array}\right. \tag{10.24}$$

$$\boxed{\begin{array}{ll}\text{Fundamental equation} \\ \text{of hydrostatics} & \left\{\begin{array}{l}-\nabla p_0+\rho_0\,\mathbf{b}=\mathbf{0}\\ -\dfrac{\partial p_0}{\partial x_i}+\rho_0\,b_i=0 \qquad i\in\{1,2,3\}\end{array}\right.\end{array}} \tag{10.25}$$

10.2.2 Gravitational Force. Triangular Pressure Distribution

Consider the particular case, which is in fact very common, of the body forces $\mathbf{b}\,(\mathbf{x},t)$ corresponding to the gravitational force (assumed constant in space and along time, and oriented in the negative direction of the x_3-axis, as shown in Figure 10.4).

Since the acceleration is null (see (10.20)) it is a quasi-static problem and, because the actions $\mathbf{b}\,(\mathbf{x},t)\equiv const.$ are independent of time, so are the responses, in particular, the hydrostatic pressure. Consequently,

$$p_0\,(\mathbf{x},t)\equiv p_0\,(\mathbf{x})=p_0\,(x,y,z)\ , \tag{10.26}$$

and (10.25) can be integrated as follows

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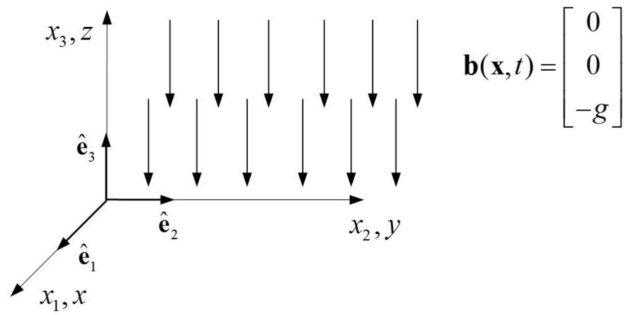

Figure 10.4: Gravitational Force.

$$\begin{cases} -\dfrac{\partial p_0(x,y,z)}{\partial x}=0 & \Longrightarrow \quad p_0(x,y,z)\equiv p_0(y,z) \\ -\dfrac{\partial p_0(y,z)}{\partial y}=0 & \Longrightarrow \quad p_0(y,z)\equiv p_0(z) \\ -\dfrac{\partial p_0(z)}{\partial z}-\rho_0\, g=0 & \Longrightarrow \quad p_0=-\rho_0\, g\, z+C \end{cases} \tag{10.27}$$

For a case such as the one shown in Figure 10.5, in which the surface pressure (height $z=h$) is considered null, the solution (10.26) results in

$$p_0\Big|_{z=h}=0 \;\Rightarrow\; -\rho_0\, g\, h+C=0 \;\Rightarrow\; C=\rho_0\, g\, h \;\Rightarrow\; \boxed{p_0=\rho_0\, g\,(h-z)}\,, \tag{10.28}$$

which corresponds to a triangular pressure distribution, as shown in Figure 10.5.

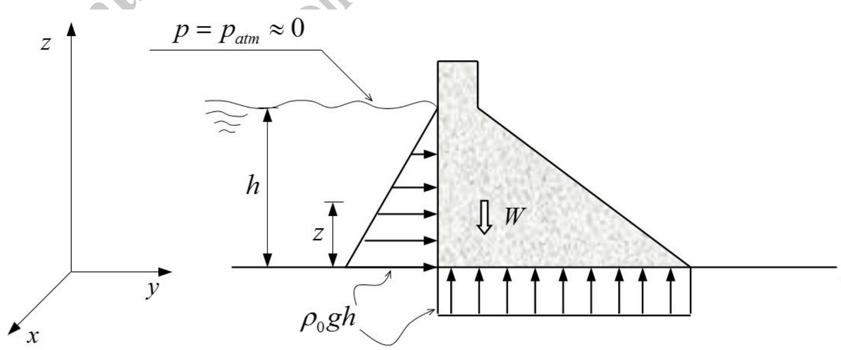

Figure 10.5: Pressure distribution on a gravity dam.

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10.2.3 Archimedes' Principle

Definition 10.1. Archimedes' principle:

1) *The upward buoyant force experienced by a body submerged in a fluid is equal to the weight of the fluid displaced by said body.*

The classical principle is complemented with:

2) *The resultant of the aforementioned buoyant force acts at the center of gravity of the volume of the displaced fluid.*

To prove Archimedes' principle, consider the situations in Figure 10.6. On the one hand, Figure 10.6 *a)* illustrates a solid with volume V and density ρ in the interior of a fluid of density ρ_0. The solid *is not necessarily in equilibrium*, even though its velocity and acceleration are assumed to be small enough to ensure a hydrostatic state in the fluid. On the other hand, Figure 10.6 *b)* shows the same fluid without the solid, such that the volume occupied by said solid in Figure 10.6 *a)* is occupied here by an identical volume of fluid.

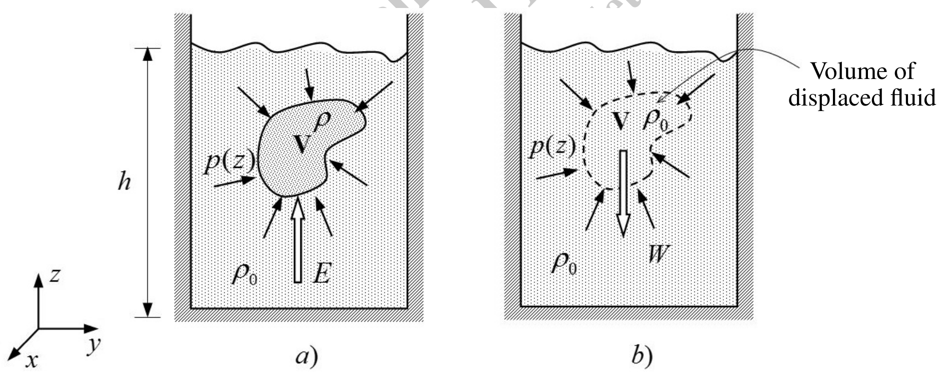

Figure 10.6: *a)* Solid submerged in a fluid and *b)* volume of the displaced fluid.

a) Pressure and stress distributions in the fluid

Using the fundamental equation of hydrostatics (10.25) and considering that the gravitational forces act in the negative direction of the z-axis, the situation corresponding with (10.26) and (10.27) is achieved. Thus, the result (10.28) is valid for both cases *a)* and *b)* of Figure 10.6.

$$\boxed{\begin{array}{l} p_0(z) = \rho_0 \, g \, (h-z) \\ \boldsymbol{\sigma} = -p_0 \, \mathbf{1} \end{array}} \tag{10.29}$$

Note that the hydrostatic pressure and the stress state in the fluid are the same for equivalent points of the fluid in the cases *a)* and *b)* of Figure 10.6.

b) Buoyant force on the submerged solid

The traction vector on the boundary of the submerged solid in Figure 10.6 *a)* is

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} = -p_0 \, \mathbf{1} \cdot \mathbf{n} = -p_0 \, \mathbf{n} \tag{10.30}$$

and the resultant $\mathbf{R}$ of the forces the fluid exerts on the solid is

$$\mathbf{R} = \int\limits_{\partial V} \mathbf{t} \, dS = \int\limits_{\partial V} -p_0 \, \mathbf{n} \, dS \, . \tag{10.31}$$

Note now that, since the hydrostatic pressure distribution is the same in both cases of Figure 10.6, this resultant is the same as the one obtained in case *b)* for the forces that the rest of the fluid exerts *on the volume of displaced fluid*, with the particularity that, because the pressure distribution is constant in space (with value p_0), the Divergence Theorem (Stokes' Theorem) can be applied on (10.30), resulting in

$$\mathbf{R} = \int\limits_{\partial V} -p_0 \, \mathbf{n} \, dS = \int\limits_{V} -\nabla p_0 \, dV \, . \tag{10.32}$$

Introducing (10.25) in (10.32) yields

$$\mathbf{R} = \int\limits_{V} -\nabla p_0 \, dV = \int\limits_{V} -\rho_0 \, \mathbf{b} \, dV = \underbrace{-\int\limits_{V} \rho_0 \, \mathbf{b} \, dV}_{W \, \hat{\mathbf{e}}_z} = W \, \hat{\mathbf{e}}_z = E \, \hat{\mathbf{e}}_z \, , \tag{10.33}$$

where E is the upward buoyant force acting on the submerged solid and W is the weight of the displaced fluid (see Figure 10.6 *b)*). That is,

$$\underbrace{\text{upward buoyant force}}_{E} = \underbrace{\text{weight of the displaced fluid}}_{W} \, , \tag{10.34}$$

whereby the first part of Archimedes' Principle is proven.

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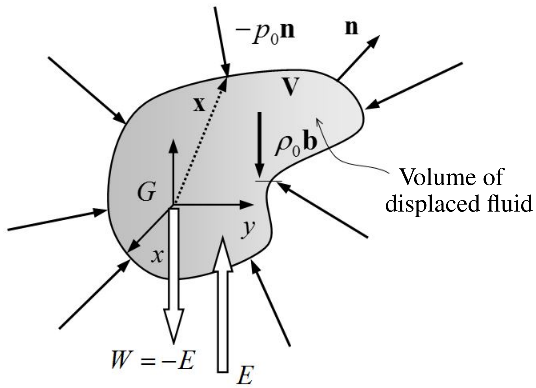


Figure 10.7: Forces acting on the volume of displaced fluid.

c) Vertical line of application of the upward buoyant force

Consider now the moment $\mathbf{M}_E^G$ of the upward buoyant force E with respect to the center of gravity, G, of the volume of displaced fluid (see Figure 10.7[2]),

$$\left\{\begin{aligned}
\mathbf{M}_E^G \quad &= \int_{\partial V} \mathbf{x}\times(-p_0\,\mathbf{n})\,dS \overset{\substack{\text{Divergence}\\ \text{Theorem}}}{=} \int_V \mathbf{x}\times(-p_0\nabla)\,dV = -\int_V \mathbf{x}\times\nabla p_0\,dV \\
\left[\mathbf{M}_E^G\right]_i &= -\int_{\partial V} e_{ijk}\,x_j\,p_0\,n_k\,dS = -\int_V \frac{\partial}{\partial x_k}\left(e_{ijk}\,x_j\,p_0\right)dV = \\
&= -\int_V \underbrace{e_{ijk}\frac{\partial x_j}{\partial x_k}}_{\substack{e_{ijk}\delta_{jk}= \\ e_{ijj}=0}}\,p_0\,dV - \int_V e_{ijk}\,x_j\frac{\partial p_0}{\partial x_k}\,dV = -\int_V e_{ijk}\,x_j\frac{\partial p_0}{\partial x_k}\,dV \\
&\qquad\qquad i\in\{1,2,3\}
\end{aligned}\right. \tag{10.35}$$

and replacing the fundamental equation of hydrostatics (10.25) in (10.35) finally yields

$$\mathbf{M}_E^G = -\int_V (\mathbf{x}\times\nabla p_0)\,dV = -\underbrace{\int_V (\mathbf{x}\times\rho_0\,\mathbf{b})\,dV}_{\mathbf{M}_W^G} = -\mathbf{M}_W^G = \mathbf{0}\,, \tag{10.36}$$

[2] Without loss of generality, the origin of the system of Cartesian axes can be placed at G.

where $\mathbf{M}_W^G$ is the moment of the weight of the displaced fluid with respect to its center of gravity G, which, considering the definition of the center of gravity, is null. Consequently, the moment of the upward buoyant force E *with respect to the center of gravity of the volume of displaced fluid* is also null. Then, it is concluded that the vertical line of application of the upward buoyant force crosses said center of gravity, as established by the second part of Archimedes' principle.

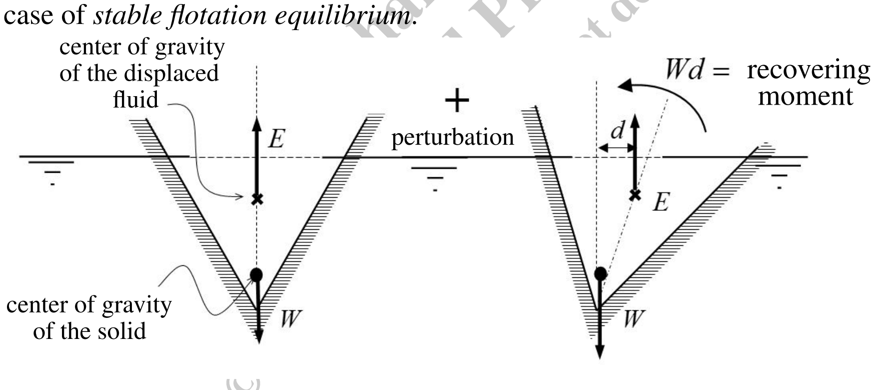
Example 10.1 – *Apply Archimedes' principle to the study of stability of the equilibrium in floating solids to determine how the relative position of the centers of gravity of the solid and the corresponding volume of displaced fluid affect the nature of this equilibrium.*

Solution

Consider a floating medium, in equilibrium, and the following two situations:

a) The center of gravity of the solid (center of thrust) is below the center of gravity of the displaced fluid (center of buoyancy).

In this case, any perturbation (inclination) tends to create a moment $M = Wd$ in the sense that tends to recover the initial state of equilibrium. It is, thus, a case of *stable flotation equilibrium*.



b) The center of gravity of the solid (center of thrust) is above the center of gravity of the displaced fluid (center of buoyancy).

In this case, any perturbation (inclination) tends to create a moment $M = Wd$ in the sense that tends to capsize the floating solid, that is, it tends to move the solid further away from the initial state of equilibrium. It is, thus, a case of *unstable flotation equilibrium*.

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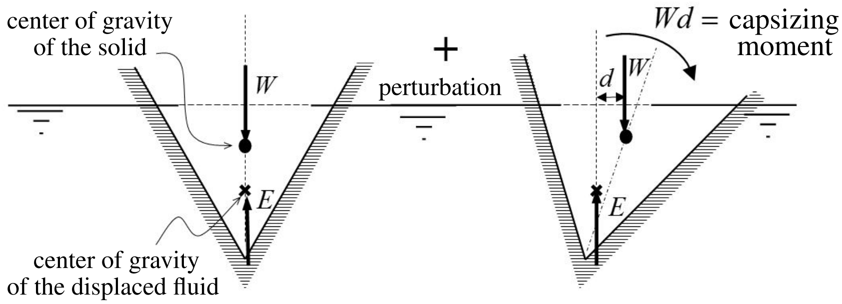

Placing weights (*ballasts*) on the keel of a boat responds to the search of improved flotation stability of this boat.

10.3 Fluid Dynamics: Barotropic Perfect Fluids

In the most common case, the velocity is not uniform nor stationary ($\mathbf{v} \equiv \mathbf{v}(\mathbf{x},t)$), and, therefore, in general, the acceleration will not be null ($\mathbf{a}(\mathbf{x},t) \neq \mathbf{0}$). In consequence, the divergence of the velocity ($\nabla \cdot \mathbf{v} \neq 0$) and the gradient of the velocity $\left(\nabla \otimes \mathbf{v} \overset{not}{=} \nabla \mathbf{v} \neq \mathbf{0}\right)$ will not be null either.

Definition 10.2. A *perfect fluid* is a Newtonian fluid characterized by the fact that the viscosities λ and μ (see (10.14)) are null.

The mechanical constitutive equation (10.14) of a perfect fluid becomes

$$\left.\begin{array}{l} \boldsymbol{\sigma} = -p\,\mathbf{1} + \lambda\,\mathrm{Tr}(\mathbf{d}) + 2\mu\,\mathbf{d} \\ \lambda = \mu = 0 \end{array}\right\} \Longrightarrow \boxed{\boldsymbol{\sigma} = -p\,\mathbf{1}}$$

$$\Longrightarrow \left\{\begin{array}{l} \nabla \cdot \boldsymbol{\sigma} = -\nabla p \\ \boldsymbol{\sigma} : \mathbf{d} = -p\,\mathbf{1} : \mathbf{d} = -p\,\mathrm{Tr}(\mathbf{d}) \end{array}\right. \tag{10.37}$$

which results in a hydrostatic stress state[3].

[3] A hydrostatic stress state (the stress tensor is spherical) should not be confused with a hydrostatic motion regime (the velocity is uniform or null).

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Definition 10.3. A *barotropic fluid* is characterized by a kinetic equation of state (10.9) in which the temperature does not intervene.

$$F(\rho,p,\theta) \equiv F(\rho,p) = 0 \quad \Longrightarrow \quad \rho = \rho(p)$$

10.3.1 Equations of the Problems

Taking into account the hypotheses of a perfect and a barotropic fluid, the equations governing a fluid dynamics problem are reduced to:

a) **Mechanical problem**

1) *Mass continuity equation*

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0 \qquad \text{(1 equation)} \qquad (10.38)$$

2) *Balance of linear momentum* (Euler's equation)

$$-\nabla p + \rho\mathbf{b} = \rho\frac{d\mathbf{v}}{dt} \qquad \text{(3 equations)} \qquad (10.39)$$

3) *Kinetic equation of state*

$$\rho = \rho(p) \qquad \text{(1 equation)} \qquad (10.40)$$

The mechanical problem is composed of 5 equations and 5 unknowns $\big(\rho(\mathbf{x},t), \mathbf{v}(\mathbf{x},t), p(\mathbf{x},t)\big)$ that can be solved uncoupled from the thermal problem.

b) **Thermal problem**

1) *Fourier's law*

$$\mathbf{q} = -K\nabla\theta \;\Rightarrow\; \nabla\cdot\mathbf{q} = -K\nabla\cdot(\nabla\theta) = -K\nabla^2\theta \qquad \text{(3 equations)} \qquad (10.41)$$

2) *Energy balance*

$$\rho\frac{du}{dt} = \underbrace{-p\nabla\cdot\mathbf{v}}_{\boldsymbol{\sigma}:\mathbf{d}} + \rho r + \underbrace{K\nabla^2\theta}_{-\nabla\cdot\mathbf{q}} \qquad \text{(1 equation)} \qquad (10.42)$$

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3) *Caloric equation of state*

$$u = u(\rho, \theta) \qquad \text{(1 equation)} \qquad (10.43)$$

The thermal problem is defined by 5 equations and 5 unknowns $\big(\mathbf{q}(\mathbf{x},t),\ \theta(\mathbf{x},t),\ u(\mathbf{x},t)\big)$ and can be solved *once the mechanical problem has been solved* and the velocity field $\mathbf{v}(\mathbf{x},t)$, the density $\rho(\mathbf{x},t)$ and the pressure $p(\mathbf{x},t)$ are known.

Remark 10.2. A general format of the fluid mechanics problem includes the thermal conductivity K between the viscosities (in a generalized sense) of the problem. The definition of a *perfect fluid* as a *fluid without viscosity* results, in this context, in the cancellation of the thermal conductivity $(K = 0)$, therefore (10.41) leads to $\mathbf{q} = -K\nabla\theta = \mathbf{0}$ and the thermal problem is reduced to the equations (10.42) and (10.43).

10.3.2 Resolution of the Mechanical Problem under Potential Body Forces. Bernoulli's Trinomial

Consider now the mechanical problem for the particular case of *potential body forces* (the body forces derive from a potential ϕ),

$$\boxed{\text{Potential body forces:} \quad \mathbf{b}(\mathbf{x},t) = -\nabla\phi(\mathbf{x},t)} \,. \qquad (10.44)$$

In the particular case of a gravitational potential with the line of action along the negative direction of the z-axis, the potential is

$$\boxed{\phi(x,y,z,t) = g\,z \quad \Longrightarrow \quad \mathbf{b} = -\nabla\phi \stackrel{not}{\equiv} \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}} \,. \qquad (10.45)$$

Lemma 10.1. *For a barotropic fluid* $(\rho = \rho\,(p))$ *there exists a function* $\mathbb{P}(\mathbf{x},t) = \hat{\mathbb{P}}(p(\mathbf{x},t))$ *that satisfies*

$$\nabla p = \rho\,\nabla\mathbb{P} \,.$$

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Proof
Defining the function $\mathbb{P}(\mathbf{x},t)$ as

$$\mathbb{P}(\mathbf{x},t) = \hat{\mathbb{P}}(p(\mathbf{x},t)) = \int_0^p \frac{1}{\rho(\bar{p})} d\bar{p} \,, \tag{10.46}$$

then, it will satisfy

$$\begin{cases} \dfrac{\partial \mathbb{P}(\mathbf{x},t)}{\partial x_i} = \dfrac{\partial \hat{\mathbb{P}}}{\partial p} \dfrac{\partial p}{\partial x_i} \\ [\nabla \mathbb{P}]_i = \dfrac{\partial \hat{\mathbb{P}}}{\partial p} [\nabla p]_i = \dfrac{1}{\rho(p)} [\nabla p]_i \qquad i \in \{1,2,3\} \end{cases} \tag{10.47}$$

leading to

$$\boxed{\nabla \mathbb{P} = \frac{1}{\rho(p)} \nabla p} \,. \tag{10.48}$$

> **Lemma 10.2.** *The convective term of the acceleration can be written as*
> $$\mathbf{v} \cdot \nabla \mathbf{v} = 2\boldsymbol{\omega} \times \mathbf{v} + \nabla \left(\frac{1}{2} \mathbf{v}^2 \right),$$
> *where* $2\boldsymbol{\omega} = \nabla \times \mathbf{v}$ *is the vorticity vector.*

Proof
Expanding the right-hand term in the Lemma[4],

$$\begin{aligned} [\mathbf{v} \cdot \nabla \mathbf{v}]_j &= \mathrm{v}_i \frac{\partial \mathrm{v}_j}{\partial x_i} = \mathrm{v}_i \underbrace{\left(\frac{\partial \mathrm{v}_j}{\partial x_i} - \frac{\partial \mathrm{v}_i}{\partial x_j} \right)}_{2 w_{ji}} + \mathrm{v}_i \frac{\partial \mathrm{v}_i}{\partial x_j} = 2\,\mathrm{v}_i \underbrace{w_{ji}}_{-w_{ij}} + \mathrm{v}_i \frac{\partial \mathrm{v}_i}{\partial x_j} = \\ &= -2\,\mathrm{v}_i w_{ij} + \mathrm{v}_i \frac{\partial \mathrm{v}_i}{\partial x_j} = 2 \underbrace{e_{ijk}\, \mathrm{v}_i\, \omega_k}_{e_{jki}\, \mathrm{v}_i\, \omega_k} + \mathrm{v}_i \frac{\partial \mathrm{v}_i}{\partial x_j} = \\ &= \underbrace{2\, e_{jki}\, \mathrm{v}_i\, \omega_k}_{[2\boldsymbol{\omega} \times \mathbf{v}]_j} + \frac{\partial}{\partial x_j} \left(\frac{1}{2} \underbrace{\mathrm{v}_i \mathrm{v}_i}_{\mathbf{v} \cdot \mathbf{v} = \mathrm{v}^2} \right) = [2\boldsymbol{\omega} \times \mathbf{v}]_j + \left[\nabla \left(\frac{1}{2} \mathrm{v}^2 \right) \right]_j \,, \\ & \qquad \qquad j \in \{1,2,3\} \end{aligned} \tag{10.49}$$

[4] The following results, previously obtained in Chapter 2, are used here:
$w_{ji} = -w_{ij} = [\nabla^a \mathbf{v}]_{ji} = (\partial \mathrm{v}_j / \partial x_i - \partial \mathrm{v}_i / \partial x_j)/2$, $w_{ij} = -e_{ijk}\omega_k$ and $\mathrm{v}^2 = |\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$.

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which leads to

$$\boxed{\mathbf{v}\cdot\nabla\mathbf{v} = 2\boldsymbol{\omega}\times\mathbf{v} + \nabla\left(\frac{1}{2}\mathrm{v}^2\right)} \;. \tag{10.50}$$

Rearranging now Euler's equation (10.39),

$$-\nabla p + \rho\mathbf{b} = \rho\frac{d\mathbf{v}}{dt} \qquad \Longrightarrow \qquad -\frac{1}{\rho}\nabla p + \mathbf{b} = \frac{d\mathbf{v}}{dt}\,, \tag{10.51}$$

and replacing (10.45) and (10.48) in (10.51) produces

$$-\nabla\mathbb{P} - \nabla\phi = \frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \frac{\partial\mathbf{v}}{\partial t} + 2\boldsymbol{\omega}\times\mathbf{v} + \nabla\left(\frac{1}{2}\mathrm{v}^2\right)\,, \tag{10.52}$$

where the result (10.50) has been taken into account. Finally, (10.52) is rewritten as

$$-\left(\nabla\mathbb{P} + \nabla\phi + \nabla\left(\frac{1}{2}\mathrm{v}^2\right)\right) = \frac{\partial\mathbf{v}}{\partial t} + 2\boldsymbol{\omega}\times\mathbf{v}\,. \tag{10.53}$$

Equation of motion of a barotropic perfect fluid under potential body forces

$$-\nabla\underbrace{\left(\mathbb{P} + \phi + \frac{1}{2}\mathrm{v}^2\right)}_{\text{Bernoulli's trinomial}} = \frac{\partial\mathbf{v}}{\partial t} + 2\boldsymbol{\omega}\times\mathbf{v} \tag{10.54}$$

Equation (10.54) is the particular form adopted by the balance of linear momentum (Euler's equation (10.39)) in barotropic perfect fluids subjected to potential body forces.

10.3.3 Solution in a Steady-State Regime

The solution to the mechanical problem defined by (10.38) to (10.40) has, in general, a *transient regime*, in which the spatial description of the mechanical variables evolves along time, and a *steady-state regime*, in which said spatial description is, approximately, constant along time (see Figure 10.8).

Consider now the equation of motion (10.54) in a *steady-state regime*,

$$\frac{\partial\mathbf{v}}{\partial t} = \mathbf{0} \qquad \Longrightarrow \qquad -\nabla\left(\mathbb{P} + \phi + \frac{1}{2}\mathrm{v}^2\right) = 2\boldsymbol{\omega}\times\mathbf{v}\,, \tag{10.55}$$

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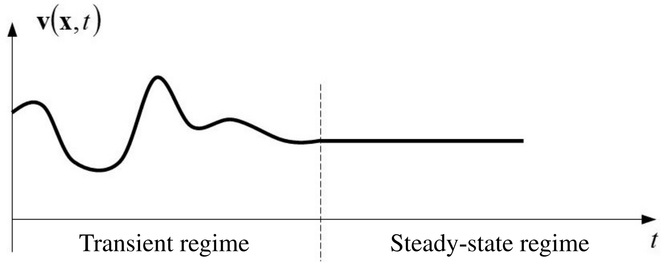

Figure 10.8: Transient and steady-state regimes.

and a streamline[5] $\Gamma : \ \mathbf{x} = \mathbf{x}(s)$ parametrized in terms of its arc-length s (see Figure 10.9). Projecting (multiplying) equation (10.53) in the direction tangent to the streamline, $\mathbf{t}$, results in

$$-\underbrace{\nabla\left(\mathbb{P}+\phi+\frac{1}{2}\mathrm{v}^2\right)}_{M(\mathbf{x})} = 2\boldsymbol{\omega}\times\mathbf{v} \implies -(\nabla M)\cdot\underbrace{\mathbf{v}}_{\frac{d\mathbf{x}}{ds}\frac{ds}{dt}} = \underbrace{(2\boldsymbol{\omega}\times\mathbf{v})\cdot\mathbf{v}}_{=\mathbf{0}} = \mathbf{0} \quad (10.56)$$

$$\left.\begin{aligned} \nabla M(\mathbf{x}(s))\cdot\frac{d\mathbf{x}}{ds} = \frac{dM}{ds} = 0 \\ \frac{\partial M(\mathbf{x}(s))}{\partial x_i}\frac{dx_i}{ds} = \frac{dM}{ds} = 0 \end{aligned}\right\} \ \forall \mathbf{x}\in\Gamma \implies \boxed{M(\mathbf{x}) = const. \quad \forall \mathbf{x}\in\Gamma} \quad (10.57)$$

and (10.57) is written as

$$\boxed{\left[\mathbb{P}+\phi+\frac{1}{2}\mathrm{v}^2\right](\mathbf{x}) = const. \quad \forall \mathbf{x}\in\Gamma}\ , \quad (10.58)$$

which establishes that *Bernoulli's trinomial remains constant along a same streamline Γ.*

> *Remark 10.3.* Note that (10.58) is no longer a partial differential equation but a (scalar) algebraic equation, already integrated. This equation allows, thus, determining one of the unknowns of the mechanical problem once the others are known.

[5] In a steady-state (stationary) regime, trajectories and streamlines coincide.

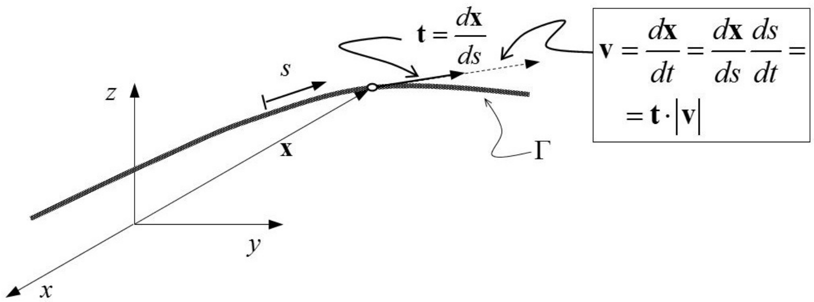

Figure 10.9: Parametrized streamline.

10.3.3.1 Solution in Steady-State Regime for an Incompressible Fluid under Gravitational Forces

Consider now the particular case of a barotropic fluid with the following characteristics:

a) **The fluid is *incompressible***

$$\rho = \rho(p) = \rho_0 = const. \tag{10.59}$$

In this case, the function $\mathbb{P}(p)$ in (10.46) can be integrated as follows.

$$\mathbb{P}(\mathbf{x},t) = \int_0^p \frac{1}{\rho(\bar{p})} d\bar{p} = \frac{1}{\rho_0} \int_0^p d\bar{p} = \frac{p}{\rho_0} \tag{10.60}$$

b) **The body forces are *gravitational***

In accordance with (10.45),

$$\phi = g\,z \qquad\qquad \mathbf{b} = -\nabla\phi \overset{not}{\equiv} \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} \tag{10.61}$$

Introducing (10.60) and (10.61) in Bernouilli's trinomial (10.58) yields

$$\frac{p}{\rho_0} + g\,z + \frac{1}{2}\mathrm{v}^2 = const. \implies z + \frac{p}{\rho_0\,g} + \frac{1}{2}\frac{\mathrm{v}^2}{g} \overset{def}{=} H = const. \qquad \forall \mathbf{x} \in \Gamma \tag{10.62}$$

The terms in (10.61) have dimensions of length (height) and may be interpreted in the following manner.

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Bernoulli's theorem

$$\underbrace{z}_{\text{elevation}} + \underbrace{\frac{p}{\rho_0\, g}}_{\substack{\text{pressure}\\ \text{head}}} + \underbrace{\frac{1}{2}\frac{\mathrm{v}^2}{g}}_{\substack{\text{velocity}\\ \text{head}}} \stackrel{def}{=} \underbrace{H}_{\substack{\text{total}\\ \text{height}}} = const. \qquad \forall \mathbf{x} \in \Gamma \tag{10.63}$$

Remark 10.4. The expression in (10.63) constitutes the so-called **Bernoulli's theorem** (for an incompressible perfect fluid under gravitational forces and in steady-state regime), which establishes that *the sum of the elevation, the pressure head and the velocity head* is constant *in every point belonging to a same streamline* (see Figure 10.10).

Remark 10.5. In engineering, water is generally considered an incompressible and perfect fluid, and the science that studies it is named *hydraulics*. Since, in general, the body forces are of the gravitational type, Bernoulli's Theorem is generally applicable in the *resolution of steady-state problems in hydraulics*.

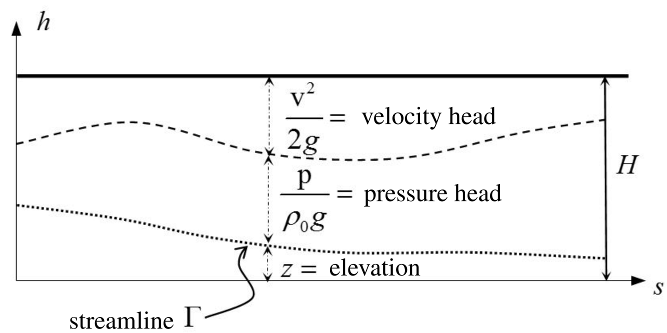

Figure 10.10: Physical interpretation of Benoulli's theorem.

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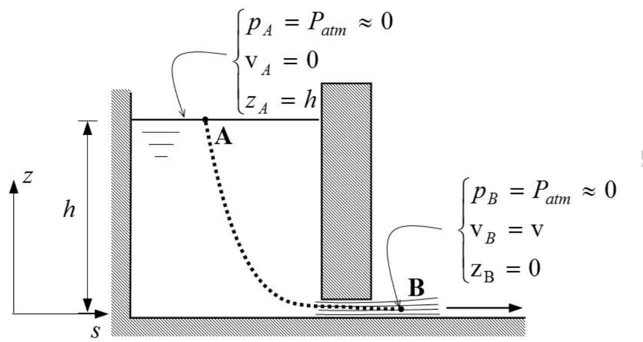
Example 10.2 – *Determine the velocity of the water exiting the tank through a small lateral hole placed at a distance h below the top surface of the water. Consider the top of the tank is open and neglect the atmospheric pressure. Assume a steady-state regime.*



Solution

The fluid in this problem (water) is an incompressible perfect fluid in steady-state regime under gravitational forces and, thus, Bernoulli's theorem can be applied.
Consider a streamline originating at point **A** of the water surface and ending at point **B** of the exit hole (shown in the figure above). Applying Bernoulli's theorem between points **A** and **B**, and taking into account that the velocity of the free surface in the tank is practically null and that its cross-section is much larger than that of the exit hole, then

$$\underbrace{z_A}_{=h} + \underbrace{\frac{p_A}{\rho_0 g}}_{=0} + \underbrace{\frac{1}{2}\frac{v_A^2}{g}}_{=0} = \underbrace{z_B}_{=0} + \underbrace{\frac{p_B}{\rho_0 g}}_{=0} + \frac{1}{2}\frac{v_B^2}{g}$$

$$h + 0 + 0 = 0 + 0 + \frac{1}{2}\frac{v^2}{g} \quad \Longrightarrow \quad \boxed{v = \sqrt{2gh}} \, .$$

10.3.4 Solution in Transient Regime

In a transient regime, the mechanical variables (in their spatial description) are time-dependent (see Figure 10.8). The starting point to solve the problem is the balance of linear momentum (10.54),

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$$\boxed{-\nabla\left(\mathbb{P}+\phi+\frac{1}{2}\mathrm{v}^2\right)=\frac{\partial\mathbf{v}}{\partial t}+2\boldsymbol{\omega}\times\mathbf{v}\ .} \tag{10.64}$$

In some cases, the solution to this equation in transient regime is particularly simple. In the following subsections, several of these cases will be studied.

10.3.4.1 (Irrotational) Potential Flow

Consider the case of
- a perfect fluid
- with potential body forces
- and irrotational flow.

> **Definition 10.4.** The motion (or flow) of a fluid is said to be *irrotational* (or potential) if the rotational of the velocity field is null at any point of this fluid.

In other words, an irrotational flow has a null vorticity vector.

$$\boxed{\text{Irrotational flow}\quad \begin{cases}\nabla\times\mathbf{v}(\mathbf{x},t)=\mathbf{0}\\ \boldsymbol{\omega}(\mathbf{x},t)=\dfrac{1}{2}\nabla\times\mathbf{v}(\mathbf{x},t)=\mathbf{0}\end{cases}\quad \forall\mathbf{x}\ \forall t} \tag{10.65}$$

If the flow is irrotational, it is inferred from (10.65) that there exists a scalar function (denoted as velocity potential $\chi(\mathbf{x},t)$) that satisfies[6]

$$\mathbf{v}(\mathbf{x},t)=\nabla\chi(\mathbf{x},t)\ . \tag{10.66}$$

Note that, in this case, the *vector* field $\mathbf{v}(\mathbf{x},t)$ is determined in terms of the *scalar* velocity potential $\chi(\mathbf{x},t)$ (which becomes the main unknown of the problem). Replacing the conditions (10.65) and (10.66) in (10.64) yields

$$-\nabla\left(\mathbb{P}+\phi+\frac{1}{2}\mathrm{v}^2\right)=\frac{\partial\mathbf{v}}{\partial t}+2\underbrace{\boldsymbol{\omega}}_{=\mathbf{0}}\times\mathbf{v}=\frac{\partial\mathbf{v}}{\partial t}=\frac{\partial}{\partial t}\Big(\nabla\chi(\mathbf{x},t)\Big)=\nabla\left(\frac{\partial\chi}{\partial t}\right)\implies \tag{10.67}$$

[6] It can be proven that, given an irrotational vector field $\mathbf{v}(\mathbf{x},t)$, that is, a vector field that satisfies $\nabla\times\mathbf{v}=\mathbf{0}$, there exists a scalar function $\chi(\mathbf{x},t)$ (potential function) such that $\mathbf{v}=\nabla\chi(\mathbf{x},t)$. Obviously, since $\nabla\times\nabla(\bullet)\equiv\mathbf{0}$, then $\nabla\times\mathbf{v}=\nabla\times\nabla\chi(\mathbf{x},t)=\mathbf{0}$ is satisfied.

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$$\boxed{\begin{array}{ll} \nabla \underbrace{\left(\mathbb{P} + \phi + \frac{1}{2}\mathrm{v}^2 + \frac{\partial \chi}{\partial t} \right)}_{M(\mathbf{x},t)} = \nabla M(\mathbf{x},t) = \mathbf{0} & \\ & \forall \mathbf{x} \ \forall t \\ \dfrac{\partial M(\mathbf{x},t)}{\partial x_i} = 0 \qquad i \in \{1,2,3\} & \end{array}} \tag{10.68}$$

This equation can be trivially integrated, resulting in

$$M(\mathbf{x},t) = \mathbb{P} + \phi + \frac{1}{2}\mathrm{v}^2 + \frac{\partial \chi}{\partial t} = \varphi(t) \ . \tag{10.69}$$

Defining a modified velocity potential $\bar{\chi}(\mathbf{x},t)$ of the form

$$\bar{\chi}(\mathbf{x},t) \overset{def}{=} \chi(\mathbf{x},t) - \int_0^t \varphi(\tau)\, d\tau \quad \Rightarrow \quad \begin{cases} \nabla \bar{\chi} = \nabla \chi = \mathbf{v}(\mathbf{x},t) \\ \dfrac{\partial \bar{\chi}}{\partial t} = \dfrac{\partial \chi}{\partial t} - \varphi(t) \end{cases} \tag{10.70}$$

and replacing (10.70) in (10.69) produces

$$\mathbb{P} + \phi + \frac{1}{2}\mathrm{v}^2 + \underbrace{\frac{\partial \chi}{\partial t} - \varphi(t)}_{\frac{\partial \bar{\chi}}{\partial t}} = 0 \implies \boxed{\mathbb{P} + \phi + \frac{1}{2}(\nabla \bar{\chi})^2 + \frac{\partial \bar{\chi}}{\partial t} = 0 \quad \forall \mathbf{x} \ \forall t} \ , \tag{10.71}$$

which is the differential equation of *hydraulic transients*.

The mechanical problem is then defined by:

1) *Mass continuity equation*

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = \frac{d\rho}{dt} + \rho \underbrace{\nabla \cdot (\nabla \bar{\chi})}_{\nabla^2 \bar{\chi}} = 0 \quad \Longrightarrow \quad \boxed{\frac{d\rho}{dt} + \rho \nabla^2 \bar{\chi} = 0} \tag{10.72}$$

2) *Balance of linear momentum (hydraulic transients equation)*

$$\boxed{\mathbb{P}(\rho, p) + \phi + \frac{1}{2}(\nabla \bar{\chi})^2 + \frac{\partial \bar{\chi}}{\partial t} = 0 \qquad \forall \mathbf{x} \ \forall t} \tag{10.73}$$

3) *Kinetic equation of state*

$$\rho = \rho(p) \tag{10.74}$$

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These constitute a system of 3 scalar equations and 3 unknowns ($p(\mathbf{x},t)$, $\rho(\mathbf{x},t)$ and $\bar{\chi}(\mathbf{x},t)$) that can be integrated in the $\mathbb{R}^3 \times \mathbb{R}^+$ domain. Once the potential $\bar{\chi}(\mathbf{x},t)$ is known, the velocity field is obtained through

$$\mathbf{v}(\mathbf{x},t) = \nabla \bar{\chi}(\mathbf{x},t) \ . \tag{10.75}$$

10.3.4.2 Incompressible and Potential Flow

Consider the case of
- a perfect fluid
- with potential body forces,
- irrotational flow
- and incompressible flow.

Since the flow is incompressible, (10.46) and (10.72) allow determining[7]

$$\frac{d\rho}{dt} = 0 \quad \Longrightarrow \quad \rho = \rho_0 \quad \Longrightarrow \quad \begin{cases} \mathbb{P}(p) = \int\limits_0^p \dfrac{1}{\rho(\bar{p})} d\bar{p} = \dfrac{p}{\rho_0} \\ \nabla^2 \bar{\chi} \overset{not}{=} \Delta \bar{\chi} = 0 \end{cases} \tag{10.76}$$

and the mechanical problem (10.72) to (10.74) is reduced to:

1) *Mass continuity equation*

$$\boxed{\Delta \bar{\chi} = \frac{\partial^2 \bar{\chi}}{\partial x_i \partial x_i} = 0} \tag{10.77}$$

2) *Balance of linear momentum (hydraulic transients equation)*

$$\boxed{\frac{p}{\rho_0} + \phi + \frac{1}{2}(\nabla \bar{\chi})^2 + \frac{\partial \bar{\chi}}{\partial t} = 0 \qquad \forall \mathbf{x} \ \forall t} \tag{10.78}$$

These constitute a system of *2 scalar equations* and 2 unknowns ($p(\mathbf{x},t)$ and $\bar{\chi}(\mathbf{x},t)$) that can be integrated in the $\mathbb{R}^3 \times \mathbb{R}^+$ domain. In a steady-state regime, the term $\partial \bar{\chi}/\partial t = 0$ and any time derivative in the system disappears, such that the problem can be integrated in $\mathbb{R}^3$.

[7] Here, the differential operator named Laplace operator or *Laplacian of* $(\bullet)$ is defined as $\Delta(\bullet) = \nabla \cdot \nabla(\bullet) \overset{not}{=} \nabla^2(\bullet) = \partial^2(\bullet)/\partial x_i \partial x_i$.

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10.4 Fluid Dynamics: (Newtonian) Viscous Fluids

Consider now the general problem described by (10.1) to (10.9),

$\dfrac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0$	Mass continuity equation	(1 eqn.)
$\nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\dfrac{d\mathbf{v}}{dt}$	Balance of linear momentum	(3 eqns.)
$\rho\dfrac{du}{dt} = \boldsymbol{\sigma}:\mathbf{d} + \rho\, r - \nabla\cdot\mathbf{q}$	Energy balance	(1 eqn.)
$\boldsymbol{\sigma} = -p\mathbf{1} + \lambda\,\mathrm{Tr}(\mathbf{d})\,\mathbf{1} + 2\mu\,\mathbf{d}$	Mechanical constitutive equation	(6 eqns.)
$s = s(\mathbf{d},\theta,\rho)$	Entropy constitutive equation	(1 eqn.)
$\mathbf{q} = -K\nabla\theta$	Heat conduction equation	(3 eqns.)
$u = u(\rho,\theta)$	Caloric equation of state	(1 eqn.)
$F(\rho,p,\theta) = 0$	Kinetic equation of state	(1 eqn.)

(10.79)

which constitute a system of 17 equations and 17 unknowns. This system is too large to be treated efficiently and a reduced system of equations that allows a simpler resolution will be sought.

10.4.1 Navier-Stokes Equation

The Navier-Stokes equation is essentially the balance of linear momentum of (10.79) expressed solely in terms of the velocity field $\mathbf{v}(\mathbf{x},t)$ and the pressure $p(\mathbf{x},t)$.

Lemma 10.3. *The divergence of the strain rate tensor* $\mathbf{d}(\mathbf{x},t)$ *is related to the velocity field* $\mathbf{v}(\mathbf{x},t)$ *by*

$$\nabla\cdot\mathbf{d} = \frac{1}{2}\Delta\mathbf{v} + \frac{1}{2}\nabla(\nabla\cdot\mathbf{v}) \ .$$

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Proof

$$
\begin{aligned}
\left[\nabla\cdot\mathbf{d}\right]_j &= \frac{\partial}{\partial x_i}d_{ij} = \frac{\partial}{\partial x_i}\left(\frac{1}{2}\left(\frac{\partial \mathrm{v}_i}{\partial x_j}+\frac{\partial \mathrm{v}_j}{\partial x_i}\right)\right) = \frac{1}{2}\frac{\partial^2 \mathrm{v}_i}{\partial x_i \partial x_j}+\frac{1}{2}\frac{\partial^2 \mathrm{v}_j}{\partial x_i \partial x_i} = \\
&= \frac{1}{2}\frac{\partial}{\partial x_j}\underbrace{\frac{\partial \mathrm{v}_i}{\partial x_i}}_{\nabla\cdot\mathbf{v}}+\frac{1}{2}\underbrace{\frac{\partial^2 \mathrm{v}_j}{\partial x_i \partial x_i}}_{\Delta \mathrm{v}_j} = \underbrace{\frac{1}{2}\frac{\partial}{\partial x_j}(\nabla\cdot\mathbf{v})}_{[\nabla(\nabla\cdot\mathbf{v})]_j}+\frac{1}{2}\underbrace{\Delta \mathrm{v}_j}_{[\Delta\mathbf{v}]_j} = \\
&= \left[\frac{1}{2}\Delta\mathbf{v}+\frac{1}{2}\nabla(\nabla\cdot\mathbf{v})\right]_j \qquad j\in\{1,2,3\}
\end{aligned}
\tag{10.80}
$$

$$
\boxed{\nabla\cdot\mathbf{d} = \frac{1}{2}\Delta\mathbf{v}+\frac{1}{2}\nabla(\nabla\cdot\mathbf{v})}
\tag{10.81}
$$

Lemma 10.4. *Given a scalar function $\alpha(\mathbf{x},t)$, the following is satisfied.*

$$
\nabla\cdot(\alpha\mathbf{1}) = \nabla\alpha
$$

Proof

$$
\left[\nabla\cdot(\alpha\mathbf{1})\right]_i = \frac{\partial(\alpha\delta_{ij})}{\partial x_j} = \delta_{ij}\frac{\partial\alpha}{\partial x_j} = \frac{\partial\alpha}{\partial x_i} = \left[\nabla\alpha\right]_i \qquad i\in\{1,2,3\}
\tag{10.82}
$$

$$
\boxed{\nabla\cdot(\alpha\mathbf{1}) = \nabla\alpha}
\tag{10.83}
$$

Replacing the mechanical constitutive equation of (10.79) into the balance of linear momentum of (10.79), and taking into account (10.81) and (10.83) leads to

$$
\left.\begin{array}{l}
\boldsymbol{\sigma} = -p\mathbf{1}+\lambda\,\mathrm{Tr}(\mathbf{d})\,\mathbf{1}+2\mu\,\mathbf{d} \\
\nabla\cdot\boldsymbol{\sigma}+\rho\mathbf{b} = \rho\dfrac{d\mathbf{v}}{dt}
\end{array}\right\} \quad\Longrightarrow
$$

$$
\left\{\begin{array}{l}
\nabla\cdot\boldsymbol{\sigma} = -\nabla p+\lambda\underbrace{\nabla(\mathrm{Tr}(\mathbf{d}))}_{\nabla(\nabla\cdot\mathbf{v})}+\mu\Delta\mathbf{v}+\mu\nabla(\nabla\cdot\mathbf{v}) \\
\nabla\cdot\boldsymbol{\sigma}+\rho\mathbf{b} = -\nabla p+(\lambda+\mu)\nabla(\nabla\cdot\mathbf{v})+\mu\Delta\mathbf{v}+\rho\mathbf{b} = \rho\dfrac{d\mathbf{v}}{dt}
\end{array}\right.
\tag{10.84}
$$

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which results in the Navier-Stokes equation.

Navier-Stokes equation

$$
\begin{aligned}
&-\nabla p+(\lambda+\mu)\nabla(\nabla\cdot\mathbf{v})+\mu\Delta\mathbf{v}+\rho\mathbf{b}=\rho\frac{d\mathbf{v}}{dt} \\
&-\frac{\partial p}{\partial x_i}+(\lambda+\mu)\frac{\partial^2 \mathrm{v}_j}{\partial x_i\partial x_j}+\mu\frac{\partial^2 \mathrm{v}_i}{\partial x_j\partial x_j}+\rho b_i=\rho\frac{d\mathrm{v}_i}{dt}\ ;\ i\in\{1,2,3\}
\end{aligned}
\tag{10.85}
$$

10.4.2 Energy Equation

The aim is to eliminate $\boldsymbol{\sigma}$ and $\mathbf{q}$ from the energy balance of (10.79) by replacing in this equation the mechanical constitutive equation and the entropy equation of (10.79). To this aim, the definition of stress power in a Newtonian fluid (see Chapter 9) is recovered,

$$
\boldsymbol{\sigma}:\mathbf{d}=W_R+2W_D=-p\nabla\cdot\mathbf{v}+\mathcal{K}\,\mathrm{Tr}^2(\mathbf{d})+2\mu\,\mathbf{d}':\mathbf{d}'\,, \tag{10.86}
$$

where $\mathbf{d}'$ is the deviatoric part of the strain rate tensor. Fourier's law is also recovered,

$$
\mathbf{q}=-K\nabla\theta \quad\Longrightarrow\quad \nabla\cdot\mathbf{q}=-\nabla\cdot(K\nabla\theta)\ . \tag{10.87}
$$

Replacing now in the energy balance of (10.79) yields

$$
\rho\frac{du}{dt}=\boldsymbol{\sigma}:\mathbf{d}+\rho r-\nabla\cdot\mathbf{q} \quad\Longrightarrow \tag{10.88}
$$

Energy equation

$$
\begin{aligned}
&\rho\frac{du}{dt}=-p\nabla\cdot\mathbf{v}+\rho r+\nabla\cdot(K\nabla\theta)+\underbrace{\mathcal{K}\,\mathrm{Tr}^2(\mathbf{d})+2\mu\,\mathbf{d}':\mathbf{d}'}_{2W_D} \\
&\rho\frac{du}{dt}=-p\frac{\partial \mathrm{v}_i}{\partial x_i}+\rho r+\frac{\partial}{\partial x_i}\left(K\frac{\partial\theta}{\partial x_i}\right)+\mathcal{K}\left(\frac{\partial \mathrm{v}_i}{\partial x_i}\right)^2+2\mu\,d'_{ij}d'_{ij}
\end{aligned}
\tag{10.89}
$$

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10.4.3 Governing Equations of the Fluid Mechanics Problem

Considering the simplified versions of the balance of linear momentum (Navier-Stokes equation (10.85)) and the energy balance (energy equation (10.89)) the problem defined in (10.79) can be reduced to the following system of 7 PDEs and 7 unknowns $(\rho(\mathbf{x},t),\ \mathbf{v}(\mathbf{x},t),\ p(\mathbf{x},t),\ u(\mathbf{x},t),\ \theta(\mathbf{x},t))$, which must be solved in the $\mathbb{R}^3 \times \mathbb{R}^+$ domain.

$$
\begin{array}{lcc}
\dfrac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 & \text{Mass continuity equation} & \text{(1 eqn.)} \\[2ex]
\begin{array}{l} -\nabla p + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{v}) + \\ \qquad + \mu \Delta \mathbf{v} + \rho \mathbf{b} = \rho \dfrac{d\mathbf{v}}{dt} \end{array} & \begin{array}{c}\text{Balance of linear momentum} \\ \text{(Navier-Stokes)}\end{array} & \text{(3 eqns.)} \\[4ex]
\begin{array}{l} \rho \dfrac{du}{dt} = -p \nabla \cdot \mathbf{v} + \rho\, r + \nabla \cdot (K \nabla \theta) + \\ \qquad + \mathcal{K}\, \text{Tr}^2(\mathbf{d}) + 2\mu\, \mathbf{d}' : \mathbf{d}' \end{array} & \text{Energy balance} & \text{(1 eqn.)} \\[4ex]
u = u(\rho, \theta) & \text{Caloric equation of state} & \text{(1 eqn.)} \\[4ex]
F(\rho, p, \theta) = 0 & \text{Kinetic equation of state} & \text{(1 eqn.)}
\end{array}
\tag{10.90}
$$

In the particular case of a *barotropic regime* $(\rho = \rho(p))$, the mechanical part can be uncoupled from the thermal part in the set of equations of (10.79), resulting in the *mechanical problem* defined by the following system of 5 equations and 5 unknowns $(\rho(\mathbf{x},t),\ \mathbf{v}(\mathbf{x},t),\ p(\mathbf{x},t))$.

$$
\begin{array}{lcc}
\dfrac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 & \text{Mass continuity equation} & \text{(1 eqn.)} \\[2ex]
\begin{array}{l} -\nabla p + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{v}) + \\ \qquad + \mu \Delta \mathbf{v} + \rho \mathbf{b} = \rho \dfrac{d\mathbf{v}}{dt} \end{array} & \begin{array}{c}\text{Balance of linear momentum} \\ \text{(Navier-Stokes)}\end{array} & \text{(3 eqns.)} \\[4ex]
\rho = \rho(p) & \begin{array}{c}\text{Kinetic equation of state} \\ \text{(barotropic)}\end{array} & \text{(1 eqn.)}
\end{array}
\tag{10.91}
$$

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10.4.4 Physical Interpretation of the Navier-Stokes and Energy Equations

Each of the terms in the Navier-Stokes equation (10.85),

$$\begin{cases} -\nabla p + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{v}) + \mu\Delta\mathbf{v} + \rho\mathbf{b} - \rho\underbrace{\dfrac{d\mathbf{v}}{dt}}_{\mathbf{a}} = \mathbf{0} \\[2ex] -\dfrac{\partial p}{\partial x_i} + \Big[(\lambda+\mu)\nabla(\nabla\cdot\mathbf{v}) + \mu\Delta\mathbf{v}\Big]_i + \rho\, b_i - \rho\, a_i = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad i\in\{1,2,3\} \end{cases} \tag{10.92}$$

can be interpreted as a component of the system of forces (per unit of volume) that acts on a volume differential of the fluid in motion as follows.

$$\boxed{\underbrace{-\nabla p}_{\substack{\text{forces due to}\\ \text{the pressure}\\ \text{gradient}}} \;-\; \underbrace{-\Big((\lambda+\mu)\nabla(\nabla\cdot\mathbf{v}) + \mu\Delta\mathbf{v}\Big)}_{\substack{\text{viscous forces exerted by}\\ \text{the contact between particles}\\ (=0 \text{ when } \lambda=\mu=0)}} + \underbrace{\rho\mathbf{b}}_{\substack{\text{body}\\ \text{forces}}} + \underbrace{\rho\mathbf{a}}_{\substack{\text{inertial}\\ \text{forces}}} = \mathbf{0}} \tag{10.93}$$

Figure 10.11 shows the projection of each of these components in the x_i-direction.

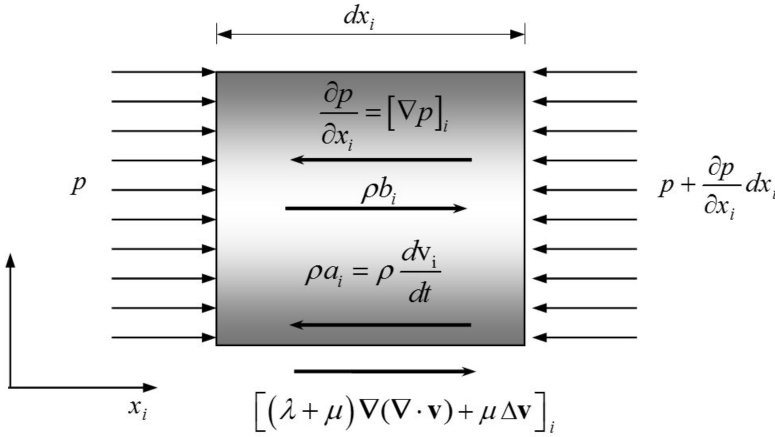

Figure 10.11: Projection of the components of the Navier-Stokes equation in the x_i-direction.

Each of the terms in the energy equation (10.89) can also be given a physical interpretation, as indicated in Table 10.1.

$\rho \dfrac{du}{dt} = -p\nabla\cdot\mathbf{v} + \rho\, r + \nabla\cdot(k\nabla\theta) + \underbrace{\mathcal{K}\,\mathrm{Tr}^2(\mathbf{d}) + 2\mu\,\mathbf{d}':\mathbf{d}'}_{2W_D}$	
$\rho \dfrac{du}{dt}$	$= \dfrac{\text{variation of internal energy}}{\text{unit of volume and unit of time}}$
$\nabla\cdot\mathbf{v} = \dfrac{\dfrac{d(dV)}{dt}}{dV}$	$= \dfrac{\text{variation of volume}}{\text{unit of volume and unit of time}}$
$-p\nabla\cdot\mathbf{v} = -\dfrac{p\dfrac{d(dV)}{dt}}{dV}$	$= \dfrac{\text{mechanical work of the thermodynamic pressure}}{\text{unit of volume and unit of time}}$ (see Figure 10.12 and footnote[8])
$\rho\, r + \nabla\cdot(k\nabla\theta)$	$= \dfrac{\text{heat generated by the internal sources and conduction}}{\text{unit of volume and unit of time}}$
$2W_D = \boldsymbol{\sigma}_D : \mathbf{d}$	$= \text{dissipative power} = \dfrac{\text{mechanical work of the viscous forces}}{\text{unit of volume and unit of time}}$

Table 10.1: Physical interpretation of the energy equation.

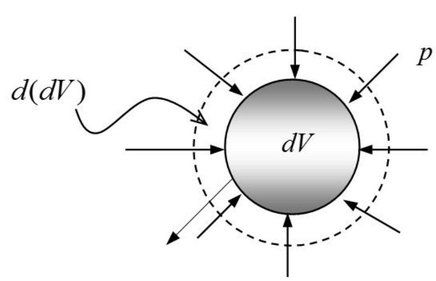

Figure 10.12: Mechanical work of the thermodynamic pressure.

[8] Here, the relation $d(dV)/dt = (\nabla\cdot\mathbf{v})\,dV$ is used (see Section 2.14.3 in Chapter 2).

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10.4.5 Reduction of the General Problem to Particular Cases

The governing equations in fluid mechanics (10.90) can be simplified for certain cases which are of particular interest in engineering applications.

10.4.5.1 Incompressible Fluids

In this case,

$$\left.\begin{array}{l} \dfrac{d\rho}{dt}=0 \\[2ex] \dfrac{d\rho}{dt}+\rho\nabla\cdot\mathbf{v}=0 \end{array}\right\} \quad\Longrightarrow\quad \left\{\begin{array}{l} \rho=\rho_0=const. \\ \nabla\cdot\mathbf{v}=\mathrm{Tr}\,(\mathbf{d})=0 \end{array}\right. \tag{10.94}$$

and introducing (10.94) in (10.90) results in the governing equations detailed in Table 10.2.

Mechanical Problem	Mass continuity equation	$\nabla\cdot\mathbf{v}=0$
	Navier-Stokes equation	$-\nabla p+\mu\,\Delta\mathbf{v}+\rho_0\,\mathbf{b}=\rho_0\dfrac{d\mathbf{v}}{dt}$
Thermal Problem	Energy balance	$\rho_0\dfrac{du}{dt}=\rho_0\,r+\nabla\cdot(K\,\nabla\theta)+2\mu\,\mathbf{d}':\mathbf{d}'$
	Caloric equation of state	$u=u\,(\rho_0,\theta)$
Constitutive equation		$\boldsymbol{\sigma}=-p\,\mathbf{1}+2\mu\,\mathbf{d}$

Table 10.2: Governing equations in incompressible Newtonian fluids

10.4.5.2 Fluids with Null Bulk Viscosity (Stokes Fluids)

In this case,

$$\mathcal{K}=\lambda+\frac{2}{3}\mu=0 \quad\Longrightarrow\quad \lambda=-\frac{2}{3}\mu \quad\Longrightarrow\quad \lambda+\mu=\frac{1}{3}\mu \tag{10.95}$$

$$2W_D=\underbrace{\mathcal{K}}_{=0}\,\mathrm{Tr}^2\,(\mathbf{d})+2\mu\,\mathbf{d}':\mathbf{d}'=2\mu\,\mathbf{d}':\mathbf{d}' \tag{10.96}$$

and replacing (10.95) and (10.96) in (10.90) yields the governing equations given in Table 10.3.

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Mass continuity equation	$\dfrac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0$
Navier-Stokes equation	$-\nabla p + \dfrac{1}{3}\mu \nabla(\nabla \cdot \mathbf{v}) + \mu \Delta \mathbf{v} + \rho \mathbf{b} = \rho \dfrac{d\mathbf{v}}{dt}$
Energy balance	$\rho \dfrac{du}{dt} = -p\nabla \cdot \mathbf{v} + \rho\, r + \nabla \cdot (K \nabla \theta) + 2\mu\, \mathbf{d}' : \mathbf{d}'$
Caloric equation of state	$u = u(\rho, \theta)$
Kinetic equation of state	$F(\rho, p, \theta) = 0$
Constitutive equation	$\boldsymbol{\sigma} = -p\,\mathbf{1} - \dfrac{2}{3}\mu\, \mathrm{Tr}(\mathbf{d})\,\mathbf{1} + 2\mu\, \mathbf{d}$

Table 10.3: Governing equations in Stokes fluids.

10.4.5.3 Perfect Fluids

Perfect fluids have null viscosity, $\lambda = \mu = \mathcal{K} = 0$, and no heat conductivity, $K = 0$. Introducing these conditions in (10.90) results in the problem shown in Table 10.4.

Mass continuity equation	$\dfrac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0$
Euler's equation	$-\nabla p + \rho \mathbf{b} = \rho \dfrac{d\mathbf{v}}{dt}$
Energy balance	$\rho \dfrac{du}{dt} = -p\nabla \cdot \mathbf{v} + \rho\, r$
Caloric equation of state	$u = u(\rho, \theta)$
Kinetic equation of state	$F(\rho, p, \theta) = 0$
Constitutive equation	$\boldsymbol{\sigma} = -p\,\mathbf{1}$

Table 10.4: Governing equations in perfect fluids.

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10.4.5.4 Hydrostatics

In this case, the following conditions apply (see (10.20)):

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \mathbf{0}\,, \quad \nabla\cdot\mathbf{v} = 0\,, \quad \rho = \rho_0\,, \quad p = p_0 \quad \text{and} \quad \boldsymbol{\sigma} = -p_0\mathbf{1}\,, \tag{10.97}$$

and, thus, (10.90) is reduced to the equations described in Table 10.5.

Mechanical Problem	Hydrostatics fundamental equation	$-\nabla p_0 + \rho_0\,\mathbf{b} = \mathbf{0}$
Thermal Problem	Energy balance	$\rho_0 \dfrac{du}{dt} = \rho_0\, r + \nabla\cdot(K\nabla\theta)$
	Caloric equation of state	$u = u(\rho_0, \theta)$
Constitutive equation		$\boldsymbol{\sigma} = -p_0\,\mathbf{1}$

Table 10.5: Governing equations in hydrostatics.

10.5 Boundary Conditions in Fluid Mechanics

The governing equations of the fluid mechanics problem presented in the previous sections require adequate boundary conditions to be solved correctly. In general, the *spatial (or Eulerian) description* is used in fluid mechanics problems, and a specific *control volume* (fixed in space) is analyzed, on whose boundary the aforementioned spatial boundary conditions are applied. Even though there are different boundary conditions, and these often depend on the type of problem being studied, the most common types of boundary conditions are summarized below.

10.5.1 Velocity Boundary Conditions

a) *Prescribed velocity*

In certain parts $\Gamma_{\bar{v}}$ of the boundary of the control volume V being analyzed, the velocities are known (see Figure 10.13).

$$\boxed{\mathbf{v}(\mathbf{x},t) = \bar{\mathbf{v}}(\mathbf{x},t) \qquad \forall \mathbf{x}\in\Gamma_{\mathrm{v}}} \tag{10.98}$$

b) *Impermeability condition*

Usually, part of the boundary of the control volume V is composed of impermeable walls, Γ_{v_n}, which are assumed to be impervious to fluid, that is, they

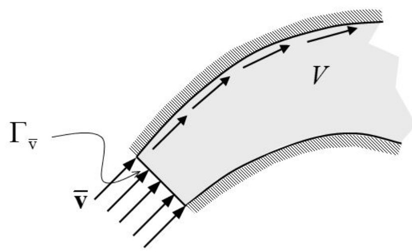

Figure 10.13: Velocity boundary conditions: prescribed velocity.

cannot be penetrated by said fluid. The mathematical expression of this condition is denoted as *impermeability condition* and it establishes that the relative velocity of the fluid, $\mathbf{v}_r$, with respect to the impermeable wall (assumed mobile and with a velocity $\mathbf{v}^*$) *in the direction normal to the boundary* must be null (see Figure10.14),

$$\begin{array}{l} \mathrm{v}_n(\mathbf{x},t) = \underbrace{\mathbf{v}\cdot\mathbf{n}}_{\text{fluid}} = \underbrace{\mathbf{v}^*\cdot\mathbf{n}}_{\text{wall}} \qquad \forall \mathbf{x}\in\Gamma_{\mathrm{v}_n} \qquad \Longrightarrow \\ \boxed{\mathbf{v}_r\cdot\mathbf{n} = (\mathbf{v}-\mathbf{v}^*)\cdot\mathbf{n} = 0 \qquad \forall \mathbf{x}\in\Gamma_{\mathrm{v}_n}} \; . \end{array} \tag{10.99}$$

In the particular case of a fixed boundary, this condition is reduced to $(\mathbf{v}^* = \mathbf{0}) \;\Rightarrow\; \mathbf{v}\cdot\mathbf{n} = 0 \quad \forall \mathbf{x}\in\Gamma_{\mathrm{v}_n}$.

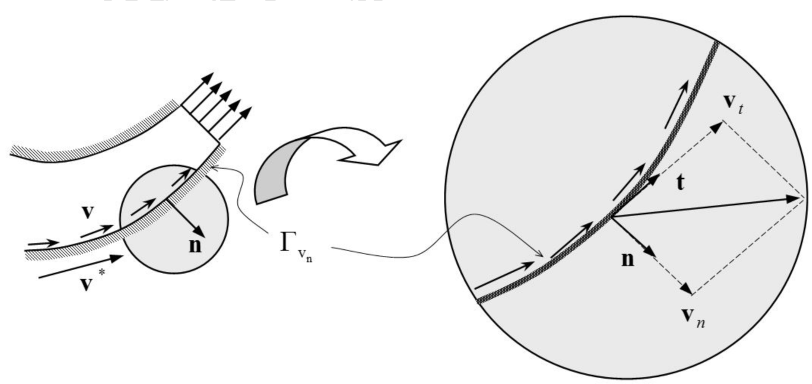

Figure 10.14: Velocity boundary conditions: impermeability condition.

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Remark 10.6. The impermeability condition is usually applied for *perfect fluids* (fluids without viscosity) in which the tangential component of the relative velocity between the fluid and the wall $\mathbf{v}_t$ (see Figure 10.14) is assumed to be non-null.

c) Adherence condition

In *viscous fluids* in contact with an impermeable wall, due to the effect of viscosity, the fluid is assumed to *adhere* to the wall (see Figure 10.15) and, thus, the relative velocity between the fluid and the wall $\mathbf{v}_r$ is null.

$$\mathbf{v}_r(\mathbf{x},t) = \mathbf{v} - \mathbf{v}^* = \mathbf{0} \qquad \forall \mathbf{x} \in \Gamma_{\text{v}} \implies \boxed{\mathbf{v} = \mathbf{v}^* \qquad \forall \mathbf{x} \in \Gamma_{\text{v}}} \tag{10.100}$$

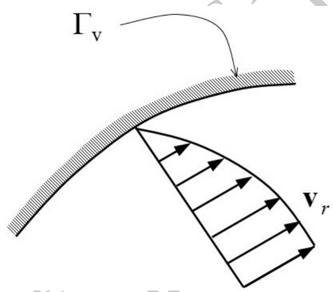

Figure 10.15: Velocity boundary conditions: adherence condition.

10.5.2 Pressure Boundary Conditions

In certain parts Γ_σ of the boundary, the traction vector $\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n}$ can be prescribed (see Figure 10.16).

$$\boxed{\mathbf{t}(\mathbf{x},t) = \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t}^*(\mathbf{x},t) \qquad \forall \mathbf{x} \in \Gamma_\sigma} \tag{10.101}$$

Under certain circumstances, only a part of the traction vector such as the *thermodynamic pressure* is prescribed. In effect, for a Newtonian fluid,

$$\begin{gathered} \boldsymbol{\sigma} = -p\mathbf{1} + \lambda \operatorname{Tr}(\mathbf{d})\,\mathbf{1} + 2\mu\,\mathbf{d} \quad \implies \\ \mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} = -p\,\mathbf{n} + \lambda \operatorname{Tr}(\mathbf{d})\,\mathbf{n} + 2\mu\,\mathbf{d} \cdot \mathbf{n} \,, \end{gathered} \tag{10.102}$$

doi:10.13140/RG.2.2.25821.20961

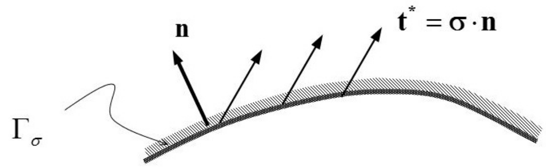

Figure 10.16: Pressure boundary conditions: prescribed traction vector.

which exposes how the thermodynamic pressure p is a part of the normal component of the traction vector $\mathbf{t}$. The prescription of the thermodynamic pressure on a part of the boundary Γ_p is written as

$$\boxed{p(\mathbf{x},t) = p^*(\mathbf{x},t) \qquad \forall \mathbf{x} \in \Gamma_p} \ . \tag{10.103}$$

10.5.3 Mixed Boundary Conditions

In certain cases (such as the entrance and exit sections of pipes) the pressure (a part of the *normal* component of the traction vector) and the *tangential* components of the velocity (which are assumed to be null, see Figure 10.17) are prescribed.

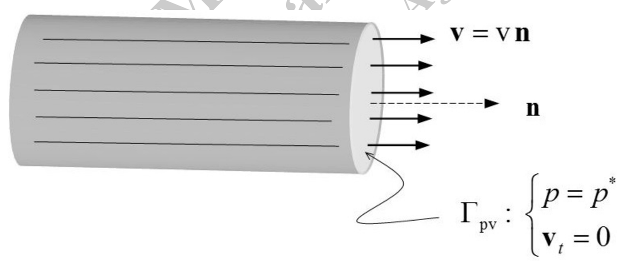

Figure 10.17: Mixed boundary conditions.

10.5.4 Boundary Conditions on Free Surfaces

Definition 10.5. A *free surface* is a contact surface between the air (atmosphere) and a fluid (generally water).

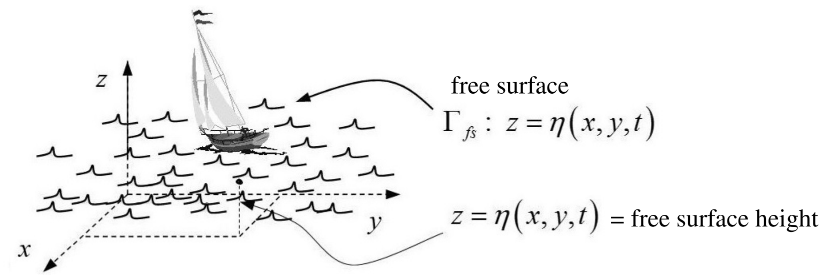

Figure 10.18: Free surface of the sea.

Examples of free surface[9] are the surface of the sea (see Figure 10.18) or the surface that separates the saturated and unsaturated parts of an embankment dam (see Figure 10.19).

A hypothesis with a clear physical sense that is frequently used in relation to a free surface is that such a surface is *a material surface* (constituted always by the same particles). This hypothesis implicitly establishes certain boundary conditions on the velocity field in the material surface Γ_{fs}. In effect, considering the free surface in Figure 10.18,

$$\Gamma_{fs} := \{\mathbf{x} \mid \phi(x,y,z,t) \equiv z - \eta(x,y,t) = 0\} \ , \qquad (10.104)$$

and imposing the material character of the free surface (null material derivative, see Section 1.11 in Chapter 1),

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \mathbf{v}\cdot\nabla\phi = -\frac{\partial \eta}{\partial t} - \mathrm{v}_x \frac{\partial \eta}{\partial x} - \mathrm{v}_y \frac{\partial \eta}{\partial y} + \mathrm{v}_z \underbrace{\frac{\partial \phi}{\partial z}}_{=1} = 0 \ , \qquad (10.105)$$

$$\boxed{\mathrm{v}_z(\mathbf{x},t) = \frac{\partial \eta}{\partial t} + \mathrm{v}_x \frac{\partial \eta}{\partial x} + \mathrm{v}_y \frac{\partial \eta}{\partial y} \qquad \forall \mathbf{x} \in \Gamma_{fs}} \ . \qquad (10.106)$$

This condition establishes the dependency of the vertical component of the velocity v_z on the other components v_x and v_y.

Another boundary condition frequently imposed on free surfaces is that, in these surfaces, the thermodynamic pressure is known and equal to the atmospheric pressure[10],

$$\boxed{p(\mathbf{x},t) = P_{atm} \qquad \forall \mathbf{x} \in \Gamma_{fs}} \qquad (10.107)$$

[9] In general, in fluid mechanics problems in which free surfaces appear, the position of these surfaces is not known and their geometrical characteristics become an unknown of the problem.

[10] The value of the atmospheric pressure is generally neglected ($P_{atm} \approx 0$).

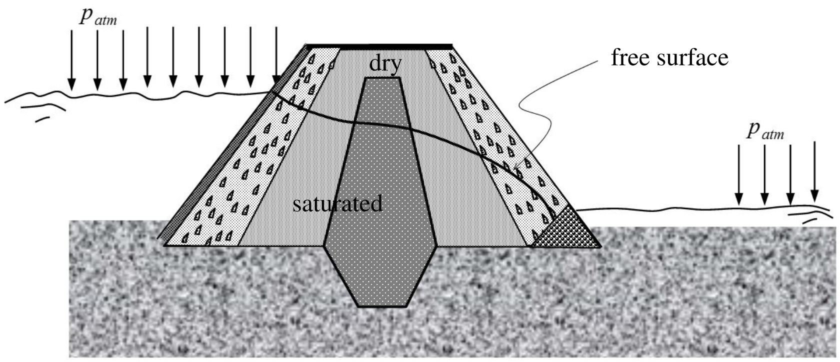

Figure 10.19: Free surface of an embankment dam.

Equation (10.107) allows identifying, in certain cases, the position of the free surface (once the pressure field is known) as the locus of points in the fluid in which the pressure is equal to the atmospheric pressure.

Equation of the free surface	$\Gamma_{fs} := \{ \mathbf{x} \mid p(\mathbf{x},t) - P_{atm} = 0 \}$	(10.108)

10.6 Laminar and Turbulent Flows

10.6.1 Laminar Flow

The equations governing a fluid mechanics problem, described in the previous sections, are valid for a certain range of motion of the fluids, named *laminar flow* (or regime). Basically, laminar flow is physically characterized by the fact that the fluid moves in parallel layers that do not mix (see Figure 10.20).

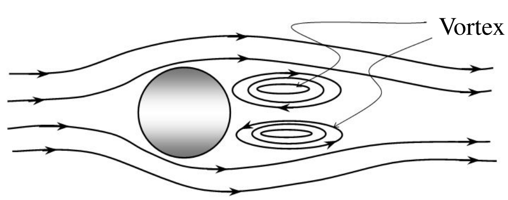

Figure 10.20: Laminar flow around an obstacle.

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The character of a laminar flow is identified by the Reynolds number R_e

$$\boxed{\begin{array}{l} \text{Reynolds number: } R_e \overset{def}{=} \dfrac{V \times L}{\nu} \\ \begin{cases} V = \text{characteristic velocity of the fluid} \\ L = \text{characteristic length of the domain} \\ \nu = \text{kinematic viscosity } (\nu = \mu/\rho) \end{cases} \end{array}} \tag{10.109}$$

such that small values of the Reynolds number characterize laminar flows.

10.6.2 Turbulent Flow

When the velocity increases and the viscosity decreases, the Reynolds number (10.109) increases. For increasing values of this number, the initially laminar flow is seen to distort and become highly unstable. The flow can then be understood as being in a situation in which the velocity $\mathbf{v}(\mathbf{x},t)$, at a given point in space, randomly and rapidly fluctuates along time about a mean value $\bar{\mathrm{v}}(\mathbf{x},t)$ (see Figure 10.21). This situation is defined as *turbulent flow* (or regime).

Even though the equations of the fluid mechanics problem in general, and the Navier-Stokes equation in particular, are still valid in turbulent regime, certain circumstances (such as the difficulty in treating the mathematical problem and the impossibility of experimentally characterizing the rapid fluctuations of the variables of this problem) impose a singular treatment for turbulent flow. The mathematical characterization of turbulent regime is done, then, by means of the so-called *turbulence models*. These models are based on isolating the mean values of the velocity and pressure fields from their fluctuations and, then, the governing equations of the problem are obtained in terms of these mean values.

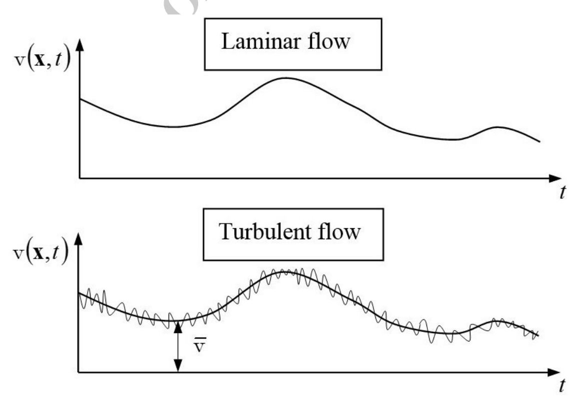

Figure 10.21: Variation of the velocity along time in laminar and turbulent flows.

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10.7 Fluid Mechanics Formulas

10.7.1 Stress tensor for Newtonian fluids

(incompressible fluid, $\nabla \cdot \mathbf{v} = 0$)

Cartesian coordinates

$$
\begin{aligned}
\sigma_x &= 2\mu \frac{\partial \mathrm{v}_x}{\partial x} - p & \qquad \tau_{xy} = \tau_{yx} &= \mu \left(\frac{\partial \mathrm{v}_x}{\partial y} + \frac{\partial \mathrm{v}_y}{\partial x} \right) \\
\sigma_y &= 2\mu \frac{\partial \mathrm{v}_y}{\partial y} - p & \qquad \tau_{yz} = \tau_{zy} &= \mu \left(\frac{\partial \mathrm{v}_y}{\partial z} + \frac{\partial \mathrm{v}_z}{\partial y} \right) \\
\sigma_z &= 2\mu \frac{\partial \mathrm{v}_z}{\partial z} - p & \qquad \tau_{zx} = \tau_{xz} &= \mu \left(\frac{\partial \mathrm{v}_z}{\partial x} + \frac{\partial \mathrm{v}_x}{\partial z} \right)
\end{aligned}
\tag{10.110}
$$

Cylindrical coordinates

$$
\begin{aligned}
\sigma_r &= 2\mu \frac{\partial \mathrm{v}_r}{\partial r} - p & \qquad \tau_{r\theta} = \tau_{\theta r} &= \mu \left(r \frac{\partial}{\partial r} \left(\frac{\mathrm{v}_\theta}{r} \right) + \frac{1}{r} \frac{\partial \mathrm{v}_r}{\partial \theta} \right) \\
\sigma_\theta &= 2\mu \left(\frac{1}{r} \frac{\partial \mathrm{v}_\theta}{\partial \theta} + \frac{\mathrm{v}_r}{r} \right) - p & \qquad \tau_{\theta z} = \tau_{z\theta} &= \mu \left(\frac{\partial \mathrm{v}_\theta}{\partial z} + \frac{1}{r} \frac{\partial \mathrm{v}_z}{\partial \theta} \right) \\
\sigma_z &= 2\mu \frac{\partial \mathrm{v}_z}{\partial z} - p & \qquad \tau_{zr} = \tau_{rz} &= \mu \left(\frac{\partial \mathrm{v}_z}{\partial r} + \frac{\partial \mathrm{v}_r}{\partial z} \right)
\end{aligned}
\tag{10.111}
$$

$$
\nabla \cdot \mathbf{v} = \frac{1}{r} \frac{\partial}{\partial r} (r \mathrm{v}_r) + \frac{1}{r} \frac{\partial \mathrm{v}_\theta}{\partial \theta} + \frac{\partial \mathrm{v}_z}{\partial z}
\tag{10.112}
$$

Spherical coordinates

$$
\begin{aligned}
\sigma_r &= 2\mu \frac{\partial \mathrm{v}_r}{\partial r} - p \\
\sigma_\theta &= 2\mu \left(\frac{1}{r} \frac{\partial \mathrm{v}_\theta}{\partial \theta} + \frac{\mathrm{v}_r}{r} \right) - p \\
\sigma_\phi &= 2\mu \left(\frac{1}{r \sin\theta} \frac{\partial \mathrm{v}_\phi}{\partial \phi} + \frac{\mathrm{v}_r}{r} + \frac{\mathrm{v}_\theta \cot\theta}{r} \right) - p
\end{aligned}
\tag{10.113}
$$

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$$
\begin{aligned}
\tau_{r\theta} &= \tau_{\theta r} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{\mathrm{v}_\theta}{r}\right)+\frac{1}{r}\frac{\partial \mathrm{v}_r}{\partial\theta}\right)\\
\tau_{\theta\phi} &= \tau_{\phi\theta} = \mu\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\mathrm{v}_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial \mathrm{v}_\theta}{\partial\phi}\right)\\
\tau_{\phi r} &= \tau_{r\phi} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial \mathrm{v}_r}{\partial\phi}+r\frac{\partial}{\partial r}\left(\frac{\mathrm{v}_\phi}{r}\right)\right)
\end{aligned}
\qquad (10.113\ \ \text{(cont.)})
$$

$$
\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\mathrm{v}_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\mathrm{v}_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial \mathrm{v}_\phi}{\partial\phi} \qquad (10.114)
$$

10.7.2 Continuity Equation

Cartesian coordinates

$$
\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}\left(\rho\mathrm{v}_x\right)+\frac{\partial}{\partial y}\left(\rho\mathrm{v}_y\right)+\frac{\partial}{\partial z}\left(\rho\mathrm{v}_z\right)=0 \qquad (10.115)
$$

Cylindrical coordinates

$$
\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}\left(\rho r\mathrm{v}_r\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\rho\mathrm{v}_\theta\right)+\frac{\partial}{\partial z}\left(\rho\mathrm{v}_z\right)=0 \qquad (10.116)
$$

Spherical coordinates

$$
\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^2\mathrm{v}_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\rho\mathrm{v}_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\rho\mathrm{v}_\phi\right)=0 \qquad (10.117)
$$

10.7.3 Navier-Stokes Equation

(incompressible fluid, $\nabla\cdot\mathbf{v}=0$; ρ and μ const.)

Cartesian coordinates

$$
\begin{aligned}
-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2\mathrm{v}_x}{\partial x^2}+\frac{\partial^2\mathrm{v}_x}{\partial y^2}+\frac{\partial^2\mathrm{v}_x}{\partial z^2}\right)+\rho b_x &= \rho\left(\frac{\partial \mathrm{v}_x}{\partial t}+\mathrm{v}_x\frac{\partial \mathrm{v}_x}{\partial x}+\mathrm{v}_y\frac{\partial \mathrm{v}_x}{\partial y}+v_z\frac{\partial \mathrm{v}_x}{\partial z}\right)\\
-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2\mathrm{v}_y}{\partial x^2}+\frac{\partial^2\mathrm{v}_y}{\partial y^2}+\frac{\partial^2\mathrm{v}_y}{\partial z^2}\right)+\rho b_y &= \rho\left(\frac{\partial \mathrm{v}_y}{\partial t}+\mathrm{v}_x\frac{\partial \mathrm{v}_y}{\partial x}+\mathrm{v}_y\frac{\partial \mathrm{v}_y}{\partial y}+\mathrm{v}_z\frac{\partial \mathrm{v}_y}{\partial z}\right)\\
-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2\mathrm{v}_z}{\partial x^2}+\frac{\partial^2\mathrm{v}_z}{\partial y^2}+\frac{\partial^2\mathrm{v}_z}{\partial z^2}\right)+\rho b_z &= \rho\left(\frac{\partial \mathrm{v}_z}{\partial t}+\mathrm{v}_x\frac{\partial \mathrm{v}_z}{\partial x}+\mathrm{v}_y\frac{\partial \mathrm{v}_z}{\partial y}+\mathrm{v}_z\frac{\partial \mathrm{v}_z}{\partial z}\right)
\end{aligned}
\qquad (10.118)
$$

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Cylindrical coordinates

$$
\begin{aligned}
&-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\mathrm{v}_r)\right)+\frac{1}{r^2}\frac{\partial^2\mathrm{v}_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial\mathrm{v}_\theta}{\partial\theta}+\frac{\partial^2\mathrm{v}_r}{\partial z^2}\right)+\rho b_r=\\
&\qquad=\rho\left(\frac{\partial\mathrm{v}_r}{\partial t}+\mathrm{v}_r\frac{\partial\mathrm{v}_r}{\partial r}+\frac{\mathrm{v}_\theta}{r}\frac{\partial\mathrm{v}_r}{\partial\theta}-\frac{\mathrm{v}_\theta^2}{r}+\mathrm{v}_z\frac{\partial\mathrm{v}_r}{\partial z}\right)\\
&-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\mathrm{v}_\theta)\right)+\frac{1}{r^2}\frac{\partial^2\mathrm{v}_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial\mathrm{v}_r}{\partial\theta}+\frac{\partial^2\mathrm{v}_\theta}{\partial z^2}\right)+\rho b_\theta=\\
&\qquad=\rho\left(\frac{\partial\mathrm{v}_\theta}{\partial t}+\mathrm{v}_r\frac{\partial\mathrm{v}_\theta}{\partial r}+\frac{\mathrm{v}_\theta}{r}\frac{\partial\mathrm{v}_\theta}{\partial\theta}+\frac{\mathrm{v}_r\mathrm{v}_\theta}{r}+\mathrm{v}_z\frac{\partial\mathrm{v}_\theta}{\partial z}\right)\\
&-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\mathrm{v}_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\mathrm{v}_z}{\partial\theta^2}+\frac{\partial^2\mathrm{v}_z}{\partial z^2}\right)+\rho b_z=\\
&\qquad=\rho\left(\frac{\partial\mathrm{v}_z}{\partial t}+\mathrm{v}_r\frac{\partial\mathrm{v}_z}{\partial r}+\frac{\mathrm{v}_\theta}{r}\frac{\partial\mathrm{v}_z}{\partial\theta}+\mathrm{v}_z\frac{\partial\mathrm{v}_z}{\partial z}\right)
\end{aligned}
\tag{10.119}
$$

Spherical coordinates

$$
\begin{aligned}
&-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\mathrm{v}_r)\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathrm{v}_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\mathrm{v}_r}{\partial\phi^2}+\right.\\
&\qquad\left.-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\mathrm{v}_\theta\sin\theta)-\frac{2}{r^2\sin\theta}\frac{\partial\mathrm{v}_\phi}{\partial\phi}\right)+\rho b_r=\\
&\qquad=\rho\left(\frac{\partial\mathrm{v}_r}{\partial t}+\mathrm{v}_r\frac{\partial\mathrm{v}_r}{\partial r}+\frac{\mathrm{v}_\theta}{r}\frac{\partial\mathrm{v}_r}{\partial\theta}+\frac{\mathrm{v}_\phi}{r\sin\theta}\frac{\partial\mathrm{v}_r}{\partial\phi}-\frac{\mathrm{v}_\theta^2+\mathrm{v}_\phi^2}{r}\right)\\
&-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\mathrm{v}_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\mathrm{v}_\theta\sin\theta)\right)+\right.\\
&\qquad\left.+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\mathrm{v}_\theta}{\partial\phi^2}+\frac{2}{r^2}\frac{\partial\mathrm{v}_r}{\partial\theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial\mathrm{v}_\phi}{\partial\phi}\right)+\rho b_\theta=\\
&\qquad=\rho\left(\frac{\partial\mathrm{v}_\theta}{\partial t}+\mathrm{v}_r\frac{\partial\mathrm{v}_\theta}{\partial r}+\frac{\mathrm{v}_\theta}{r}\frac{\partial\mathrm{v}_\theta}{\partial\theta}+\frac{\mathrm{v}_\phi}{r\sin\theta}\frac{\partial\mathrm{v}_\theta}{\partial\phi}+\frac{\mathrm{v}_r\mathrm{v}_\theta}{r}-\frac{\mathrm{v}_\phi^2\cot\theta}{r}\right)
\end{aligned}
\tag{10.120}
$$

doi:10.13140/RG.2.2.25821.20961

$$
\begin{aligned}
&-\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \mathrm{v}_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\mathrm{v}_\phi\sin\theta\right)\right)+\right.\\
&\qquad\left.+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\mathrm{v}_\phi}{\partial\phi^2}+\frac{2}{r^2\sin\theta}\frac{\partial\mathrm{v}_r}{\partial\phi}+\frac{2\cot\,\theta}{r^2\sin\theta}\frac{\partial\mathrm{v}_\theta}{\partial\phi}\right)+\rho b_\varphi=\\
&\quad=\rho\left(\frac{\partial\mathrm{v}_\phi}{\partial t}+\mathrm{v}_r\frac{\partial\mathrm{v}_\phi}{\partial r}+\frac{\mathrm{v}_\theta}{r}\frac{\partial\mathrm{v}_\phi}{\partial\theta}+\frac{\mathrm{v}_\phi}{r\sin\theta}\frac{\partial\mathrm{v}_\phi}{\partial\phi}+\frac{\mathrm{v}_\phi\mathrm{v}_r}{r}+\frac{\mathrm{v}_\theta\mathrm{v}_\phi}{r}\cot\theta\right)
\end{aligned}
\tag{10.120 (cont.)}
$$

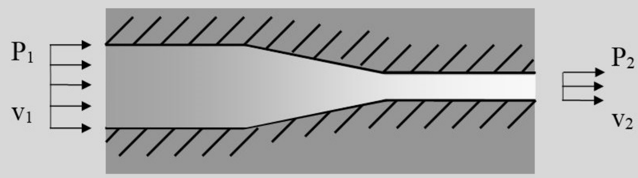
doi:10.13140/RG.2.2.25821.20961

PROBLEMS

Problem 10.1 – *The barotropic fluid flowing inside the pipe shown in the figure below has the following kinetic equation of state.*

$$p = \beta \ln\left(\frac{\rho}{\rho_0}\right) \qquad (\beta \textit{ and } \rho_0 \textit{ const.})$$

Determine, for a steady-state regime, the exit pressure P_2 *in terms of the other variables shown in the figure.*



Solution

The global spatial form of the *mass continuity equation* (5.22) states

$$\frac{d}{dt}\int_V \rho \, dV = 0 \, .$$

Using the *Reynolds Transport Theorem* (5.37) on this expression results in

$$\frac{d}{dt}\int_V \rho \, dV = \frac{\partial}{\partial t}\int_V \rho \, dV + \int_{\partial V} \rho \mathbf{v}\cdot\mathbf{n}\, dS \quad \Longrightarrow \quad \frac{\partial}{\partial t}\int_V \rho \, dV + \int_{\partial V} \rho \mathbf{v}\cdot\mathbf{n}\, dS = 0 \, ,$$

and introducing the conditions associated with a steady-state regime yields

$$\frac{\partial}{\partial t}\int_V \rho \, dV = 0 \quad \Longrightarrow \quad \int_{\partial V} \rho \mathbf{v}\cdot\mathbf{n}\, dS = 0 \, .$$

Applying this last expression to the problem described in the statement produces

$$-\rho_1 \mathrm{v}_1 S_1 + \rho_2 \mathrm{v}_2 S_2 = 0 \quad \Longrightarrow \quad \rho_1 \mathrm{v}_1 S_1 = \rho_2 \mathrm{v}_2 S_2 \, .$$

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Finally, isolating the density from the given *kinetic equation of state*,

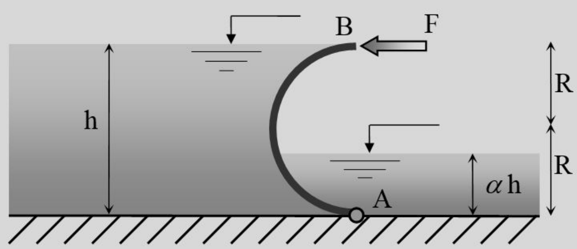
$$p = \beta \ln \left(\frac{\rho}{\rho_0} \right) \qquad \Longrightarrow \qquad \rho = \rho_0 \, \mathrm{e}^{p/\beta} \ ,$$

and introducing it into the previous one produces

$$\rho_0 \, \mathrm{e}^{\mathrm{P}_1/\beta} \, \mathrm{v}_1 S_1 = \rho_0 \, \mathrm{e}^{\mathrm{P}_2/\beta} \, \mathrm{v}_2 S_2 \quad \Longrightarrow \quad \mathrm{e}^{(\mathrm{P}_2 - \mathrm{P}_1)/\beta} = \frac{\mathrm{v}_1 S_1}{\mathrm{v}_2 S_2} \quad \Longrightarrow$$

$$\boxed{\mathrm{P}_2 = \mathrm{P}_1 + \beta \ln \left(\frac{\mathrm{v}_1 S_1}{\mathrm{v}_2 S_2} \right)} \ .$$

Problem 10.2 – *Determine the value per unit of thickness of the horizontal force* F *that must be applied on point* B *of the semicircular floodgate shown in the figure such that the straight line* AB *remains vertical. The floodgate can rotate around the hinge* A *and separates two different height levels,* h *and* αh, *of a same fluid.*



HYPOTHESES:

1) *The weight of the floodgate can be neglected.*
2) *The atmospheric pressure is negligible.*

Solution

The only forces acting on the floodgate are the pressure forces of the fluids, the force F and the reaction in A (horizontal component H and vertical component V).

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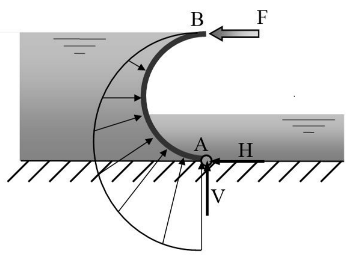


Figure A

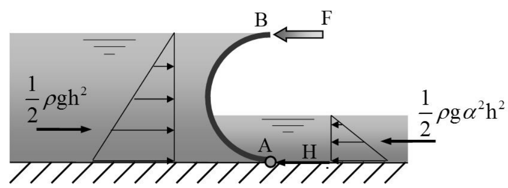


Figure B

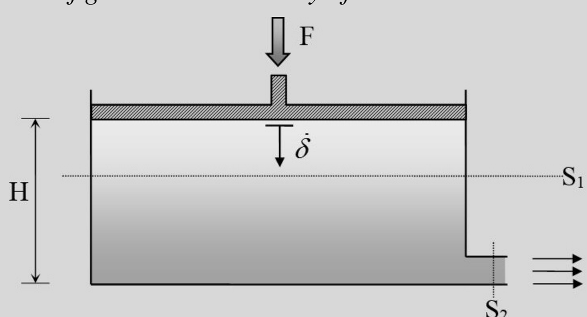
Since the pressure exerted by the fluids is perpendicular to the surface of the floodgate and the floodgate is circular, the resultant force obtained by integrating the pressures on the surface are applied at the center of the circumference defined by the floodgate. Thus, posing the equilibrium of momentum with respect to the center of this circumference (see Figure A) results in

$$\mathrm{FR} = \mathrm{HR} \quad \Longrightarrow \quad \mathrm{H} = \mathrm{F}\,.$$

Imposing now the equilibrium of horizontal forces, knowing that the fluids exert a horizontal pressure with a triangular distribution (see Figure B), yields

$$2\mathrm{F} + \frac{1}{2}(\rho g \alpha \mathrm{h})(\alpha \mathrm{h}) = \frac{1}{2}(\rho g \mathrm{h})\mathrm{h} \quad \Longrightarrow \quad \boxed{\mathrm{F} = \frac{1}{4}\rho g \mathrm{h}^2\left(1-\alpha^2\right)}\ .$$

Problem 10.3 – *Determine the relation between the force* F *applied on the piston shown in the figure and its velocity of descent* $\dot{\delta}$.



HYPOTHESES:

1) *Assume the fluid is an incompressible perfect fluid in steady-state regime.*
2) *The atmospheric pressure is negligible.*
3) S_1 *and* S_2 *are the cross-sections.*
4) *The density of the fluid is* ρ.

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Solution

The stress state of a perfect fluid is known to be of the form $\boldsymbol{\sigma} = -p\mathbf{1}$ (see Section 9.3 in Chapter 9). The mass continuity equation (5.22) is applied to obtain the relation between the velocities of the fluid,

$$\mathrm{v}_1 \mathrm{S}_1 = \mathrm{v}_2 \mathrm{S}_2 \quad \Longrightarrow \quad \mathrm{v}_2 = \frac{\mathrm{S}_1}{\mathrm{S}_2}\mathrm{v}_1 = \frac{\mathrm{S}_1}{\mathrm{S}_2}\dot{\delta} \ .$$

Taking into account *Bernoulli's theorem* (10.63) between an arbitrary point in contact with the piston and another at the exit cross-section, both belonging to a same streamline, results in

$$\mathrm{H} + \frac{p}{\rho g} + \frac{\dot{\delta}^2}{2g} = 0 + 0 + \left(\frac{\mathrm{S}_1}{\mathrm{S}_2}\dot{\delta}\right)^2 \frac{1}{2g} \quad \Longrightarrow \quad p = \frac{\rho}{2}\left(\left(\frac{\mathrm{S}_1}{\mathrm{S}_2}\right)^2 - 1\right)\dot{\delta}^2 - \rho g \mathrm{H}$$

Therefore, p must be constant for any point in contact with the piston $(\mathrm{x} = \mathrm{H})$. Then,

$$p = const. \qquad \forall \mathrm{x} = \mathrm{H} \quad \Longrightarrow \quad \mathrm{F} = p\,\mathrm{S}_1 \ .$$

Finally, the force F is related to $\dot{\delta}$ in the following manner.

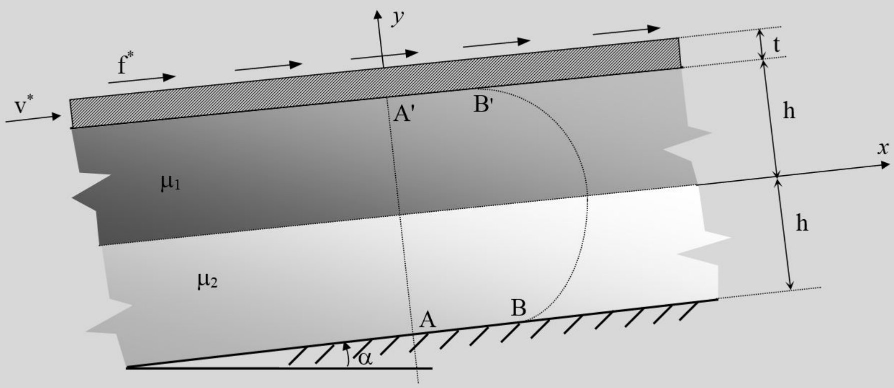
$$\boxed{\mathrm{F} = \frac{\rho}{2}\mathrm{S}_1\left(\left(\frac{\mathrm{S}_1}{\mathrm{S}_2}\right)^2 - 1\right)\dot{\delta}^2 - \rho \mathrm{g} \mathrm{H} \mathrm{S}_1}$$

Problem 10.4 – *A shear force* f^* *per unit of surface acts on an rigid plate of indefinite size with density* ρ^* *and thickness* t. *The plate slides at a velocity* v^* *in the longitudinal direction on a plane inclined at an angle* α *with respect to the horizontal longitudinal direction. Between the plate and the inclined plane are two distinct and immiscible Newtonian fluids with viscosities* μ_1 *and* μ_2, *which are distributed into two layers with the same thickness* h.

a) *Establish the generic form of the pressure and velocity fields and argue the hypotheses used to determine them.*
b) *Integrate the corresponding differential equations and obtain, except for the integration constants, the distribution of pressures and velocities in each fluid.*
c) *Indicate and justify the boundary conditions that must be applied to determine the above integration constants.*
d) *Completely determine the pressure and velocity fields as well as the stresses in each fluid. Plot the distribution of each variable (velocities, pressure and*

stresses) on a cross-section such as $\mathrm{A}-\mathrm{A}'$*, indicating the most significant values.*

e) Obtain the value of v^* *in terms of* f^* *and the volume flow rate* q *that flows through a semicircular section such as* $\mathrm{B}-\mathrm{B}'$.



HYPOTHESES:

1) *Both fluids are incompressible.*
2) *Assume a steady-state regime.*
3) *The body forces of the fluids can be neglected.*
4) *The atmospheric pressure is negligible.*

Solution

a) Note that the z-dimension, perpendicular to the plane of the paper, does not intervene in the problem. Thus, the pressure and velocity fields are reduced to

$$p = p(x,y) \quad \text{and} \quad \mathbf{v} = \mathbf{v}(x,y) \stackrel{not}{\equiv} \left[\mathrm{v}_x(x,y)\ ,\ 0,\ 0 \right]^T .$$

In fact, v_x does not depend on x either since the velocity should be the same in all the cross-sections of the type $\mathrm{A}-\mathrm{A}'$. If this is not acknowledged *a priori*, the *mass continuity equation* (5.22) may be imposed, considering the incompressible nature of the fluids, as follows.

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0 \quad \Longrightarrow \quad \nabla \cdot \mathbf{v} = 0 \quad \Longrightarrow \quad \frac{\partial \mathrm{v}_x}{\partial x} + \frac{\partial \mathrm{v}_y}{\partial y} = 0\ , \ \text{but} \ \ \mathrm{v}_y = 0$$

$$\Longrightarrow \quad \frac{\partial \mathrm{v}_x}{\partial x} = 0 \quad \Longrightarrow \quad \mathrm{v}_x = \mathrm{v}_x(y) \quad \Longrightarrow \quad \mathbf{v} \stackrel{not}{\equiv} \left[\mathrm{v}_x(y)\ ,\ 0,\ 0 \right]^T$$

Therefore, the pressure and velocity fields are

$$\boxed{p = p(x,y) \qquad \text{and} \qquad \mathbf{v} \stackrel{not}{\equiv} \left[\mathrm{v}_x(y)\ ,\ 0\ ,\ 0 \right]^T} \ .$$

b) The components of the Navier-Stokes differential equation (10.85) in Cartesian coordinates must be integrated to obtain the expressions of $\mathbf{v}$ and p,

$$\begin{aligned} x-\text{component} &\implies 0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 \mathrm{v}_x}{\partial y^2} \ , \\ y-\text{component} &\implies 0 = -\frac{\partial p}{\partial y} \quad \Rightarrow \quad p = p(x) \ . \end{aligned}$$

The pressure p only depends on x and the component v_x of $\mathbf{v}$ only depends on y. Therefore, the partial derivatives in the equation for the x-component can be replaced by total derivatives. In this way, an equality of functions is obtained in which the pressure term depends solely on x whilst the velocity term depends exclusively on y. Consequently, these terms must be constant.

$$\left. \begin{array}{l} \dfrac{dp}{dx} = \mu \dfrac{d^2 \mathrm{v}_x}{dy^2} \\ f(x) = f(y) \end{array} \right\} \implies \frac{dp}{dx} = \mu \frac{d^2 \mathrm{v}_x}{dy^2} = k = const.$$

$$\frac{dp}{dx} = k \implies \boxed{p(x) = kx + \mathrm{A}}$$

$$\frac{d^2 \mathrm{v}_x}{dy^2} = \frac{k}{\mu} \implies \boxed{\mathrm{v}_x(y) = \frac{k}{2\mu} y^2 + \mathrm{B}y + \mathrm{C}}$$

To determine the stresses, the constitutive equation in Cartesian coordinates of Table10.2 is used,

$$\left. \begin{array}{l} \sigma_x = \sigma_y = \sigma_z = -p(x) \\ \tau_{xy} = \tau_{yx} = \mu \dfrac{\partial \mathrm{v}_x(y)}{\partial y} \end{array} \right\} \implies$$

$$\boxed{\boldsymbol{\sigma}(x,y) \stackrel{not}{\equiv} \begin{bmatrix} -kx - \mathrm{A} & \mu \left(\dfrac{k}{\mu} y + \mathrm{B} \right) & 0 \\ \mu \left(\dfrac{k}{\mu} y + \mathrm{B} \right) & -kx - \mathrm{A} & 0 \\ 0 & 0 & -kx - \mathrm{A} \end{bmatrix}}$$

where the constants in these expressions (k, A, B, C) are different for each fluid.

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c) The boundary conditions that must be applied in this problem are:

VELOCITY BOUNDARY CONDITIONS

1. $\left. \mathrm{v}_x^1(y) \right|_{y=\mathrm{h}} = \mathrm{v}^*$, since the plate moves at a velocity v^* and $\mu > 0$.
2. $\left. \mathrm{v}_x^2(y) \right|_{y=-\mathrm{h}} = 0$, since the inclined plane does not move and $\mu > 0$.
3. $\left. \mathrm{v}_x^1(y) \right|_{y=0} = \left. \mathrm{v}_x^2(y) \right|_{y=0}$, which is the continuity condition for $\mathbf{v}$ at the interface between the two fluids.

PRESSURE BOUNDARY CONDITIONS

In the fluid with density μ_1, the pressure is prescribed for $y = \mathrm{h}$ or, directly, since p does not depend on y (because the weight of the fluid is neglected), the pressure p^1 is prescribed in the whole domain of this fluid. The value of p^1 corresponds to the pressure that the plate exerts on the fluid with density μ_1, which is the projection of the plate's weight in the direction of the y-axis.

4. $W = \rho^* g t$ is the weight of a section of the plate with unit length, according to the x- and z-axis. Here, $p_{atm} = 0$ has been considered.
5. $p^1 = \rho^* g t \cos\alpha$, $\forall \mathbf{x}$ is the projection on the y-axis. Since a unit length has been considered, the weight is directly the exerted pressure.
6. $\left. p^1 \right|_{y=0} = \left. p^2 \right|_{y=0}$ $\forall \mathbf{x}$ is the continuity condition for the pressure in the interface between the two fluids.

STRESS BOUNDARY CONDITIONS

The continuity condition for stresses that must be imposed in the interface between the two fluids does not affect the complete tensor $\boldsymbol{\sigma}$. Instead, only the traction vector $\mathbf{t}$ is affected. The condition

$$\left. \mathbf{t}^1 \right|_{y=0} = - \left. \mathbf{t}^2 \right|_{y=0}$$

must be satisfied. Considering that the unit normal vector $\mathbf{n}$ is the exterior normal vector, then

$$\mathbf{n}^1 \stackrel{not}{\equiv} [0,\ -1,\ 0]^T \qquad \text{and} \qquad \mathbf{n}^2 \stackrel{not}{\equiv} [0,\ 1,\ 0]^T \ .$$

Hence, the shear stresses must satisfy:

7. $\left. \tau_{xy}^1 \right|_{y=0} = \left. \tau_{xy}^2 \right|_{y=0}$

d) Only 7 boundary conditions have been established and 8 constants must be determined, but since some equations include two constants, it suffices. Replacing the expressions of p, $\mathbf{v}$ and $\boldsymbol{\sigma}$ in the boundary conditions results in:

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$$\frac{k_1}{2\mu_1}\mathrm{h}^2 + \mathrm{B}_1\mathrm{h} + \mathrm{C}_1 = \mathrm{v}^*$$

$$\frac{k_2}{2\mu_2}\mathrm{h}^2 - \mathrm{B}_2\mathrm{h} + \mathrm{C}_2 = 0$$

$$\mathrm{C}_1 = \mathrm{C}_2$$

$$k_1 x + \mathrm{A}_1 = \rho^* g t \cos\alpha \ , \ \forall x \implies \begin{cases} k_1 = 0 \\ \mathrm{A}_1 = \rho^* g t \cos\alpha \end{cases}$$

$$k_1 x + \mathrm{A}_1 = k_2 x + \mathrm{A}_2 \ , \ \forall x \implies \begin{cases} k_1 = \mathrm{k}_2 = 0 \\ \mathrm{A}_1 = \mathrm{A}_2 = \rho^* g t \cos\alpha \end{cases}$$

$$\left.\begin{array}{c} y = 0 \\ k_1 = \mathrm{k}_2 = 0 \end{array}\right\} \implies \mu_1 \mathrm{B}_1 = \mu_2 \mathrm{B}_2$$

Solving and replacing these values in the expressions for the pressure, velocity and stress obtained in *b)* results in

$$\boxed{\begin{aligned} \mathrm{v}_x^1(y) &= \frac{\mathrm{v}^*}{1+\dfrac{\mu_1}{\mu_2}}\left(\frac{y}{\mathrm{h}} + \frac{\mu_1}{\mu_2}\right) \\ \mathrm{v}_x^2(y) &= \frac{\mu_1}{\mu_2}\frac{\mathrm{v}^*}{1+\dfrac{\mu_1}{\mu_2}}\left(\frac{y}{\mathrm{h}} + 1\right) \end{aligned}}$$

$$\boxed{\begin{aligned} & p^1 = p^2 = \rho^* g t \cos\alpha = const. \\ & \tau_{xy}^1 = \tau_{xy}^2 = \mu_1 \frac{\mathrm{v}^*}{\mathrm{h}\left(1+\dfrac{\mu_1}{\mu_2}\right)} = const. \end{aligned}}$$

e) To determine the relation between f^* and v^*, the equilibrium of forces on a unit element of the plate is posed. Three forces act on this element:

1) The force f^* that pushes the plate in the positive direction of the x-axis.
2) The projection of the plate's own weight in the direction of the x-axis. This force pulls the plate in the negative direction of the x-axis.
3) The shear force of the fluid on the plate, which resists the motion of the plate and, thus, acts in the negative direction of the x-axis.

To determine the sign criterion of this last force, the stresses acting on an element of the fluid domain are drawn:

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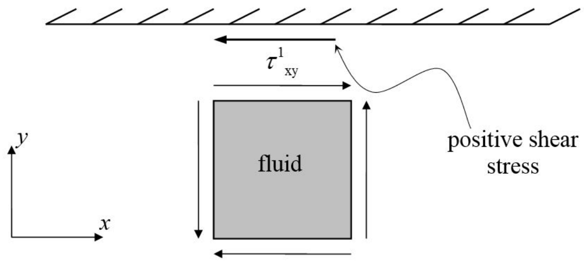

Posing the equilibrium of forces yields

$$f^* = \rho^* g t \sin\alpha + \mu_1 \frac{v^*}{\left(1+\dfrac{\mu_1}{\mu_2}\right) h}$$

and, isolating v^*, produces the velocity in terms of the shear force,

$$\boxed{v^* = \frac{h}{\mu_1}\left(1+\frac{\mu_1}{\mu_2}\right)(f^* - \rho^* g t \sin\alpha)} \; .$$

To compute the volume flow rate that flows across the surface $B-B'$, one must take into account that the fluids are incompressible and, thus, the volume flow rate crossing the curved surface is the same as if a straight segment joining B and B' was considered, that is,

$$q = \int\limits_{BB'\,curved} \mathbf{v}\cdot\mathbf{n}\, dS = \int\limits_{BB'\,straight} \mathbf{v}\cdot\mathbf{n}\, dS = \int\limits_{-h}^{h} v_x(y)\; dy \; .$$

Then, replacing the expressions found in *d)* for the velocity v_x and integrating results in the volume flow

$$\boxed{q = v^* h\left(\frac{1}{2}+\frac{\mu_1}{\mu_1+\mu_2}\right)} \; .$$

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Problem 10.5 – *Figure A shows the cross-section of a damper of indefinite length composed of a piston* $\mathrm{ABA'B'}$ *that slides inside a container filled with an incompressible Newtonian fluid with viscosity* μ. *The piston descends at a velocity* $\dot{\delta}(t)$, *producing a lateral flow of fluid between the piston and the walls (see Figure B).*

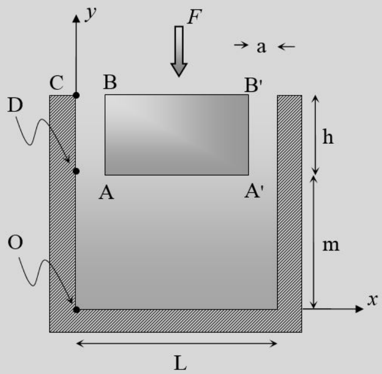

Figure A

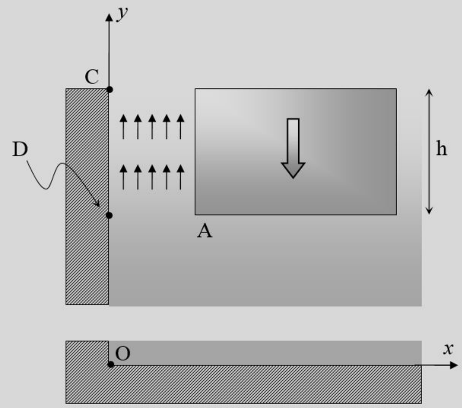

Figure B

a) *Determine the pressure and velocity fields in the zone of the fluid shown in Figure B (zone* ABCD*), except for the integration constants.*
b) *Indicate and justify the boundary conditions that must be applied to determine the above integration constants.*
c) *Completely determine the pressure and velocity fields in zone* ABCD *of the fluid.*
d) *Determine the expression of the stress tensor in zone* ABCD *of the fluid.*
e) *Assuming that the stress* σ_y *in the surface* $\mathrm{A}-\mathrm{A'}$ *is uniform and equal to the stress in point* A, *prove there exists a relation of the form* $\mathrm{F} = \eta\,\dot{\delta}(t)$, *where* F *is the force per unit of length applied on the piston and* $\dot{\delta}(t)$ *is the velocity of descent of said piston. Compute the value of* η.

HYPOTHESES:

1) *The body forces of the fluid can be neglected.*
2) *The weight of the piston can be neglected.*
3) *Assume a steady-state regime.*
4) *The atmospheric pressure is negligible.*

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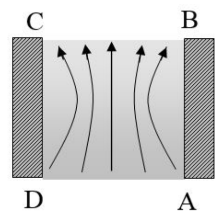
Solution

a) The problem is not defined in the z-direction, the direction perpendicular to the plane of the paper, and, thus, is independent of the z variable. Then, consider the bidimensional situation

$$\mathbf{v} \stackrel{not}{\equiv} \left[\mathrm{v}_x(x,y)\ ,\ \ \mathrm{v}_y(x,y)\ ,\ \ 0\right]^T \ .$$

On the other hand, $\mathrm{v}_x = 0$ must be satisfied on the walls AB and CD, owing to the impermeability condition (a fluid cannot penetrate through a solid).
For convenience, an additional approximate hypothesis is introduced to further simplify the problem: it will be assumed that $\mathrm{v}_x = 0$ in all the zone ABCD of the fluid. However, the streamlines have, in fact, the approximate form shown in the figure to the right. It is even possible that vortexes are formed in this region if there is a high velocity.



In short, the velocity and pressure fields are assumed to be of the form

$$\mathbf{v} \stackrel{not}{\equiv} \left[0\ ,\ \ \mathrm{v}_y(x,y)\ ,\ \ 0\right]^T \quad \text{and} \quad p = p(x,y) \ .$$

The *mass continuity equation* (5.22) for an incompressible fluid ($\rho = const.$) is reduced to $\nabla \cdot \mathbf{v} = 0$ and, for this particular problem,

$$\frac{\partial \mathrm{v}_y}{\partial y} = 0 \quad \Longrightarrow \quad \mathrm{v}_y = \mathrm{v}_y(x) \ .$$

Then, the velocity remains constant for a same vertical line since the spatial description of the velocity does not depend on y.
Now, the *Navier-Stokes equation* (10.85) in Cartesian coordinates is imposed, considering the hypotheses given in the statement of the problem and the additional assumptions described above. Since the problem is bidimensional, the z-component of the equation does not provide information.

$$\left.\begin{array}{r} 0 = -\dfrac{\partial p}{\partial x} \quad \Longrightarrow \quad p = p(y) \\ 0 = -\dfrac{\partial p}{\partial y} + \mu \dfrac{\partial^2 \mathrm{v}_y}{\partial x^2} \end{array}\right\} \quad \Longrightarrow \quad \frac{\partial p}{\partial y} = \mu \frac{\partial^2 \mathrm{v}_y}{\partial x^2}$$

The term in the right-hand side of the equation depends solely on x and the one in the left-hand side depends only on y, therefore both terms must be constant.

$$\frac{\partial p}{\partial y} = k \quad \Longrightarrow \quad \boxed{p = ky + C_1}$$

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$$\mu \frac{\partial^2 \mathrm{v}_y}{\partial x^2} = k \quad \Longrightarrow \quad \frac{\partial \mathrm{v}_y}{\partial x} = \frac{k}{\mu} x + C_2 \quad \Longrightarrow \quad \boxed{\mathrm{v}_y(x) = \frac{1}{2}\frac{k}{\mu} x^2 + C_2\, x + C_3}$$

b) The boundary conditions that must be applied in this problem are:

VELOCITY BOUNDARY CONDITIONS

1. $\mathrm{v}_y(x)\Big|_{x=0} = 0\,,\ \forall y$, since there is no relative displacement of the fluid with respect to the wall.
2. $\mathrm{v}_y(x)\Big|_{x=\mathrm{a}} = -\dot{\delta}\,,\ \forall y$, again, since there is no relative displacement.

PRESSURE BOUNDARY CONDITIONS

3. $p(y)\Big|_{y=\mathrm{m}+\mathrm{h}} = p_{atm} = 0$

VOLUME FLOW RATE BOUNDARY CONDITIONS

In an incompressible fluid the entrance and exit volume flow rates are the same, $Q_{in} = Q_{out}$, where

$$Q = \int_S \mathbf{v} \cdot \mathbf{n}\, dS\,.$$

The piston descends at a velocity $\dot{\delta}$ and, thus, its cross-section is introduced into the fluid, pushing it upwards. Then, the entrance volume flow rate can be defined as (velocity · surface),

$$Q_{in} = \dot{\delta} \cdot \mathrm{L}\,.$$

On the other hand, the exit volume flow rate, flowing in the space left between the piston and the lateral walls, is determined by means of the general expression for volume flow rate

$$Q_{out} = 2\int_{S_{\mathrm{a}}} \mathbf{v} \cdot \mathbf{n}\, dS = 2\int_0^{\mathrm{a}} \mathrm{v}_y(x)\ dx\,.$$

Finally, equating the entrance and exit volume flow rates results in:

4. $$2\int_0^a \mathrm{v}_y(x)\, dx = \dot{\delta}\,\mathrm{L}$$

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c) The constants are determined by means of the boundary conditions described in *b)* as follows:

$$\mathrm{v}_y(x)\Big|_{x=0}=0 \quad \Longrightarrow \quad \mathrm{v}_y(0)=C_3 \quad \Longrightarrow \quad \boxed{C_3=0}$$

$$\mathrm{v}_y(x)\Big|_{x=\mathrm{a}}=-\dot{\delta} \quad \Longrightarrow \quad \mathrm{v}_y(\mathrm{a})=\frac{1}{2}\frac{k}{\mu}\mathrm{a}^2+C_2\,\mathrm{a} \quad \Longrightarrow \quad C_2=-\frac{\dot{\delta}}{\mathrm{a}}-\frac{k}{2\mu}\mathrm{a}$$

$$2\int_0^{\mathrm{a}}\mathrm{v}_y(x)\,dx=2\int_0^{\mathrm{a}}\left(\frac{k}{2\mu}x^2+C_2\,x\right)dx=2\left(\frac{k}{2\mu}\frac{\mathrm{a}^3}{3}+C_2\frac{\mathrm{a}^2}{2}\right)=\dot{\delta}\mathrm{L} \quad \Longrightarrow$$

$$\boxed{k=-\frac{6\mu}{\mathrm{a}^3}\dot{\delta}\,(\mathrm{a}+\mathrm{L})} \qquad \text{and} \qquad \boxed{C_2=\frac{\dot{\delta}}{\mathrm{a}}\left(2+3\frac{\mathrm{L}}{\mathrm{a}}\right)}$$

$$p(y)\Big|_{y=\mathrm{m}+\mathrm{h}}=0 \quad \Longrightarrow \quad k(\mathrm{m}+\mathrm{h})+C_1=0 \quad \Longrightarrow \quad \boxed{C_1=\frac{6\mu}{\mathrm{a}^3}\dot{\delta}\,(\mathrm{a}+\mathrm{L})\,(\mathrm{m}+\mathrm{h})}$$

Introducing these values in the expressions for the pressure and velocity obtained in *a)* results in:

$$\boxed{\begin{aligned} p=p(y)&=\frac{6\mu}{\mathrm{a}^3}\dot{\delta}\,(\mathrm{a}+\mathrm{L})\,(\mathrm{m}+\mathrm{h}-y)\\ \mathrm{v}_y(x)&=-\frac{3}{\mathrm{a}^3}(\mathrm{a}+\mathrm{L})\,\dot{\delta}x^2+\frac{\dot{\delta}}{\mathrm{a}}\left(2+3\frac{\mathrm{L}}{\mathrm{a}}\right)x \end{aligned}}$$

d) The stresses in zone ABCD of the fluid are computed by means of the constitutive equation in Cartesian coordinates of Table 10.2. Using the expressions for the pressure and velocity fields obtained in *c)* yields

$$\boxed{\boldsymbol{\sigma}\overset{not}{\equiv}\begin{bmatrix} -p & \mu\dfrac{\partial \mathrm{v}_y}{\partial x} & 0\\ \mu\dfrac{\partial \mathrm{v}_y}{\partial x} & -p & 0\\ 0 & 0 & -p\end{bmatrix}}$$

where

$$\mu\frac{\partial \mathrm{v}_y}{\partial x}=\mu\dot{\delta}\left(-6\frac{x}{\mathrm{a}^3}(\mathrm{a}+\mathrm{L})+3\frac{\mathrm{L}}{\mathrm{a}^2}+\frac{2}{\mathrm{a}}\right).$$

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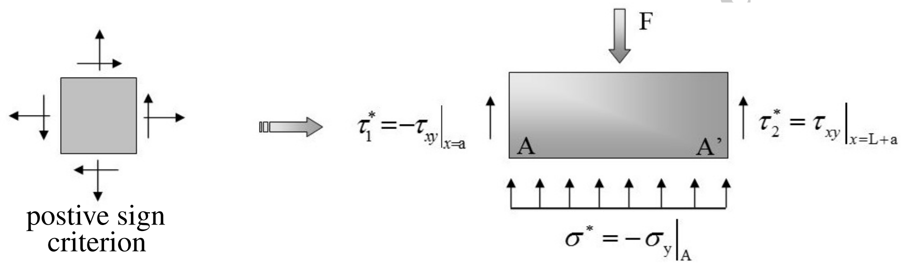
COMMENT

When the piston descends, the steady-state regime hypothesis is, in fact, not completely rigorous since, at some point, the piston will reach the lowest point of its trajectory and the flow will vary. To be able to apply this hypothesis, either $(\mathrm{m}+\mathrm{h})$ must be a very large length or $\dot{\delta}$ must be a very low velocity.

e) The stresses acting on the piston must be computed to obtain the resultant forces and, then, the equilibrium of forces is applied to determine the expression for F. These stresses are:



The stresses in the inferior surface of the piston are

$$\sigma^* = -\sigma_y\Big|_{\mathrm{A}} = p(y)\Big|_{y=\mathrm{m}} = k\,\mathrm{m} + C_1 \quad \Longrightarrow \quad \sigma^* = \frac{6\mu}{\mathrm{a}^3}(\mathrm{a}+\mathrm{L})\,\mathrm{h}\dot{\delta}\ .$$

In the lateral surfaces, due to symmetry, $\tau_1^* = \tau_2^*$ and, therefore, only τ_1^* needs to be computed,

$$\tau_1^* = \tau_2^* = -\tau_{xy}\Big|_{x=\mathrm{a}} = -\mu\dot{\delta}\left(-6\frac{\mathrm{a}}{\mathrm{a}^3}(\mathrm{a}+\mathrm{L}) + 3\frac{\mathrm{L}}{\mathrm{a}^2} + \frac{2}{\mathrm{a}}\right) \quad \Longrightarrow$$

$$\tau_1^* = \tau_2^* = \frac{\mu\dot{\delta}}{\mathrm{a}}\left(3\frac{\mathrm{L}}{\mathrm{a}} + 4\right)\ .$$

Imposing the equilibrium of forces (since $\dot{\delta}$ is a constant velocity),

$$\mathrm{F} = \mathrm{L}\sigma^* + \mathrm{h}\,\tau_1^* + \mathrm{h}\,\tau_2^* \quad \Longrightarrow$$

$$\boxed{\mathrm{F} = \eta\,\dot{\delta} \qquad \text{with} \quad \eta = \frac{2\mu\mathrm{h}}{\mathrm{a}}\left(3\frac{\mathrm{L}^2}{\mathrm{a}^2} + 6\frac{\mathrm{L}}{\mathrm{a}} + 4\right)}$$

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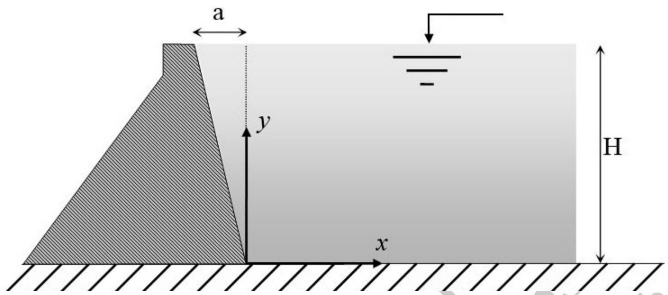
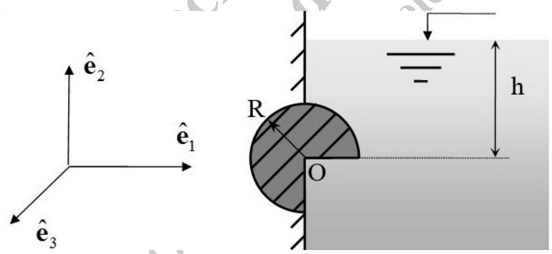
EXERCISES

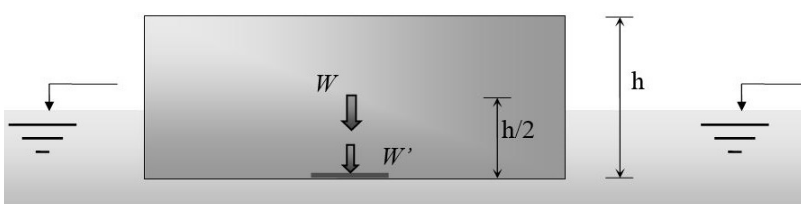
10.1 – *Compute the horizontal and vertical components of the resultant of the actions, per unit of length, exerted by the water on the gravity dam shown in the figure.*



10.2 – *The wall of a tank has a valve that rotates about point* O *as shown in the figure. Compute the resultant force and moment, per unit of thickness, that the fluid exerts on the valve. The weight of the valve can be neglected.*



10.3 – *Determine the weight of the ballast* W′ *required at the bottom of the crate shown in the figure, whose weight is* W*, such that it is maintained afloat in stable equilibrium.*



NOTE*: The water has a density* ρ *and the weights are per unit of thickness.*

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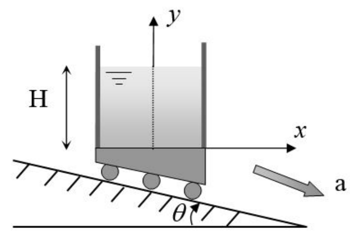
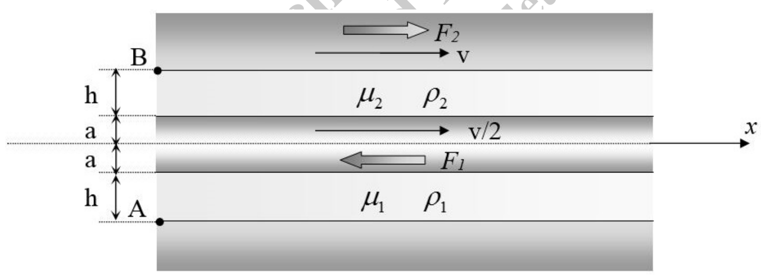
10.4 – *A container filled with water up to a height* H *is placed on an inclined plane with angle* θ *and dropped such that it slides down this plane with a constant acceleration value* a*. Determine the distribution of pressures and the equation of the free surface in terms of* a*,* H*,* θ *and the atmospheric pressure* p_a.



10.5 – *A plate of indefinite size and thickness* 2a *separates two incompressible Newtonian fluids that move between two rigid boundaries of indefinite length placed at a distance* h *from the plate, as shown in the figure. The plate and the top boundary move at velocities* v/2 *and* v*, respectively. Determine:*

a) *The pressure, velocity and stress fields in terms of the integration constants.*
b) *The integration constants, by applying the adequate boundary conditions.*
c) *The forces per unit of surface* F_1 *and* F_2 *exerted on the plate and the top boundary needed to produce the described motion.*
d) *The dissipated energy, per unit of time and of surface perpendicular to the plane of the paper, due to viscous effects.*



Additional hypotheses:

1) *The pressures at points* A *and* B *are* p_A *and* p_B*, respectively.*
2) *Consider a steady-state regime.*
3) *Due to the indefinite character of the x-direction, the flow and its properties can be considered to be invariable in this direction.*

10.6 – *A volume flow rate Q of an incompressible isotropic Newtonian fluid flows in steady-state regime between the plate and the horizontal surface shown in the figure. The plate is kept horizontal and immobile by means of a force with horizontal and vertical components* H *and* V*, respectively, acting on an appropriate point of said plate. Determine:*

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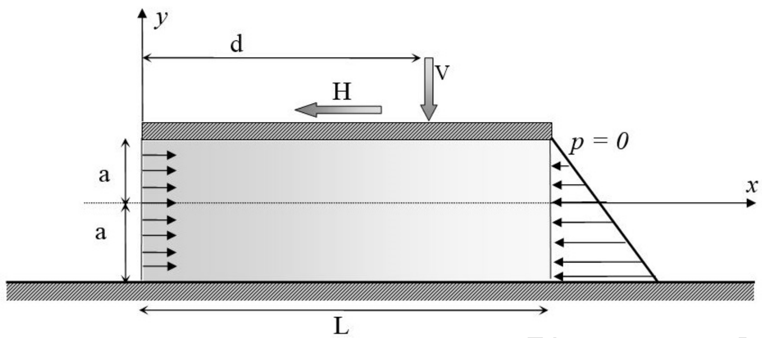
a) The pressure and velocity fields.

b) The value of the vertical component of the force and the distance d *from the origin to its application point, such that the plate does not rotate.*

c) The value of the horizontal component of the force.



Additional hypotheses:

1) The flow is assumed to be parallel to the x-y plane.

2) Inertial forces can be neglected.

3) The volume flow rate, Q, and the components of the force, V *and* H*, are considered per unit of length in the z-direction.*

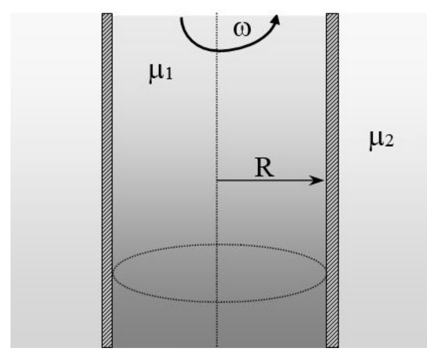
4) The weight of the plate and the atmospheric pressure are negligible.

10.7 – *A cylindrical shell of indefinite length and internal radius* R *rotates in steady-state regime at an angular velocity* ω *inside an infinite domain occupied by an incompressible Newtonian fluid with viscosity* μ_2*. A different incompressible Newtonian fluid, with viscosity* μ_1*, is contained inside the cylindrical shell. Determine:*

a) The pressure and velocity fields of the internal fluid.

b) The pressure and velocity fields of the external fluid.

c) The moment that must be applied on the cylindrical shell to maintain its velocity.



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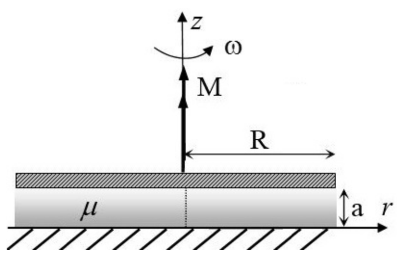
10.8 – *A disc of radius* R *rotates with a constant angular velocity* ω *at a distance* a *from a horizontal surface. Between the disc and the surface is an incompressible Newtonian fluid with viscosity* μ. *Determine:*

a) *The velocity field of the fluid in terms of the integration constants.*
b) *The value of the integration constants, by applying the appropriate boundary conditions, and the complete expression of the velocity field.*
c) *The pressure field and the shear stress* $\tau_{z\theta}$.
d) *The value of the moment* M *that must be applied on the axis of the disc to maintain the described motion.*

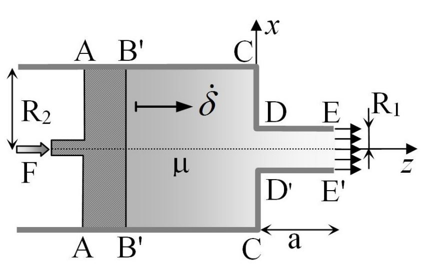


Additional hypotheses:

1) *The rotation of the disc is sufficiently slow to neglect the inertial forces.*
2) *The effect of the lateral walls (fluid-wall friction effects) can be neglected.*
3) *The velocity field varies linearly with the distance to the inferior surface.*
4) *Assume a steady-state regime.*

10.9 – *The cross-section of a cylindrical piston* $ABA'B'$ *that slides inside a container filled with an incompressible Newtonian fluid with viscosity* μ *is shown in the figure. The motion of the piston, at a velocity* $\dot{\delta}$, *causes the fluid to flow through the pipe* $DED'E'$.

a) *Determine the pressure and velocity fields of the fluid in zone* $DED'E'$ *in terms of the integration constants.*
b) *Indicate and justify the boundary conditions that must be applied to determine the value of the integration constants. Determine these constants and the complete expressions of the pressure and velocity fields.*
c) *Compute the stresses in zone* $DED'E'$ *of the fluid.*
d) *Assuming that the stress normal to the surface* BB' *in the fluid is constant and equal to the pressure in points* D *and* D', *prove there exists a relation between the force* F *applied on the piston and its velocity* $\dot{\delta}$, *and that said relation is of the form* $F = \eta\dot{\delta}$. *Determine the value of* η.



Additional hypotheses:

1) *The body forces of the fluid and weight of the piston can be neglected.*
2) *Assume a steady-state regime.*
3) *The atmospheric pressure is negligible.*

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Chapter 11
Variational Principles

11.1 Governing Equations

Variational calculus is a mathematical tool that allows working with the so-called *integral* or *weak form* of the governing differential equations of a problem. Given a system of differential equations, which must be verified in local form (point by point) for a certain domain, the variational principles allow obtaining an integral or weak formulation (global, in the domain), whose imposition, nonetheless, guarantees that the aforementioned differential equations are satisfied. Integral formulations are of particular interest when treating and solving the problem by means of numerical methods.

11.1.1 Functionals. Functional Derivatives

Definition 11.1. The *functional* $\mathbb{F}(\mathbf{u})$ is a mapping of the function space $\mathbb{X}$ onto the set of real numbers $\mathbb{R}$,

$$\mathbb{F}(\mathbf{u}) : \mathbb{X} \to \mathbb{R} \quad \text{where} \quad \mathbb{X} := \left\{ \mathbf{u}(\mathbf{x}) \;\middle|\; \mathbf{u}(\mathbf{x}) : \mathbb{R}^3 \supset \Omega \to \mathbb{R}^m \right\} .$$

In other words, the functional $\mathbb{F}(\mathbf{u})$ is a function that takes an element $\mathbf{u}(\mathbf{x})$ (a scalar, vector or tensor function defined in a domain Ω of $\mathbb{R}^3$ or, in general, $\mathbb{R}^n$) of a function space $\mathbb{X}$ as its input argument and returns a real number.

With certain language misuse, one could say that the functional $\mathbb{F}(\mathbf{u})$ is a *scalar function whose arguments are functions* $\mathbf{u}(\mathbf{x})$.

Example 11.1 – *Consider an interval* $\Omega \equiv [a,b] \in \mathbb{R}$ *and the space* $\mathbb{X}$ *constituted by all the real functions with real variables in the interval* $[a,b]$ $(\mathbf{u}(\mathbf{x}) : [a,b] \to \mathbb{R})$ *with first derivatives* $u'(x)$ *that are integrable in this interval. Examples of possible functionals are*

$$\mathbb{F}(u) = \int_a^b u(x)\,dx\,, \qquad \mathbb{G}(u) = \int_a^b u'(x)\,dx$$

$$\text{and} \quad \mathbb{H}(u) = \int_a^b F\left(x, u(x), u'(x)\right)\,dx\,.$$

Definition 11.2. Consider the (scalar, vector or tensor) function space $\mathbb{X} := \left\{ \mathbf{u}(\mathbf{x}) \;\middle|\; \mathbf{u}(\mathbf{x}) : \mathbb{R}^3 \supset \Omega \to \mathbb{R}^n \right\}$ on a domain Ω and a functional $\mathbb{F}(\bullet) : \mathbb{X} \to \mathbb{R}$.

Consider the two functions $\mathbf{u}$, $\boldsymbol{\eta} \in \mathbb{X}$ and the (perturbation) parameter $\varepsilon \in \mathbb{R}$. Then, the function $\mathbf{u} + \varepsilon\,\boldsymbol{\eta} \in \mathbb{X}$, can be interpreted as a *perturbed function* of the function $\mathbf{u}$ in the direction $\boldsymbol{\eta}$.

The *Gateaux variation (or Gateaux derivative) of the functional* $\mathbb{F}(\mathbf{u})$ in the direction $\boldsymbol{\eta}$ is defined as

$$\delta\mathbb{F}(\mathbf{u};\boldsymbol{\eta}) \stackrel{def}{=} \frac{d}{d\varepsilon}\mathbb{F}(\mathbf{u} + \varepsilon\,\boldsymbol{\eta})\Big|_{\varepsilon=0}.$$

Remark 11.1. The direction with respect to which the variation is taken is often denoted as $\boldsymbol{\eta} \stackrel{not}{=} \delta\mathbf{u}$. This notation will be used frequently in the remainder of the chapter. Do not confuse $\delta\mathbf{u}(\mathbf{x})$ with the differential $d\mathbf{u}(\mathbf{x})$ (in an infinitesimal calculus context) of a function $\mathbf{u}(\mathbf{x})$. However, obtaining the Gateaux variation of a functional has in certain cases *the same formalism* as the ordinary differentiation of functions and, thus, the risk of confusion (see Example 11.2).

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Example 11.2 – *Obtain the Gateaux derivative of the functional*

$$\mathbb{F}(\mathbf{u}) \stackrel{def}{=} \int_{\Omega} \phi(\mathbf{u})\, d\Omega + \int_{\partial\Omega} \varphi(\mathbf{u})\, d\Gamma \ .$$

Solution

$$\begin{aligned}
\delta\mathbb{F}(\mathbf{u};\,\delta\mathbf{u}) = \frac{d}{d\varepsilon}\mathbb{F}(\mathbf{u}+\varepsilon\delta\mathbf{u})\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_{\Omega}\phi(\mathbf{u}+\varepsilon\delta\mathbf{u})\, d\Omega\Big|_{\varepsilon=0} + \\
+ \frac{d}{d\varepsilon}\int_{\partial\Omega}\varphi(\mathbf{u}+\varepsilon\delta\mathbf{u})\, d\Gamma\Big|_{\varepsilon=0} =
\end{aligned}$$

$$\begin{aligned}
= \left[\int_{\Omega} \frac{\partial\phi(\mathbf{u}+\varepsilon\delta\mathbf{u})}{\partial\mathbf{u}} \cdot \underbrace{\frac{d(\mathbf{u}+\varepsilon\delta\mathbf{u})}{d\varepsilon}}_{\delta\mathbf{u}} d\Omega\right]_{\varepsilon=0} + \\
+ \left[\int_{\partial\Omega} \frac{\partial\varphi(\mathbf{u}+\varepsilon\delta\mathbf{u})}{\partial\mathbf{u}} \cdot \underbrace{\frac{d(\mathbf{u}+\varepsilon\delta\mathbf{u})}{d\varepsilon}}_{\delta\mathbf{u}} d\Gamma\right]_{\varepsilon=0} \quad \Longrightarrow
\end{aligned}$$

$$\boxed{\delta\left[\int_{\Omega}\phi(\mathbf{u})\, d\Omega + \int_{\partial\Omega}\varphi(\mathbf{u})\, d\Gamma\right] = \int_{\Omega}\frac{\partial\phi(\mathbf{u})}{\partial\mathbf{u}}\cdot\delta\mathbf{u}\, d\Omega + \int_{\partial\Omega}\frac{\partial\varphi(\mathbf{u})}{\partial\mathbf{u}}\cdot\delta\mathbf{u}\, d\Gamma}$$

Note, in this case, the formal similarity of obtaining the Gateaux derivative of the functional with the differentiation of functions.

Consider now a domain $\Omega \subset \mathbb{R}^3$, its boundary $\partial\Omega = \Gamma_u \bigcup \Gamma_\sigma$ with $\Gamma_u \bigcap \Gamma_\sigma = \emptyset$ (see Figure 11.1) and the space $\mathbb{V}$ of the functions $\mathbf{u}(\mathbf{x})$ defined on Ω and *such that they take a prescribed value* $\mathbf{u}^*(\mathbf{x})$ *at the boundary* Γ_u:

$$\mathbb{V} := \left\{ \mathbf{u}(\mathbf{x}) \;\middle|\; \mathbf{u}(\mathbf{x}) : \Omega \to \mathbb{R}^m \quad ; \quad \mathbf{u}(\mathbf{x})\Big|_{\mathbf{x}\in\Gamma_u} = \mathbf{u}^*(\mathbf{x}) \right\} \tag{11.1}$$

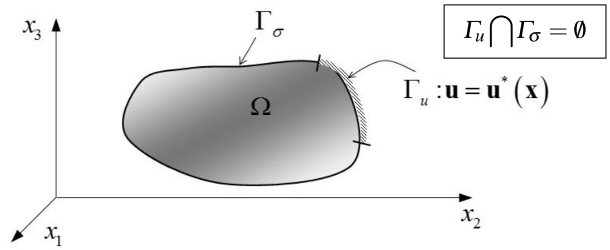

Figure 11.1: Definition of the domain $\Omega \subset \mathbb{R}^3$.

> *Remark 11.2.* When computing the Gateaux derivative, a condition, established in the definition itself, on the perturbation $\boldsymbol{\eta} \equiv \delta\mathbf{u}$ is that the perturbed function $\mathbf{u} + \varepsilon\, \delta\mathbf{u}$ must belong to the same function space $\mathbb{V}$ $(\mathbf{u} + \varepsilon\, \delta\mathbf{u} \in \mathbb{V})$. In this case, if $\mathbf{u} + \varepsilon\, \delta\mathbf{u} \in \mathbb{V}$,
>
> $$(\mathbf{u} + \varepsilon\, \delta\mathbf{u})\Big|_{\mathbf{x}\in\Gamma_u} = \mathbf{u}^* \implies \underbrace{\mathbf{u}\Big|_{\mathbf{x}\in\Gamma_u}}_{\mathbf{u}^*} + \varepsilon\, \delta\mathbf{u}\Big|_{\mathbf{x}\in\Gamma_u} = \mathbf{u}^* \implies \varepsilon\, \delta\mathbf{u}\Big|_{\mathbf{x}\in\Gamma_u} = \mathbf{0}$$
>
> and the perturbation $\delta\mathbf{u}$ must satisfy $\delta\mathbf{u}\Big|_{\mathbf{x}\in\Gamma_u} = \mathbf{0}$.

Based on the family of functions (11.1), consider now the following family of functionals

$$\boxed{\mathbb{F}(\mathbf{u}) = \int_\Omega \phi(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u})\, d\Omega + \int_{\Gamma_\sigma} \varphi(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u})\, d\Gamma \qquad \forall \mathbf{u} \in \mathbb{V}} , \quad (11.2)$$

where the functions ϕ and φ are regular enough to be integrable in the domains Ω and Γ_σ, respectively. Assume, in addition, that, through adequate algebraic operations, the Gateaux derivative of $\mathbb{F}(\mathbf{u})$ can be written as

$$\boxed{\begin{array}{r} \delta\mathbb{F}(\mathbf{u}; \delta\mathbf{u}) = \displaystyle\int_\Omega \mathbb{E}(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u}) \cdot \delta\mathbf{u}\, d\Omega + \int_{\Gamma_\sigma} \mathbb{T}(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u}) \cdot \delta\mathbf{u}\, d\Gamma \\ \forall \delta\mathbf{u};\;\; \delta\mathbf{u}\Big|_{\mathbf{x}\in\Gamma_\sigma} = \mathbf{0} \end{array}} . \quad (11.3)$$

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Example 11.3 – *Obtain the Gateaux derivative, in the format given in* (11.3)*, of the functional*

$$\mathbb{F}(\mathbf{u}) = \int_a^b \phi\left(x, u(x), u'(x)\right) dx \qquad with \quad u(x)\Big|_{x=a} = u(a) = p \, .$$

Solution

The given functional is a particular case of the functional in (11.2), reduced to a single dimension with $\varphi \equiv 0$, $\Omega \equiv (a,b)$, $\Gamma_u \equiv a$ and $\Gamma_\sigma \equiv b$.

Perturbing the function $u(x)$ and replacing in the functional yields

$$\left.\begin{array}{l} u(x) \to u(x) + \varepsilon\, \eta(x) \\ u'(x) \to u'(x) + \varepsilon\, \eta'(x) \end{array}\right\} \quad \forall \eta(x) \equiv \delta u(x) \;\Big|\; \eta(a) \overset{not}{=} \eta_a = 0 \quad \Longrightarrow$$

$$\mathbb{F}(u + \varepsilon\, \eta) = \int_a^b \phi\left(x,\; u(x) + \varepsilon\, \eta,\; u'(x) + \varepsilon\, \eta'\right) dx$$

The Gateaux derivative is then

$$\delta\mathbb{F}(u;\eta) = \frac{d}{d\varepsilon}\mathbb{F}(u + \varepsilon\, \eta)\Big|_{\varepsilon=0} = \int_a^b \left(\frac{\partial \phi}{\partial u}\eta + \frac{\partial \phi}{\partial u'}\eta'\right) dx \, .$$

On the other hand, the previous expression can be integrated by parts,

$$\begin{aligned}
\int_a^b \frac{\partial \phi}{\partial u'}\eta' \, dx &= \left[\frac{\partial \phi}{\partial u'}\eta\right]_{x=a}^{x=b} - \int_a^b \left(\frac{d}{dx}\left(\frac{\partial \phi}{\partial u'}\right)\eta\right) dx = \\
&= \frac{\partial \phi}{\partial u'}\Big|_{x=b} \eta_b \; - \; \frac{\partial \phi}{\partial u'}\Big|_{x=a} \underbrace{\eta_a}_{=0} \; - \; \int_a^b \left(\frac{d}{dx}\left(\frac{\partial \phi}{\partial u'}\right)\eta\right) dx = \\
&= \frac{\partial \phi}{\partial u'}\Big|_{x=b} \eta_b - \int_a^b \left(\frac{d}{dx}\left(\frac{\partial \phi}{\partial u'}\right)\eta\right) dx \, ,
\end{aligned}$$

producing the expression

$$\delta\mathbb{F}\Big(u; \underbrace{\delta u}_{\eta}\Big) = \int_a^b \left(\frac{\partial \phi}{\partial u} - \frac{d}{dx}\left(\frac{\partial \phi}{\partial u'}\right)\right) \delta u \, dx + \frac{\partial \phi}{\partial u'}\Big|_{x=b} \delta u_b \, ,$$

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which is a particular case of (11.3) with

$$\boxed{\begin{aligned}&\mathbb{E}(x,u,u') \equiv \frac{\partial \phi}{\partial u} - \frac{d}{dx}\left(\frac{\partial \phi}{\partial u'}\right) \qquad \forall x \in (a,b)\\ &\mathbb{T}(x,u,u') \equiv \left.\frac{\partial \phi}{\partial u'}\right|_{x=b}\end{aligned}} \ .$$

11.1.2 Extrema of the Functionals. Variational Principles. Euler-Lagrange Equations

Consider a real function of a real variable $f(x):\mathbb{R}\rightarrow\mathbb{R}$. This function is said to have a minimum at $x=x_0$ when

$$f(x_0) \leq f(x) \qquad \forall x \in \mathbb{R}\ . \tag{11.4}$$

The necessary condition for f to have an extrema (maximum, minimum or saddle point) at $x=x_0$ is known to be

$$\left.\frac{df(x)}{dx}\right|_{x=x_0} \overset{not}{\equiv} f'(x_0) = 0\ . \tag{11.5}$$

This concept can be extended to the functionals in a function space. Given a functional $\mathbb{F}(\mathbf{u}):\mathbb{V}\rightarrow\mathbb{R}$, this functional is said to have a minimum at $\mathbf{u}(\mathbf{x})$ when

$$\mathbb{F}(\mathbf{u}) \leq \mathbb{F}(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbb{V}\ , \tag{11.6}$$

and a necessary condition for the functional to have an extreme (maximum, minimum or saddle point) at $\mathbf{u}(\mathbf{x})$ is that the derivative $\delta\mathbb{F}(\mathbf{u};\ \delta\mathbf{u})$ be null in every direction $\delta\mathbf{u}$,

$$\boxed{\delta\mathbb{F}(\mathbf{u};\delta\mathbf{u}) = 0 \qquad \forall \delta\mathbf{u} \quad \Big| \quad \left.\delta\mathbf{u}\right|_{\mathbf{x}\in\Gamma_u} = \mathbf{0}} \ . \tag{11.7}$$

Expressing (11.7) in the same format as (11.3) results in

$$\boxed{\begin{aligned}&\text{Variational principle}\\ &\delta\mathbb{F}(\mathbf{u};\delta\mathbf{u}) = \int_{\Omega} \mathbb{E}\cdot\delta\mathbf{u}\, d\Omega + \int_{\Gamma_\sigma} \mathbb{T}\cdot\delta\mathbf{u}\, d\Gamma = 0\\ &\qquad\qquad\qquad\qquad \forall \delta\mathbf{u} \ \Big| \ \left.\delta\mathbf{u}\right|_{\mathbf{x}\in\Gamma_\mathbf{u}} = \mathbf{0}\end{aligned}} \tag{11.8}$$

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Theorem 11.1. Fundamental Theorem of Variational Calculus*:*
Given $\mathbb{E}(\mathbf{x}) : \Omega \to \mathbb{R}^m$ *and* $\mathbb{T}(\mathbf{x}) : \Gamma_\sigma \to \mathbb{R}^m$ *that satisfy*

$$\int\limits_{\Omega} \mathbb{E}(\mathbf{x}) \cdot \delta\mathbf{u}\, d\Omega + \int\limits_{\Gamma_\sigma} \mathbb{T}(\mathbf{x}) \cdot \delta\mathbf{u}\, d\Gamma = 0 \qquad \forall \delta\mathbf{u} \;\Big|\; \delta\mathbf{u}\Big|_{\mathbf{x}\in\Gamma_u} = \mathbf{0}$$

$$\Longleftrightarrow \quad \boxed{\begin{array}{ll} \mathbb{E}(\mathbf{x}) = \mathbf{0} & \forall \mathbf{x} \in \Omega \\ \mathbb{T}(\mathbf{x}) = \mathbf{0} & \forall \mathbf{x} \in \Gamma_\sigma \end{array}}$$

Proof (indicative[1])
Consider the following choice for $\delta\mathbf{u}(\mathbf{x})$.

$$\delta\mathbf{u}(\mathbf{x}) = \begin{cases} \mathbb{E}(\mathbf{x}) & \forall \mathbf{x} \in \Omega \\ \mathbf{0} & \forall \mathbf{x} \in \Gamma_u \\ \mathbb{T}(\mathbf{x}) & \forall \mathbf{x} \in \Gamma_\sigma \end{cases}$$

Replacing in the theorem results in

$$\int\limits_{\Omega} \underbrace{\mathbb{E}(\mathbf{x}) \cdot \mathbb{E}(\mathbf{x})}_{\geq 0}\, d\Omega + \int\limits_{\Gamma_\sigma} \underbrace{\mathbb{T}(\mathbf{x}) \cdot \mathbb{T}(\mathbf{x})}_{\geq 0}\, d\Gamma = 0 \quad \Longleftrightarrow \quad \mathbb{E}(\mathbf{x}) = \mathbb{T}(\mathbf{x}) = \mathbf{0}\,.$$

Q.E.D.

Equation (11.8) is denoted as *variational principle*[2] and, since $\delta\mathbf{u}$ is arbitrary, in accordance with Theorem 11.1 it is completely equivalent to

Euler-Lagrange equations
$$\mathbb{E}(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x})) = \mathbf{0} \qquad \forall \mathbf{x} \in \Omega \tag{11.9}$$

Natural boundary conditions
$$\mathbb{T}(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla\mathbf{u}(\mathbf{x})) = \mathbf{0} \qquad \forall \mathbf{x} \in \Gamma_\sigma \tag{11.10}$$

[1] This proof is not rigorous and is provided solely as an intuitive indication of the line of reasoning followed by the theorem's proof.
[2] Strictly speaking, (11.8) is a *variational equation* or the weak form of a differential problem.

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Remark 11.3. Equations (11.9),

$$\mathbb{E}\left(\mathbf{x},\mathbf{u}\left(\mathbf{x}\right),\nabla\mathbf{u}\left(\mathbf{x}\right)\right)=\mathbf{0} \qquad \forall\mathbf{x}\in\Omega \ ,$$

are, in general, a set of partial differential equations (PDEs) known as *Euler-Lagrange equations* of the variational principle (11.8).

Equations (11.10),

$$\mathbb{T}\left(\mathbf{x},\mathbf{u}\left(\mathbf{x}\right),\nabla\mathbf{u}\left(\mathbf{x}\right)\right)=\mathbf{0} \qquad \forall\mathbf{x}\in\Gamma_\sigma \ ,$$

constitute a set of boundary conditions on these differential equations denoted as *natural or Neumann boundary conditions*. Together with the conditions (11.1),

$$\mathbf{u}\left(\mathbf{x}\right)=\mathbf{u}^*\left(\mathbf{x}\right) \qquad \forall\mathbf{x}\in\Gamma_u \ ,$$

named *essential or Dirichlet boundary conditions*, they define a system whose solution $\mathbf{u}\left(\mathbf{x}\right)$ is an extreme of the functional $\mathbb{F}$.

Example 11.4 – *Obtain the Euler-Lagrange equations and the corresponding natural and essential boundary conditions of the functional in Example 11.3,*

$$\mathbb{F}\left(u\right)=\int_a^b\phi\left(x,u\left(x\right),u'\left(x\right)\right)dx \qquad with \quad u\left(x\right)\Big|_{x=a}=u\left(a\right)=p\ .$$

Solution

From the result of Example 11.3,

$$\delta\mathbb{F}\Big(u;\underbrace{\delta u}_{\eta}\Big)=\int_a^b\left(\frac{\partial\phi}{\partial u}-\frac{d}{dx}\left(\frac{\partial\phi}{\partial u'}\right)\right)\delta u\,dx+\frac{\partial\phi}{\partial u'}\bigg|_{x=b}\delta u_b\ ,$$

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one directly obtains:

$$
\begin{array}{l}
\text{Euler-Lagrange equations :} \\
\qquad \mathbb{E}(x,u,u') \equiv \dfrac{\partial \phi}{\partial u} - \dfrac{d}{dx}\left(\dfrac{\partial \phi}{\partial u'}\right) = 0 \quad \forall x \in (a,b) \\[2ex]
\text{Natural boundary conditions :} \\
\qquad \mathbb{T}(x,u,u') \equiv \left.\dfrac{\partial \phi}{\partial u'}\right|_{x=b} = 0 \\[2ex]
\text{Essential boundary conditions :} \\
\qquad u(x)\Big|_{x=a} = u(a) = p
\end{array}
$$

11.2 Virtual Work Principle (Theorem)

Consider a material volume of the continuous medium V_t, occupying at time t the volume in space V, subjected to the body forces $\mathbf{b}(\mathbf{x},t)$ and the surface forces $\mathbf{t}^*(\mathbf{x},t)$ on the boundary Γ_σ (see Figure 11.2). Consider also the functional space $\mathbb{V}$ of all the *admissible displacements*, which satisfy the boundary condition $\mathbf{u}\big|_{\mathbf{x}\in\Gamma_u} = \mathbf{u}^*$.

$$
\boxed{\begin{array}{l}\text{Space of admissible displacements} \\ \mathbb{V} := \left\{ \mathbf{u}_t(\mathbf{x}) : V \to \mathbb{R}^3 \;\middle|\; \mathbf{u}_t(\mathbf{x})\Big|_{\mathbf{x}\in\Gamma_u} = \mathbf{u}_t^*(\mathbf{x}) \right\}\end{array}} \tag{11.11}
$$

Two of the equations governing the behavior of the medium are

$$
\text{Cauchy's equation:} \quad \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}) + \rho\,(\mathbf{b} - \mathbf{a}(\mathbf{u})) = \mathbf{0} \quad \forall \mathbf{x} \in V\ , \tag{11.12}
$$

$$
\begin{array}{c}\text{Equilibrium condition} \\ \text{at the boundary } \Gamma_\sigma\text{:}\end{array} \quad \boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n} - \mathbf{t}^* = \mathbf{0} \quad \forall \mathbf{x} \in \Gamma_\sigma\ , \tag{11.13}
$$

where the implicit dependency of the stresses on the displacements (through the strains and the constitutive equation $\boldsymbol{\sigma}(\mathbf{u}) = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}(\mathbf{u}))$) and of the accelerations on the displacements (through equation $\mathbf{a}(\mathbf{x},t) = \partial^2\mathbf{u}(\mathbf{x},t)/\partial t^2$) has been taken into account.

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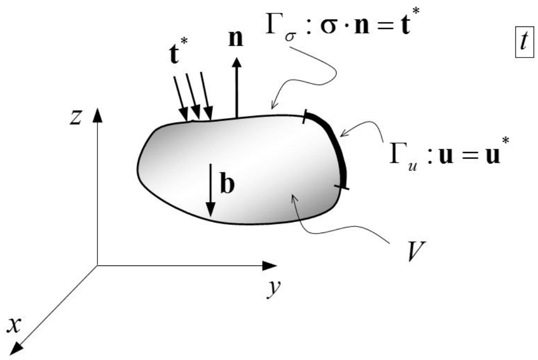

Figure 11.2: Definition of the material volume V_t.

Consider now the variational principle

$$
\boxed{
\begin{aligned}
&\delta \mathbb{W}(\mathbf{u};\delta\mathbf{u}) = \int_V \underbrace{\Big(\nabla\cdot\boldsymbol{\sigma}(\mathbf{u}) + \rho\,(\mathbf{b} - \mathbf{a}(\mathbf{u}))\Big)}_{\mathbb{E}} \cdot \delta\mathbf{u}\, dV + \\
&\quad + \int_{\Gamma_\sigma} \underbrace{\Big(\mathbf{t}^* - \boldsymbol{\sigma}(\mathbf{u})\cdot\mathbf{n}\Big)}_{\mathbb{T}} \cdot \delta\mathbf{u}\, d\Gamma = 0 \;\; ; \quad \forall \delta\mathbf{u}(\mathbf{x}) \;\Big|\; \delta\mathbf{u}\Big|_{\mathbf{x}\in\Gamma_u} = \mathbf{0} \, ,
\end{aligned}
}
\tag{11.14}
$$

where the displacement perturbations $\delta\mathbf{u}$ are denoted as *virtual displacements*.

$$
\boxed{\text{Virtual displacements:} \quad \delta\mathbf{u} : V \to \mathbb{R}^3 \;\Big|\; \delta\mathbf{u}\Big|_{\mathbf{x}\in\Gamma_u} = \mathbf{0}}
\tag{11.15}
$$

In view of (11.8) and (11.9), the Euler-Lagrange equations of the variational principle (11.14) and their natural boundary conditions are

$$
\boxed{
\begin{array}{lll}
\text{Euler-Lagrange equations:} & \mathbb{E} \equiv \nabla\cdot\boldsymbol{\sigma} + \rho\,(\mathbf{b} - \mathbf{a}) = \mathbf{0} & \forall \mathbf{x} \in \Omega \, , \\
\text{Natural boundary conditions:} & \mathbb{T} \equiv \mathbf{t}^* - \boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{0} & \forall \mathbf{x} \in \Gamma_\sigma \, ,
\end{array}
}
\tag{11.16}
$$

that is, Cauchy's equation (11.12) and the equilibrium condition at the boundary (11.13).

The variational principle (11.14) can be rewritten in a totally equivalent form as follows. Consider the term

$$\left\{\begin{array}{l}(\nabla\cdot\boldsymbol{\sigma})\cdot\delta\mathbf{u}=\nabla\cdot(\boldsymbol{\sigma}\cdot\delta\mathbf{u})-\boldsymbol{\sigma}:(\nabla\otimes\delta\mathbf{u})=\nabla\cdot(\boldsymbol{\sigma}\cdot\delta\mathbf{u})-\boldsymbol{\sigma}:(\delta\mathbf{u}\otimes\nabla)\\ \dfrac{\partial\sigma_{ij}}{\partial x_i}\delta u_j=\dfrac{\partial(\sigma_{ij}\delta u_j)}{\partial x_i}-\sigma_{ij}\dfrac{\partial(\delta u_j)}{\partial x_i}=\dfrac{\partial(\sigma_{ij}\delta u_j)}{\partial x_i}-\sigma_{ji}\dfrac{\partial(\delta u_j)}{\partial x_i}\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i,j\in\{1,2,3\}\end{array}\right. \tag{11.17}$$

and the splitting of $\delta\mathbf{u}\otimes\nabla$ into its symmetrical part, $\nabla^s\delta\mathbf{u}$, and its skew-symmetric part $\nabla^a\delta\mathbf{u}$,

$$\begin{gathered}\delta\mathbf{u}\otimes\nabla=\nabla^s\delta\mathbf{u}+\nabla^a\delta\mathbf{u}\qquad\text{with}\\ \nabla^s\delta\mathbf{u}\overset{def}{=}\frac{1}{2}(\delta\mathbf{u}\otimes\nabla+\nabla\otimes\delta\mathbf{u})\quad\text{and}\quad\nabla^a\delta\mathbf{u}\overset{def}{=}\frac{1}{2}(\delta\mathbf{u}\otimes\nabla-\nabla\otimes\delta\mathbf{u})\,.\end{gathered} \tag{11.18}$$

Introducing (11.18) in (11.17)[3] produces

$$\begin{aligned}(\nabla\cdot\boldsymbol{\sigma})\cdot\delta\mathbf{u}&=\nabla\cdot(\boldsymbol{\sigma}\cdot\delta\mathbf{u})-\boldsymbol{\sigma}:(\delta\mathbf{u}\otimes\nabla)=\\ &=\nabla\cdot(\boldsymbol{\sigma}\cdot\delta\mathbf{u})-\boldsymbol{\sigma}:\nabla^s\delta\mathbf{u}-\underbrace{\boldsymbol{\sigma}:\nabla^a\delta\mathbf{u}}_{=\mathbf{0}}\quad\Longrightarrow\end{aligned} \tag{11.19}$$

$$\boxed{(\nabla\cdot\boldsymbol{\sigma})\cdot\delta\mathbf{u}=\nabla\cdot(\boldsymbol{\sigma}\cdot\delta\mathbf{u})-\boldsymbol{\sigma}:\nabla^s\delta\mathbf{u}}\,. \tag{11.20}$$

Integrating now (11.20) over the domain V and applying the Divergence Theorem yields

$$\begin{aligned}\int\limits_V(\nabla\cdot\boldsymbol{\sigma})\cdot\delta\mathbf{u}\,dV&=\int\limits_V\nabla\cdot(\boldsymbol{\sigma}\cdot\delta\mathbf{u})\,dV-\int\limits_V\boldsymbol{\sigma}:\nabla^s\delta\mathbf{u}\,dV=\\ &=\int\limits_{\partial V=\Gamma_u\bigcup\Gamma_\sigma}\mathbf{n}\cdot(\boldsymbol{\sigma}\cdot\delta\mathbf{u})\,d\Gamma-\int\limits_V\boldsymbol{\sigma}:\nabla^s\delta\mathbf{u}\,dV=\\ &=\int\limits_{\Gamma_u}(\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\underbrace{\delta\mathbf{u}}_{=\mathbf{0}}\,d\Gamma+\int\limits_{\Gamma_\sigma}(\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\delta\mathbf{u}\,d\Gamma-\int\limits_V\boldsymbol{\sigma}:\nabla^s\delta\mathbf{u}\,dV\quad\Longrightarrow\end{aligned} \tag{11.21}$$

[3] The tensor $\boldsymbol{\sigma}$ is symmetrical and the tensor $\nabla^a\delta\mathbf{u}$ is skew-symmetric. Consequently, their product is null, $\boldsymbol{\sigma}:\nabla^a\delta\mathbf{u}=0$.

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$$\boxed{\int_V (\nabla\cdot\boldsymbol{\sigma})\cdot\delta\mathbf{u}\,dV = \int_{\Gamma_\sigma} (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\delta\mathbf{u}\,d\Gamma - \int_V \boldsymbol{\sigma} : \nabla^s\delta\mathbf{u}\,dV}\;, \qquad (11.22)$$

where the condition $\delta\mathbf{u}\big|_{\mathbf{x}\in\Gamma_u} = \mathbf{0}$ (see (11.15)) has been taken into account. Finally, introducing (11.20) in the original form of the variational principle (11.14) results in

$$\begin{aligned}
\delta\mathbb{W}(\mathbf{u};\delta\mathbf{u}) &= \int_V \big(\nabla\cdot\boldsymbol{\sigma} + \rho\,(\mathbf{b}-\mathbf{a})\big)\cdot\delta\mathbf{u}\,dV + \int_{\Gamma_\sigma} \big(\mathbf{t}^* - (\boldsymbol{\sigma}\cdot\mathbf{n})\big)\cdot\delta\mathbf{u}\,d\Gamma = \\
&= \int_V (\nabla\cdot\boldsymbol{\sigma})\cdot\delta\mathbf{u}\,dV + \int_V \rho\,(\mathbf{b}-\mathbf{a})\cdot\delta\mathbf{u}\,dV + \int_{\Gamma_\sigma} \mathbf{t}^*\cdot\delta\mathbf{u}\,d\Gamma - \int_{\Gamma_\sigma} (\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\delta\mathbf{u}\,d\Gamma = \\
&= -\int_V \boldsymbol{\sigma} : \nabla^s\delta\mathbf{u}\,dV + \int_V \rho\,(\mathbf{b}-\mathbf{a})\cdot\delta\mathbf{u}\,dV + \int_{\Gamma_\sigma} \mathbf{t}^*\cdot\delta\mathbf{u}\,d\Gamma = 0 \quad \Longrightarrow
\end{aligned} \qquad (11.23)$$

Virtual Work Principle

$$\begin{aligned}
\delta\mathbb{W}(\mathbf{u};\delta\mathbf{u}) = \int_V \rho\,(\mathbf{b}-\mathbf{a})\cdot\delta\mathbf{u}\,dV + \int_{\Gamma_\sigma} \mathbf{t}^*\cdot\delta\mathbf{u}\,d\Gamma \\
- \int_V \boldsymbol{\sigma} : \nabla^s\delta\mathbf{u}\,dV = 0 \qquad \forall\delta\mathbf{u}(\mathbf{x}) \;\big|\; \delta\mathbf{u}\Big|_{\mathbf{x}\in\Gamma_u} = \mathbf{0}
\end{aligned} \qquad (11.24)$$

Expression (11.24), which is completely equivalent to the original variational principle and maintains the same Euler-Lagrange equations and boundary conditions (11.16), is known as the *Virtual Work Principle (or Theorem)* (VWP).

Remark 11.4. The VWP is a variational principle frequently applied in solid mechanics that can be interpreted as the search of an extrema of a functional of a displacement field $\mathbb{W}(\mathbf{u})$, not necessarily known in its explicit form, whose variation (Gateaux derivative) $\delta\mathbb{W}(\mathbf{u};\delta\mathbf{u})$ is known and is given by (11.14). Since the Euler-Lagrange equations of the VWP are the Cauchy's equation (11.12) and the equilibrium condition at the boundary (11.13), its imposition is completely equivalent (yet, more convenient when solving the problem through numerical methods) to the imposition in local form of the aforementioned equations and receives the name of *weak form* of these equations.

> *Remark 11.5.* The constitutive equation does not intervene in the VWP formulation and the type of kinematics considered (finite or infinitesimal strains) is not distinguished either. Thus, the application of the VWP is not restricted by the type of constitutive equation chosen (elastic, elastoplastic, fluid, etc.) nor by the kinematics (finite or infinitesimal strains) considered.

11.2.1 Interpretation of the Virtual Work Principle

Consider the continuous medium in the present configuration V_t at time t subjected to the *fictitious* body forces $\mathbf{b}^*(\mathbf{x},t) = \mathbf{b}(\mathbf{x},t) - \mathbf{a}(\mathbf{x},t)$ and the real surface forces $\mathbf{t}^*(\mathbf{x},t)$ (see Figure 11.3), and suffering the real stresses $\boldsymbol{\sigma}(\mathbf{x},t)$. Consider, in addition, the virtual (fictitious) configuration $V_{t+\delta t}$ corresponding to time $t+\delta t$, separated from the real configuration by a virtual displacement field (11.15)

$$\boxed{\text{Virtual displacements:} \quad \delta\mathbf{u} \ \Big| \ \delta\mathbf{u}\Big|_{\mathbf{x}\in\Gamma_u} = \mathbf{0}} \ . \qquad (11.25)$$

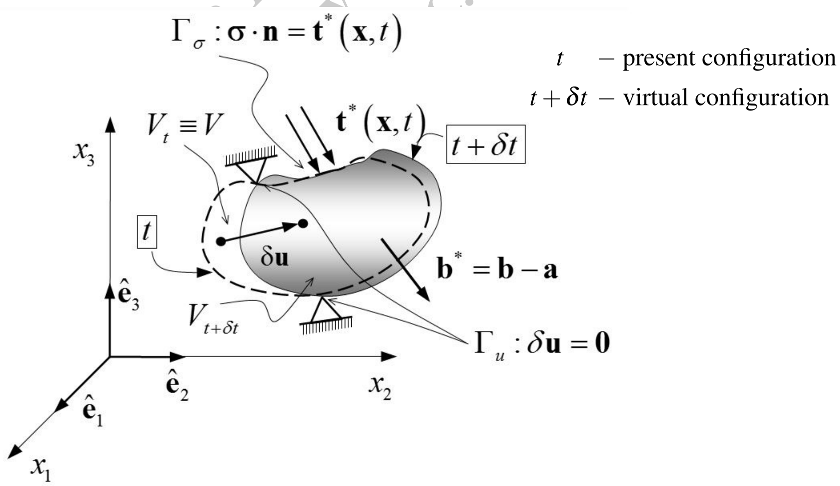

Figure 11.3: Continuous medium subjected to fictitious body forces and real surface forces.

Under *infinitesimal strain kinematics*, the virtual strains associated with the virtual displacements (11.25) are

$$\boxed{\text{Virtual strains:} \quad \delta\boldsymbol{\varepsilon} = \nabla^s \delta\mathbf{u}} \tag{11.26}$$

and, assuming that the stresses $\boldsymbol{\sigma}(\mathbf{x},t)$ remain constant along the time interval $[t, t+\delta t]$, the virtual strain work (*internal virtual work*) performed by the medium during this interval is

$$\boxed{\begin{array}{c}\text{Internal}\\ \text{virtual work:}\end{array} \quad \delta\mathbb{W}^{int} = \int_V \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\, dV = \int_V \boldsymbol{\sigma} : \nabla^s \delta\mathbf{u}\, dV} \, . \tag{11.27}$$

Likewise, assuming that both the pseudo-body forces $\mathbf{b}^*(\mathbf{x},t)$ and the surface forces $\mathbf{t}^*(\mathbf{x},t)$ remain constant during the virtual strain process in the interval $[t, t+\delta t]$, the work performed by these forces (*external virtual work*) results in

$$\boxed{\begin{array}{c}\text{External}\\ \text{virtual work:}\end{array} \quad \delta\mathbb{W}^{ext} = \int_V \rho\, \underbrace{(\mathbf{b}-\mathbf{a})}_{\mathbf{b}^*} \cdot \delta\mathbf{u}\, dV = \int_{\Gamma_\sigma} \mathbf{t}^* \cdot \delta\mathbf{u}\, dV} \tag{11.28}$$

and, comparing the VWP (11.24) with expressions (11.27) and (11.28), the VWP can be interpreted as follows.

Virtual Work Principle

$$\underbrace{\delta\mathbb{W}}_{\substack{\text{Total}\\ \text{virtual}\\ \text{work}}} = \underbrace{\int_V \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\, dV}_{\substack{\text{Internal virtual}\\ \text{work } (\delta\mathbb{W}^{int})}} - \underbrace{\int_V \rho\,(\mathbf{b}-\mathbf{a}) \cdot \delta\mathbf{u}\, dV + \int_{\Gamma_\sigma} \mathbf{t}^* \cdot \delta\mathbf{u}\, d\Gamma}_{\substack{\text{External virtual}\\ \text{work } (\delta\mathbb{W}^{ext})}} = 0$$

$$\Longrightarrow \qquad \delta\mathbb{W} = \delta\mathbb{W}^{int} - \delta\mathbb{W}^{ext} = 0 \tag{11.29}$$

for any kinematically admissible change in the virtual configuration

$$\left(\delta\mathbf{u}\Big|_{\mathbf{x}=\Gamma_u} = \mathbf{0}\right)$$

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11.2.2 Virtual Work Principle in terms of the Stress and Strain Vectors

The *vectors* of stress $\{\boldsymbol{\sigma}\}$ and virtual strain $\{\delta\boldsymbol{\varepsilon}\}$ can be extracted from the symmetrical *tensors* of stress, $\boldsymbol{\sigma}$, and virtual strain, $\delta\boldsymbol{\varepsilon} = \nabla^s \delta\mathbf{u}$, in (11.29) as follows.

$$\{\boldsymbol{\sigma}\} \in \mathbb{R}^6;\ \{\boldsymbol{\sigma}\} \overset{not}{\equiv} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} \qquad \{\delta\boldsymbol{\varepsilon}\} \in \mathbb{R}^6;\ \{\delta\boldsymbol{\varepsilon}\} \overset{not}{\equiv} \begin{bmatrix} \delta\varepsilon_x \\ \delta\varepsilon_y \\ \delta\varepsilon_z \\ \delta\gamma_{xy} \\ \delta\gamma_{xz} \\ \delta\gamma_{yz} \end{bmatrix} = \begin{bmatrix} \delta\varepsilon_x \\ \delta\varepsilon_y \\ \delta\varepsilon_z \\ 2\delta\varepsilon_{xy} \\ 2\delta\varepsilon_{xz} \\ 2\delta\varepsilon_{yz} \end{bmatrix} \tag{11.30}$$

They satisfy the equality

$$\begin{cases} \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon} = \{\boldsymbol{\sigma}\} \cdot \{\delta\boldsymbol{\varepsilon}\} = \{\delta\boldsymbol{\varepsilon}\} \cdot \{\boldsymbol{\sigma}\} \ , \\ \sigma_{ij}\delta\varepsilon_{ij} = \sigma_m \delta\varepsilon_m = \delta\varepsilon_m \sigma_m \qquad i,j \in \{1,2,3\} \ , \ m \in \{1,..6\} \ . \end{cases} \tag{11.31}$$

Replacing (11.31) in the VWP (11.29) results in

Virtual Work Principle

$$\underbrace{\delta\mathbb{W}}_{\substack{\text{Total}\\\text{virtual}\\\text{work}}} = \underbrace{\int_V \{\delta\boldsymbol{\varepsilon}\} \cdot \{\boldsymbol{\sigma}\}\, dV}_{\substack{\text{Internal virtual}\\\text{work } (\delta\mathbb{W}^{int})}} - \underbrace{\int_V \rho\,(\mathbf{b} - \mathbf{a}) \cdot \delta\mathbf{u}\, dV + \int_{\Gamma_\sigma} \mathbf{t}^* \cdot \delta\mathbf{u}\, d\Gamma}_{\substack{\text{External virtual}\\\text{work } (\delta\mathbb{W}^{ext})}} = 0$$

$$\Longrightarrow \qquad \delta\mathbb{W} = \delta\mathbb{W}^{int} - \delta\mathbb{W}^{ext} \tag{11.32}$$

for any kinematically admissible change in the virtual configuration

$$\left(\delta\mathbf{u}\Big|_{\mathbf{x}=\Gamma_u} = \mathbf{0}\right)$$

which constitutes the VWP form most commonly used in engineering.

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11.3 Potential Energy. Minimum Potential Energy Principle

The functional $\mathbb{W}$, in terms of which the variational principle (11.24) is established, can be *explicitly* formulated only under certain circumstances. One such case requires the following conditions:

1) *Linear elastic problem*

The constitutive equation can be written in terms of the elastic potential $\hat{u}(\boldsymbol{\varepsilon})$ as follows[4].

$$\boxed{\begin{array}{ll} \text{Elastic potential:} & \hat{u}(\boldsymbol{\varepsilon}) = \dfrac{1}{2}\,\boldsymbol{\varepsilon} : \underbrace{\mathbb{C} : \boldsymbol{\varepsilon}}_{\boldsymbol{\sigma}} = \dfrac{1}{2}\,\boldsymbol{\sigma} : \boldsymbol{\varepsilon} \\ & \dfrac{\partial \hat{u}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} = \mathbb{C} : \boldsymbol{\varepsilon} = \boldsymbol{\sigma} \end{array}} \tag{11.33}$$

2) *The body forces $\rho \mathbf{b}^*(\mathbf{x},t)$ are conservative*

That is, these body forces derive from a potential $\phi(\mathbf{u})$ and, thus,

$$\boxed{\frac{\partial \phi(\mathbf{u})}{\partial \mathbf{u}} = -\rho \mathbf{b}^* = -\rho(\mathbf{b} - \mathbf{a})} \;. \tag{11.34}$$

Remark 11.6. A typical case of conservative body forces is obtained for the quasi-static case ($\mathbf{a} = \mathbf{0}$) under gravitational forces and constant density,

$$\mathbf{b}(\mathbf{x},t) \overset{not}{\equiv} [0,\, 0,\, -g]^T = const. \qquad \text{and} \qquad \rho(\mathbf{x},t) = const.$$

In this case, the potential of the body forces is

$$\phi(\mathbf{u}) = -\rho \mathbf{b} \cdot \mathbf{u} \quad \Longrightarrow \quad \frac{\partial \phi(\mathbf{u})}{\partial \mathbf{u}} = -\rho \mathbf{b} \,.$$

[4] The restriction to the linear elastic problem can be made less strict and be extended to the case of *hyperelastic* materials in a finite strain regime.

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3) *The surface forces* $\mathbf{t}^*(\mathbf{x},t)$ *are conservative*
Therefore, they derive from a potential $G(\mathbf{u})$ such that

$$\boxed{\mathbf{t}^* = -\frac{\partial G(\mathbf{u})}{\partial \mathbf{u}}} \,. \tag{11.35}$$

Remark 11.7. A typical case of conservative surface forces occurs when the traction vector $\mathbf{t}^*(\mathbf{x},t)$ *is independent of the displacements*,

$$\frac{\partial \mathbf{t}^*}{\partial \mathbf{u}} = \mathbf{0} \,.$$

In this case, the potential of the surface forces is

$$G(\mathbf{u}) = -\mathbf{t}^* \cdot \mathbf{u} \quad \Longrightarrow \quad \frac{\partial G(\mathbf{u})}{\partial \mathbf{u}} = -\mathbf{t}^* \,.$$

Under the above circumstances, the following functional, named total *potential energy*, can be defined.

Potential energy

$$\underbrace{\mathbb{U}(\mathbf{u})}_{\substack{\text{Total}\\\text{potential}\\\text{energy}}} = \underbrace{\int_V \hat{u}(\boldsymbol{\varepsilon}(\mathbf{u}))\, dV}_{\substack{\text{Elastic}\\\text{energy}}} + \underbrace{\int_V \phi(\mathbf{u})\, dV}_{\substack{\text{Potential energy}\\\text{of the body forces}}} + \underbrace{\int_{\Gamma_\sigma} G(\mathbf{u})\, d\Gamma}_{\substack{\text{Potential energy of}\\\text{the surface forces}}} \tag{11.36}$$

whose Gateaux variation is

$$\begin{aligned}\delta\mathbb{U}(\mathbf{u};\delta\mathbf{u}) &= \int_V \underbrace{\frac{\partial \hat{u}}{\partial \boldsymbol{\varepsilon}}}_{\boldsymbol{\sigma}} : \underbrace{\nabla^S(\delta\mathbf{u})}_{\delta\boldsymbol{\varepsilon}}\, dV + \int_V \underbrace{\frac{\partial \phi(\mathbf{u})}{\partial \mathbf{u}}}_{-\rho(\mathbf{b}-\mathbf{a})} \cdot \delta\mathbf{u}\, dV + \int_{\Gamma_\sigma} \underbrace{\frac{\partial G(\mathbf{u})}{\partial \mathbf{u}}}_{-\mathbf{t}^*} \cdot \delta\mathbf{u}\, d\Gamma = \\ &= \int_V \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\, dV - \int_V \rho(\mathbf{b}-\mathbf{a}) \cdot \delta\mathbf{u}\, dV - \int_{\Gamma_\sigma} \mathbf{t}^* \cdot \delta\mathbf{u}\, d\Gamma \,; \quad \forall \delta\mathbf{u} \mid \delta\mathbf{u}\big|_{\mathbf{x}\in\Gamma_u} = \mathbf{0}\,,\end{aligned} \tag{11.37}$$

where (11.33) to (11.35) have been taken into account.

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Comparing (11.37) with the VWP (11.29) leads to

$$\delta\mathbb{W} \equiv \delta\mathbb{U}(\mathbf{u};\delta\mathbf{u}) = \int\limits_V \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\, dV - \int\limits_V \rho\,(\mathbf{b}-\mathbf{a})\cdot\delta\mathbf{u}\, dV - \int\limits_{\Gamma_\sigma} \mathbf{t}^*\cdot\delta\mathbf{u}\, d\Gamma = 0 \qquad \forall\delta\mathbf{u} \;\big|\; \delta\mathbf{u}\big|_{\mathbf{x}\in\Gamma_u} = \mathbf{0}\,. \tag{11.38}$$

Definition 11.3. *Minimum Potential Energy Principle*:
The variational principle (11.38), which is still the weak form of Cauchy's equation (11.12) and the equilibrium condition at the boundary (11.13), is now the Gateaux variation of the *potential energy* functional $\mathbb{U}(\mathbf{u})$ in (11.36). Consequently, this functional, which for the case of constant body and surface forces takes the form

$$\mathbb{U}(\mathbf{u}) = \int\limits_V \underbrace{\frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}}_{\hat{u}(\boldsymbol{\varepsilon})}\, dV - \int\limits_V \rho\,(\mathbf{b}-\mathbf{a})\cdot\mathbf{u}\, dV - \int\limits_{\Gamma_\sigma} \mathbf{t}^*\cdot\mathbf{u}\, d\Gamma\,,$$

presents an extreme (which can be proven to be a minimum[5]*) for the solution to the linear elastic problem.*

[5] The condition of minimum of an extreme is proven by means of the thermodynamic requirement that $\mathbb{C}$ be positive-definite (see Chapter 6).

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PROBLEMS

Problem 11.1 – *From the expression of the Virtual Work Principle,*

$$\int_{V_0} \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\, dV_0 = \int_{V_0} \rho\mathbf{b}\cdot\delta\mathbf{u}\, dV_0 + \int_{\Gamma_\sigma} \mathbf{t}\cdot\delta\mathbf{u}\, d\Gamma \qquad \forall\,\delta\mathbf{u}\;\big|\;\delta\mathbf{u}=\mathbf{0} \quad in\ \Gamma_u\,,$$

prove the Minimum Potential Energy Principle for a linear elastic material under infinitesimal strain regime.

Solution

A linear elastic material is a particular type of hyperelastic material and, thus, there must exist an elastic potential of the type

$$\exists\,\mathbb{W}(\boldsymbol{\varepsilon}) \quad \Big| \quad \frac{\partial\mathbb{W}}{\partial\varepsilon_{ij}} = \sigma_{ij} \quad\Longrightarrow\quad \delta\mathbb{W} = \sigma_{ij}\,\delta\varepsilon_{ij} = \boldsymbol{\sigma}:\delta\boldsymbol{\varepsilon}\,.$$

In addition, if the external forces are conservative, the following is satisfied:

$$\exists \quad \mathbb{G}(\mathbf{u}) \;\Big|\; \mathbf{t}^* = -\frac{\partial\mathbb{G}(\mathbf{u})}{\partial\mathbf{u}} \quad\Longrightarrow\quad \delta\mathbb{G} = -\mathbf{t}^*\cdot\delta\mathbf{u}$$

$$\exists \quad \Phi(\mathbf{u}) \;\Big|\; \rho\mathbf{b} = -\frac{\partial\Phi(\mathbf{u})}{\partial\mathbf{u}} \quad\Longrightarrow\quad \delta\Phi = -\rho\mathbf{b}\cdot\delta\mathbf{u}$$

Now, the given expression of the Virtual Work Principle can be rewritten as

$$\int_{V_0} \delta\mathbb{W}\, dV_0 + \int_{V_0} \delta\Phi\, dV_0 + \int_{\Gamma_\sigma} \delta\mathbb{G}\, d\Gamma = 0 \qquad \forall\,\delta\mathbf{u}\,\big|\;\delta\mathbf{u}=\mathbf{0} \quad in\ \Gamma_u\,,$$

$$\delta\left(\int_{V_0} \mathbb{W}\, dV_0 + \int_{V_0} \Phi\, dV_0 + \int_{\Gamma_\sigma} \mathbb{G}\, d\Gamma\right) = 0 \qquad \forall\,\delta\mathbf{u}\,\big|\;\delta\mathbf{u}=\mathbf{0} \quad in\ \Gamma_u\,.$$

Defining the *total potential energy* as

$$\mathbb{U}(\mathbf{u}) = \int_{V_0} \mathbb{W}\, dV_0 + \int_{V_0} \Phi\, dV_0 + \int_{\Gamma_\sigma} \mathbb{G}\, d\Gamma$$

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leads to

$$\delta\mathbb{U} = \mathbf{0} \qquad \forall\, \delta\mathbf{u} \,\big|\; \delta\mathbf{u} = \mathbf{0} \quad in\ \Gamma_u \ ,$$

which is the same as stating that $\mathbb{U}$ has an extreme at $\mathbf{u}$. To prove that this extreme is a minimum, consider

$$\mathbb{W}(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} \qquad \text{where} \quad \mathbb{C}_{ijkl} = \frac{\partial^2 \mathbb{W}}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \ .$$

Then, the expressions for $\mathbb{U}(\mathbf{u})$ and $\mathbb{U}(\mathbf{u}+\delta\mathbf{u})$ are computed as

$$\mathbb{U}(\mathbf{u}) = \int\limits_{V_0} \frac{1}{2}\boldsymbol{\varepsilon}(\mathbf{u}) : \mathbb{C} : \boldsymbol{\varepsilon}(\mathbf{u})\, dV_0 - \int\limits_{V_0} \rho\mathbf{b}\cdot(\mathbf{u})\, dV_0 - \int\limits_{\Gamma_\sigma} \mathbf{t}^*\cdot\mathbf{u}\, d\Gamma \qquad \text{and}$$

$$\begin{aligned}\mathbb{U}(\mathbf{u}+\delta\mathbf{u}) = & \int\limits_{V_0} \frac{1}{2}\boldsymbol{\varepsilon}(\mathbf{u}+\delta\mathbf{u}) : \mathbb{C} : \boldsymbol{\varepsilon}(\mathbf{u}+\delta\mathbf{u})\, dV_0 - \int\limits_{V_0} \rho\mathbf{b}\cdot(\mathbf{u}+\delta\mathbf{u})\, dV_0 \\ & - \int\limits_{\Gamma_\sigma} \mathbf{t}^*\cdot(\mathbf{u}+\delta\mathbf{u})\, d\Gamma \ .\end{aligned}$$

Taking into account

$$\boldsymbol{\varepsilon}(\mathbf{u}+\delta\mathbf{u}) = \boldsymbol{\varepsilon}(\mathbf{u}) + \boldsymbol{\varepsilon}(\delta\mathbf{u})$$

results in the following expression for the subtraction $\mathbb{U}(\mathbf{u}+\delta\mathbf{u}) - \mathbb{U}(\mathbf{u})$:

$$\begin{aligned}\mathbb{U}(\mathbf{u}+\delta\mathbf{u}) - \mathbb{U}(\mathbf{u}) = & \int\limits_{V_0} \frac{1}{2}\boldsymbol{\varepsilon}(\mathbf{u}) : \mathbb{C} : \boldsymbol{\varepsilon}(\delta\mathbf{u})\, dV_0 + \int\limits_{V_0} \frac{1}{2}\boldsymbol{\varepsilon}(\delta\mathbf{u}) : \mathbb{C} : \boldsymbol{\varepsilon}(\mathbf{u})\, dV_0 \\ & + \int\limits_{V_0} \frac{1}{2}\boldsymbol{\varepsilon}(\delta\mathbf{u}) : \mathbb{C} : \boldsymbol{\varepsilon}(\delta\mathbf{u})\, dV_0 - \int\limits_{V_0} \rho\mathbf{b}\cdot\delta\mathbf{u}\, dV_0 - \int\limits_{\Gamma_\sigma} \mathbf{t}^*\cdot\delta\mathbf{u}\, d\Gamma\end{aligned}$$

Introducing

$$\boldsymbol{\varepsilon}(\mathbf{u}) : \mathbb{C} : \boldsymbol{\varepsilon}(\delta\mathbf{u}) = \boldsymbol{\varepsilon}(\delta\mathbf{u}) : \mathbb{C} : \boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}$$

reduces the subtraction to

$$\begin{aligned}\mathbb{U}(\mathbf{u}+\delta\mathbf{u}) - \mathbb{U}(\mathbf{u}) = & \int\limits_{V_0} \boldsymbol{\sigma} : \delta\boldsymbol{\varepsilon}\, dV_0 + \int\limits_{V_0} \frac{1}{2}\boldsymbol{\varepsilon}(\delta\mathbf{u}) : \mathbb{C} : \boldsymbol{\varepsilon}(\delta\mathbf{u})\, dV_0 \\ & - \int\limits_{V_0} \rho\mathbf{b}\cdot\delta\mathbf{u}\, dV_0 - \int\limits_{\Gamma_\sigma} \mathbf{t}^*\cdot\delta\mathbf{u}\, d\Gamma \ .\end{aligned}$$

Now, considering the previous expression and

$$\delta \mathbb{U} = \int\limits_{V_0} \boldsymbol{\sigma} : \delta \boldsymbol{\varepsilon} \, dV_0 - \int\limits_{V_0} \rho \mathbf{b} \cdot \delta \mathbf{u} \, dV_0 - \int\limits_{\Gamma_\sigma} \mathbf{t}^* \cdot \delta \mathbf{u} \, d\Gamma$$

yields

$$\mathbb{U}(\mathbf{u} + \delta \mathbf{u}) - \mathbb{U}(\mathbf{u}) = \int\limits_{V_0} \frac{1}{2} \boldsymbol{\varepsilon}(\delta \mathbf{u}) : \mathbb{C} : \boldsymbol{\varepsilon}(\delta \mathbf{u}) \, dV_0 \, .$$

Finally, since the tensor $\mathbb{C}_{ijkl} = \partial^2 \mathbb{W} / (\partial \varepsilon_{ij} \partial \varepsilon_{kl})$ is positive-definite,

$$\boxed{\mathbb{U}(\mathbf{u} + \delta \mathbf{u}) - \mathbb{U}(\mathbf{u}) \geq 0}$$

and, thus, the potential energy is seen to have a *minimum* in the equilibrium state.

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